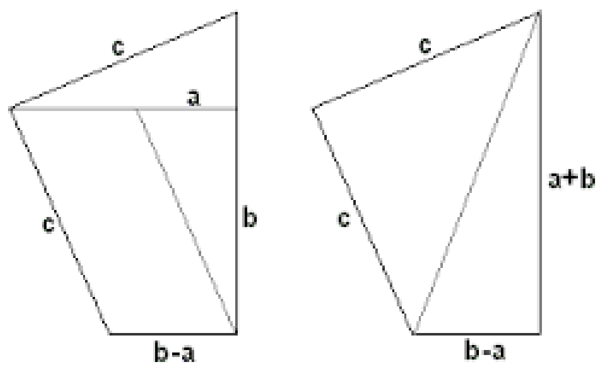
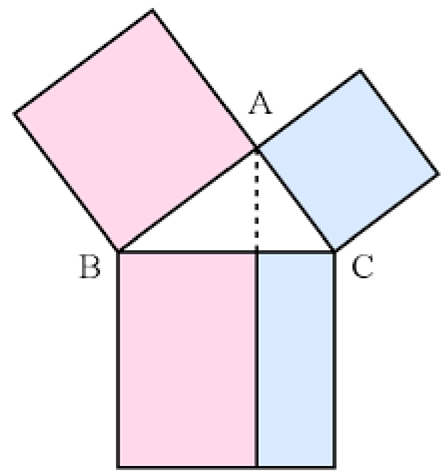
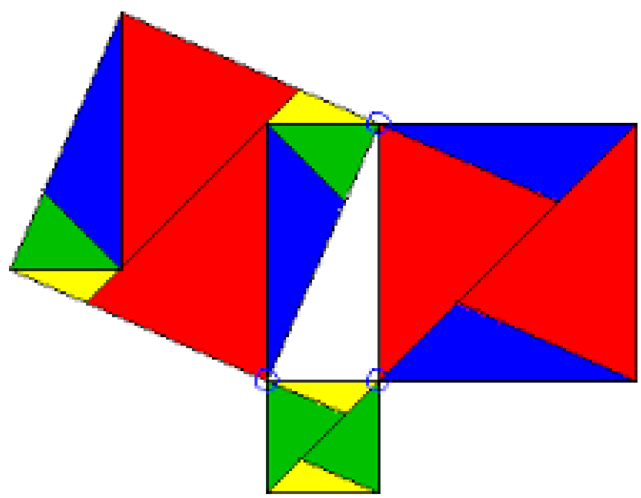
Geometry Part 2







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Advice for students:

1. This class is about problem-solving, which often involves applying material learned to unfamiliar problems. We will cover basic concepts thoroughly, practice with routine problems, and then often attack harder problems where you have to apply new and previously-learned concepts. These can be difficult, and one of my main goals this semester is to help you develop important problem-solving skills. In the face of challenges, try to be persistent, patient, and creative. Be proactive about seeking help when you need it. ***We do not expect everyone to get all homework problems correct; the important thing is to try your best!***

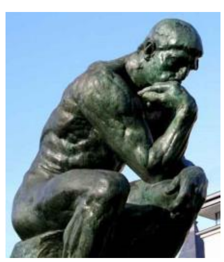
2. In sports, you have practices. In performing arts, you have rehearsals. In academics, you have homework. Homework in this class is designed to help you cement your understanding of the material by practicing straightforward problems and develop your problem-solving skills. Do your best to try all the homework questions. If you have difficulty, come to the next class with specific questions that will help you advance your understanding.

3. The problem sets in this book comes with answers. They are at the end of each individual problem set. Checking your answers is essential. We recommend that you check answers after every few problems to make sure you are on the right track. We all make mistakes; please let us know if you think you have found an error in the answers.

4. You will never need a formal textbook in class or for homework. All problems will be assigned from this book or supplemental handouts.

5. HELP!!! Get help when you need it. Some places (in no particular order):
- -Classmates
- -Parents
- -Textbook (if you have one around)
- -Your teacher
- -Internet (believe it or not, Google can help you find great explanations and practice problems)
- -Khan Academy videos: goto www.khanacademy.org

6. Think more… memorize less!

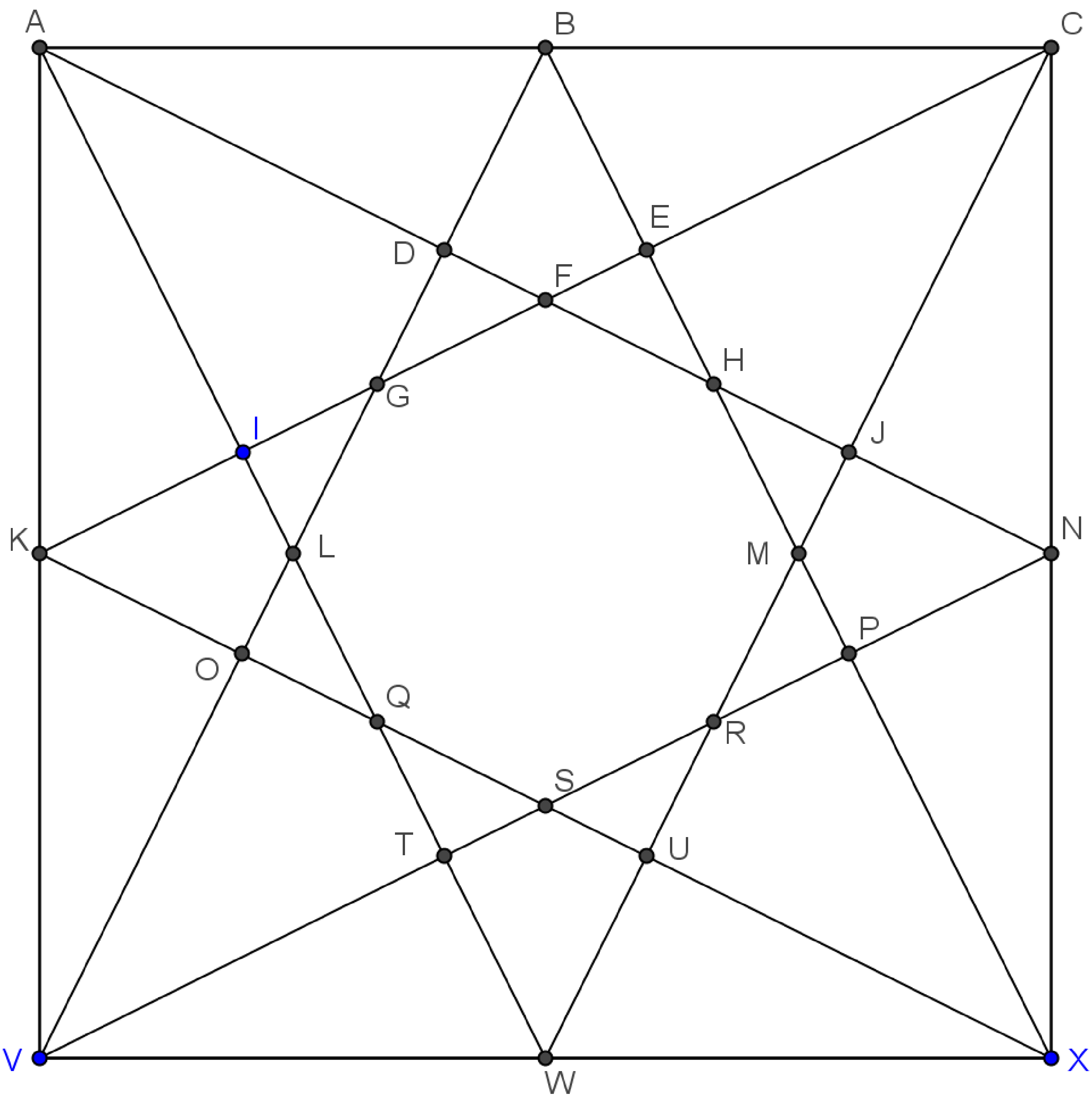


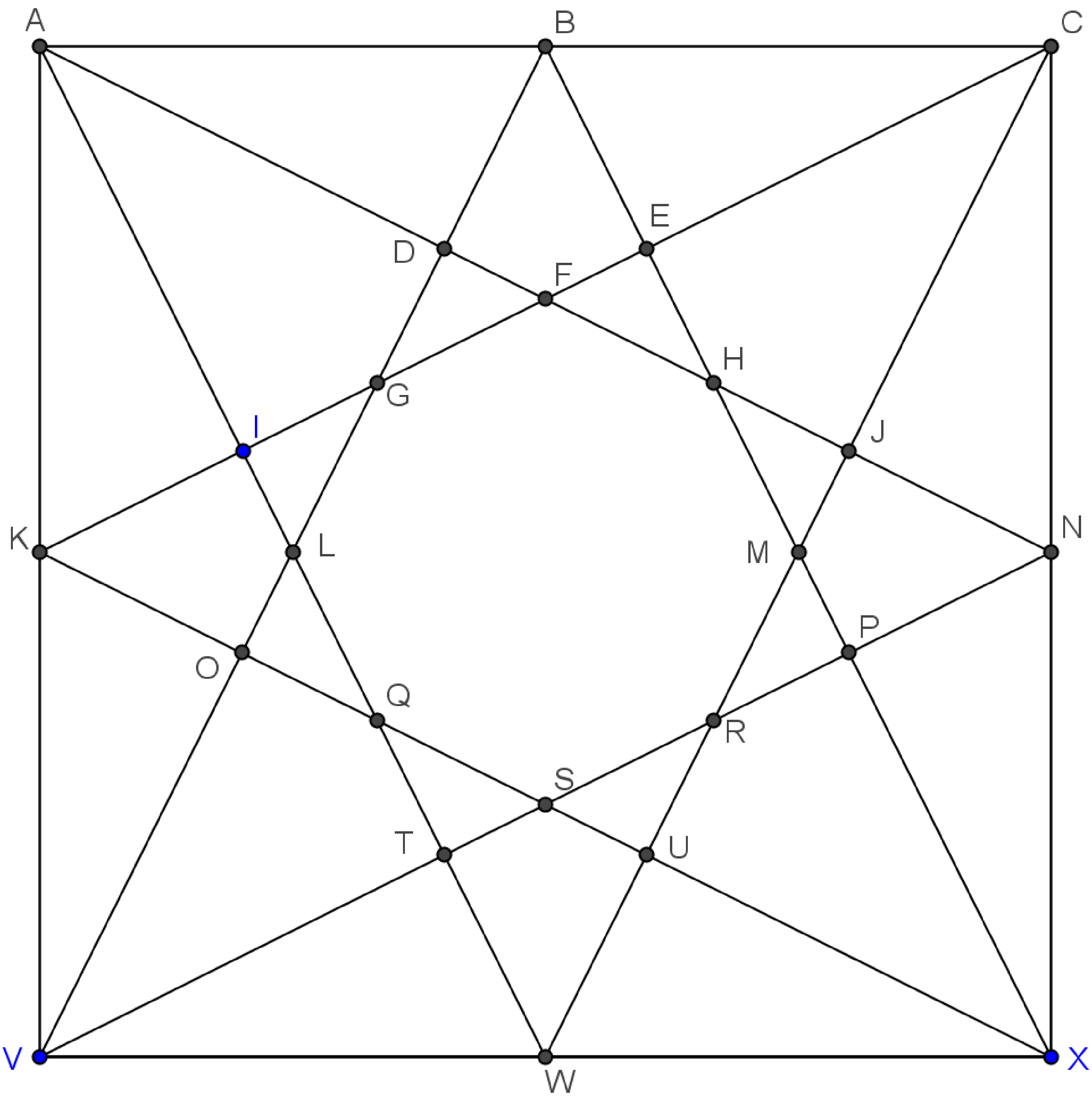
7. Problems with boxes around their numbers, like 17. or e. tend to be harder. We encourage those who are considering taking accelerated math courses to try them.

Contents

Geometry 1 Review Activity

In the diagram below, square ACXV has a side length of 2. Points B, N, W, and K are the midpoints of the square's sides. They are each connected to the opposite side corners with line segments. There are a few extra copies of this. So tear a page out if it makes things easier!

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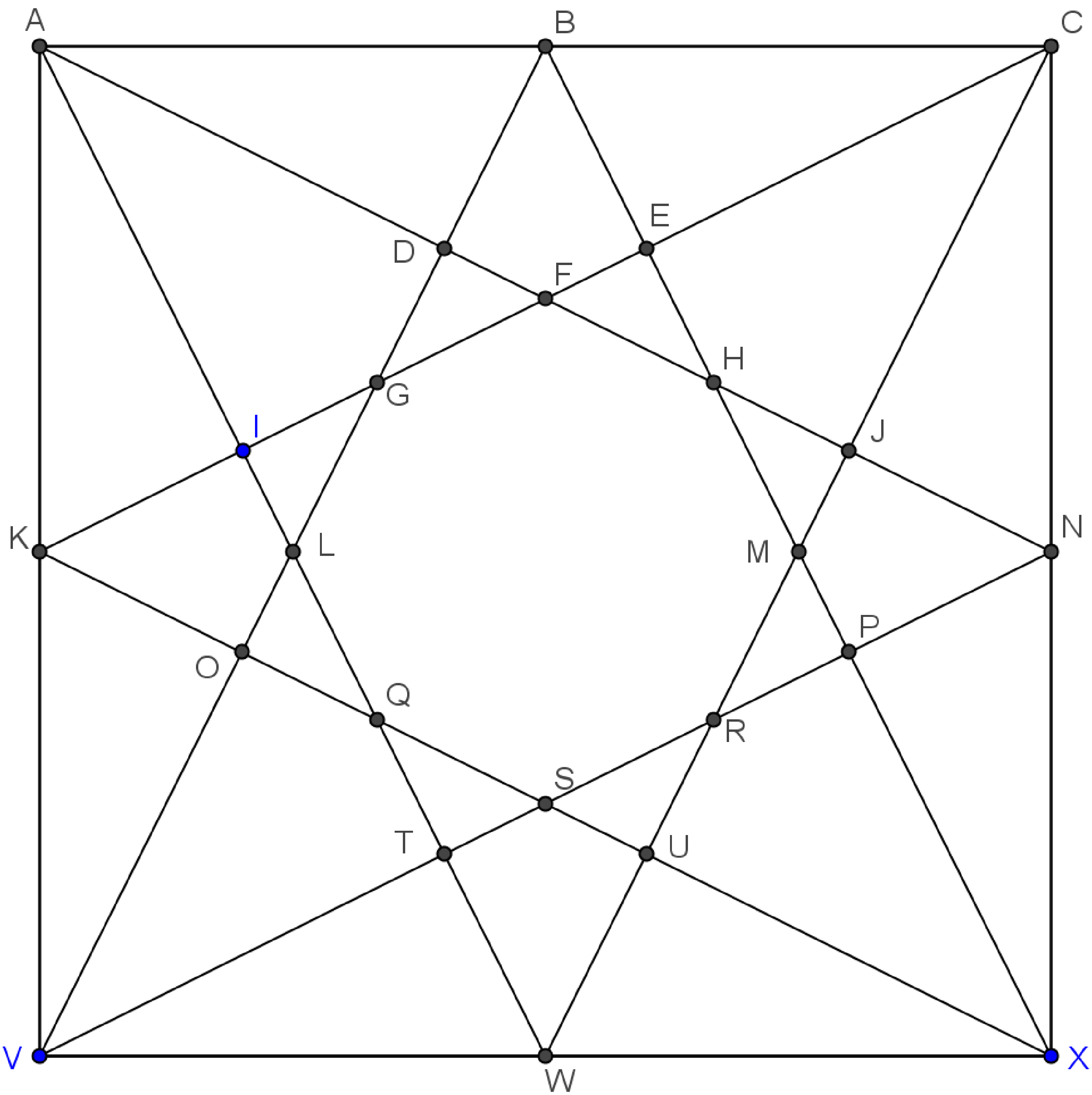
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1. Explain why $\angle ADB$ must be a right angle. (hint: use the coordinate plane, making point V the origin and finding the slopes of lines $\overline{AD}$ *and* $\overline{BD}$.)

2. Using either symmetry or parallel lines, which other points have line segments meeting at right angles? (you don't need to include the vertices of the square!)

3. Explain why $\angle BAD \cong \angle BCE$. Use CPCTC—find the correct (right?) triangle!

4. Let $\angle BAD$ measure $x°$. Now find the measure of all of the following angles, in terms of x:

a. $\angle ABD$ b. $\angle DBE$ c. $\angle DFE$ d. $\angle CNA$

e. $\angle DGF$ f. $\angle DAI$ g. $\angle IGD$

5. Explain why $\Delta CAK \cong \Delta CXW$ and thus $\angle BCE \cong \angle NCJ$.

6. Explain why $\Delta CBE \cong \Delta CNJ$ and thus $\overline{CE} \cong \overline{CJ}$.

7. Prove that $\overline{HE} \cong \overline{HJ}$. You may want to draw $\overline{CH}$ (any form of proof is ok).

8. Find all triangles congruent to ΔCAN (without drawing any more line segments).

9. Find triangles similar to—but not congruent to— ΔHEF. How many different-sized triangles are all similar to ΔHEF but not congruent to each other?

10. Assume that $\overline{EF} < \overline{EH}$ (we'll check on this later!). What possible values can x take on?

11. There is one possible value of x that would make the octagon in the center of the diagram equiangular. What value is that? Note: it turns out that this is not the value of x, so the octagon is not equiangular.

12. Must an equiangular octagon be regular? Explain.

13. Show that $\Delta ACN \sim \Delta ADB$ and use this to find $\overline{BD}$.

14. Find the area of ΔACF. It may help to think about what portion of the large square it represents.

15. Must F be the midpoint of $\overline{AN}$? Justify your answer.

16. Explain why the areas of ΔACF and ΔCFN are equal, even though the two triangles are not congruent.

17. Are squares IEPT and DJUO congruent? Explain.

18. Let V be the origin and W and X be on the positive *x*-axis. Find the coordinates of the following points. Some may require solving a system of linear equations.

a. W   b. K   c. C   d. A

e. F

f. E

g. H

h. J

19. Find the lengths of the following segments:

a. $\overline{AN}$

b. $\overline{BF}$

c. $\overline{HF}$

d. $\overline{EF}$

e. $\overline{EH}$

20. Find (without drawing any more line segments)

a. A rhombus that is not a square.

b. As many non-congruent trapezoids as you can.

c. As many non-congruent isosceles triangles as you can.

d. A parallelogram with no right angles—and find its area.

e. As many non-congruent kites as you can.

21. Is MJNP similar to LDNT? Explain.

22. Challenge: what is the area of octagon GFHMRSQL?

Some answers

1. slope of AN is -1/2 and slope of VB is 2, so they are negative reciprocals 2. J, O, U, T, I, P, E

3. ΔACK and ΔCAN are congruent by SAS since both have sides of 2 and 1 with a right angle between them

4a. 90-x b. 2x c. 180-2x d. 90-x e. 90-2x f. 90-2x g. 90+2x

5. by SAS (sides of 1 & 2 with a right angle between them)

6. ASA where all angles are same and BC=CN=1; CPCTC

7. draw CH and use HL

8. ACK, VXK, XVN, XCW, CXB, AVB, and VAW

9. CJF, HDB, CIW.. three (?)

10. x>22.5 degrees ; also x<45 or else some angles would be negative!

11. 22.5 degrees, to make 90+2x equal to 180-2x

12. no.. may have sides of different lengths

13. both have right angles and angle A so they are similar… BD=$1/\sqrt{5}$ using proportions

14. same base and $1/4^{th}$ of the height so it is $1/8^{th}$ of the square and its area = ½

15. yes.. AFC is congruent to KFN so AF=KN… .perhaps we also know diagonals of rectangle bisect each other

16. ACF has base of 2 and height of ½; for area of ½; CFN has base of 1 and height of 1 for area of ½… another way: look at CAN and draw altitude from C to AN. Both triangles have the same height and their bases are equal, so their areas are equal.

17. yes.. their sides are each the distance between parallel lines the same distance apart…

Or use CUX congruent to XOV and CJN congruent to XUW so can go from there…

18a. (1,0) b. (0,1) c. (2,2) d. (0,2) e. (1,1.5) f. (1.2, 1.6) g. (4/3, 4/3) h. (1.6, 1.2)

19a. $\sqrt{5}$ b. ½ c. $\frac{\sqrt{5}}{6}$ d. $\frac{\sqrt{5}}{10}$ e. $\frac{2\sqrt{5}}{15}$

21. Yes; draw LN and see that NJM and NDL are similar by AA, so kites are similar too

22. square ODJU has area 0.8; from #19, EH=$\frac{2\sqrt{5}}{15}$ and EF=$\frac{\sqrt{5}}{10}$ so ΔEFH has area 1/30 and 3 other triangles are congruent… so 4/5 – 4/30 = 2/3

Unit 6 Handout #1: Algebra and Similar Triangle Review

In this unit we will be solving some quadratic equations and working with radicals.

Solving quadratic equations:
-If there is only one term with an *x* in it, then isolate that term and take the square root of both sides, then solve for *x*. Be sure to take the ± square roots!
-If there is more than one term with an *x* in it, try to solve by factoring. Make one side zero and try to factor the other side. Then set each factor to zero and solve. This works because of the ***zero-product property***: if the product of two numbers is zero, then one of them must be zero.

Working with radicals:
-Simplify a radical by looking for perfect square factors.
-Add and subtract "like terms"
-Multiply or divide terms even if they are not "like terms" if they are both square roots.

Example #1: Solve $2(x-3)^2 - 5 = 27$

Isolate the $(x-3)^2$ term by adding 5 to both sides and dividing by 2: $(x-3)^2 = 16$

Take the square root of both sides: $(x-3) = \pm\sqrt{16} = \pm 4$

Solve the resulting equations: $x - 3 = 4 \text{ so } x = 7 \quad \text{or } x - 3 = -4 \text{ so } x = -1$

Example #2: Solve $x^2 + 10 = 7x$

Note that there are two terms with *x*'s, so we cannot use the method in example #1 above.

Set one side to zero, by subtracting 7*x* from both sides: $x^2 - 7x + 10 = 0$

Factor the left side: $(x-2)(x-5) = 0$

Set each factor to zero: $x - 2 = 0 \text{ or } x - 5 = 0 \quad \text{so } x = 2 \text{ or } x = 5$

Example #3: Simplify the following: a. $\sqrt{75}$ **b.** $\sqrt{12} + \sqrt{27}$ **c.** $\sqrt{12} \cdot \sqrt{15}$

a. 25 is a perfect square and a factor or 75 so $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$

b. Simplify each and see if they are like terms $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$; $\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$

they are like terms so they can be added $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$

c. Multiply under the radical and then simplify: $\sqrt{12} \cdot \sqrt{15} = \sqrt{12 \cdot 15} = \sqrt{180} = \sqrt{36}\sqrt{5} = 6\sqrt{5}$

1. Solve the following equations:

a. $x^2 - 3x - 10 = 0$

b. $x^2 = 9x + 10$

c. $x^2 + 4x = 0$

d. $(x+1)(x-4) = 14$

e. $2x^2 + 12 = 10x$

f. $2x^2 + x = 6$

g. $x^2 + 9 = 25$

h. $x^2 - 32 = 0$ (don't factor!)

i. $3x^2 - 8 = 2(x+1)^2 - 5$

j. $3x^2 - 6x = 45$

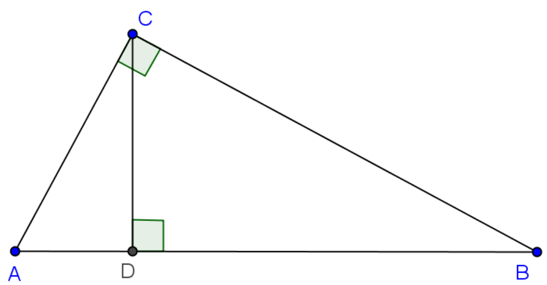
2. Simplify the following radical expressions, or state that they cannot be simplified:

a. $\sqrt{12}$ b. $\sqrt{18}+\sqrt{8}$ c. $\sqrt{75}$

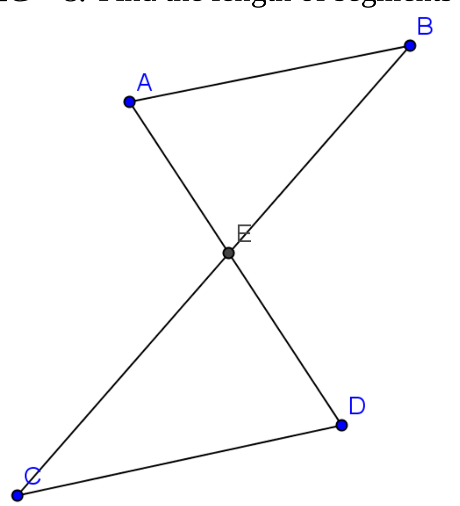
d. $\sqrt{27}-\sqrt{48}$ e. $\sqrt{8}+\sqrt{18}$ f. $\sqrt{8}\cdot\sqrt{12}$

g. $\left(2\sqrt{3}\right)^2$ h. $3\sqrt{2}\cdot 4\sqrt{6}$ i. $\dfrac{9\sqrt{3}}{3}$

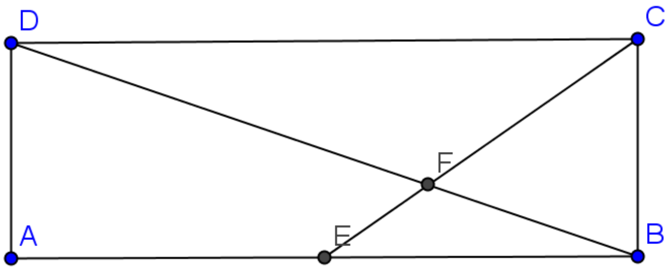
3. Identify the similar triangles below; be sure to name them appropriately.



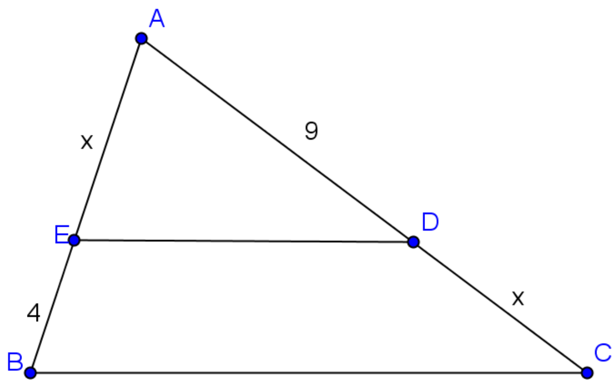
4. In the diagram below, $\overline{AB} \,//\, \overline{CD}$ and $\overline{AB}=5,\ \overline{AE}=4,\ \overline{ED}=6,\ and\ \overline{EC}=8.$ Find the length of segments $\overline{CD}\ and\ \overline{BE}.$



5. In rectangle ABCD below, E is the midpoint of side $\overline{AB}$. If CE=18, then what is $\overline{CF}$?

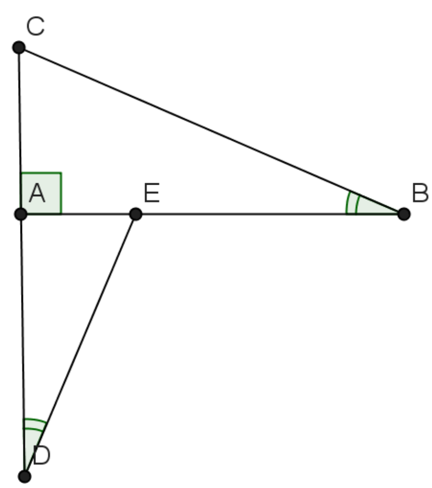


6. Find x in the diagram below, given that $\angle B \cong \angle AED$.



7. Use the diagram below to answer the following questions:

a. Given AC=6, DE=10 and BC=15, find AE.

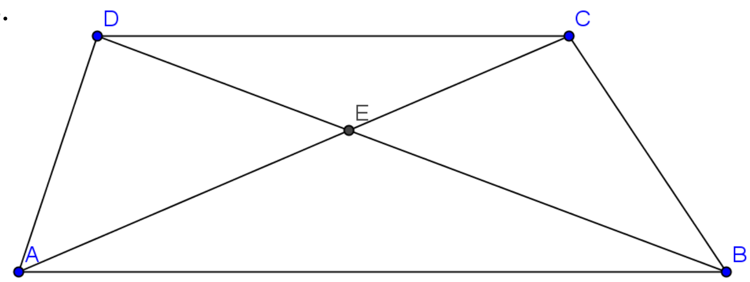


b. Instead, AD=10, CD=16, and AE=4. Find BE.

c. Instead, BE=11, AD=7, and AC=6. Find AE.

8. In the trapezoid below, CD=15, AB=20 and AC=18.

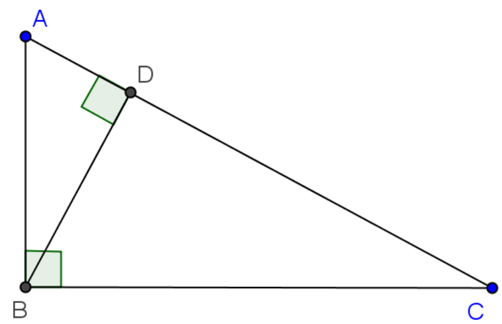
a. Find CE.



b. If $\overline{BE}$ is 3 units longer than $\overline{DE}$, then find DE.

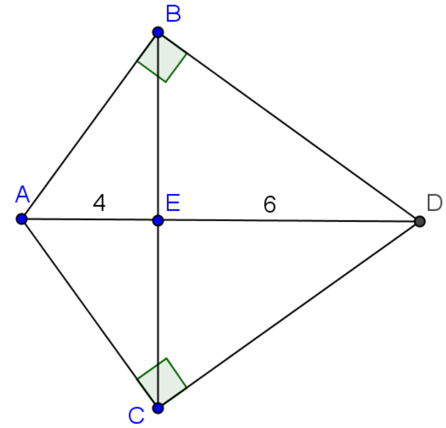
9. Altitude $\overline{BD}$ is drawn in right triangle ABC below.

a. If AB=8 and AC=18, then find the length of $\overline{AD}$.

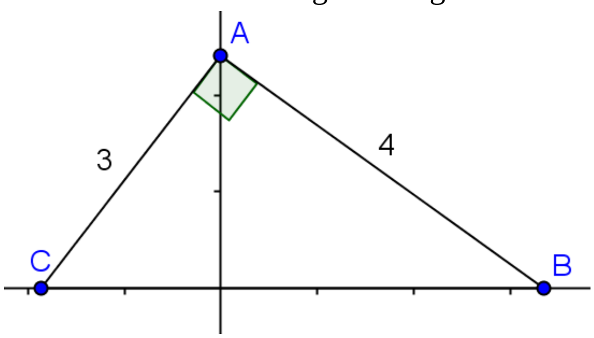


b. Instead, if AB=8 and CD=12, then find the length of $\overline{AD}$.

10. A kite has two right angles. Its longer diagonal is broken into segments of length 6 and 4 by the shorter diagonal. How long is the shorter diagonal? (remember that the diagonals of a kite are perpendicular)



11. Andi has a 3-4-5 right triangle and wants to place it on the coordinate plane such that right angle A is on the *y*-axis, the hypotenuse is on the *x*-axis, and the leg of side 4 is in the first quadrant. See the diagram below. Use similar triangles or algebra to find the coordinates of the three vertices.



Answers

1a. 5, -2 b. 10, -1 c. 0, -4 d. 6, -3 e. 2, 3 f. -2, 1.5 g. 4, -4 h. $\pm\sqrt{32} = \pm 4\sqrt{2}$ i. 5, -1j. 5, -3

2a. $2\sqrt{3}$ b. $5\sqrt{2}$ c. $5\sqrt{3}$ d. $-\sqrt{3}$ e. $5\sqrt{2}$ f. $\sqrt{96} = 4\sqrt{6}$

g. 12 h. $24\sqrt{3}$ i. $3\sqrt{3}$

3. $\Delta ACB \sim \Delta ADC \sim \Delta CDB$ 4. CD=7.5 and BE=16/3 5. 12 6. 6

7a. 4 b. 11 c. 3 (-14 makes no sense!) 8a. 54/7 b. 9 9a. 32/9 b. 4 10. $4\sqrt{6}$

11. (0,2.4), (3.2,0), and (-1.8,0)

Unit 6 Handout #2: Similar Right Triangles and the Pythagorean Theorem

An altitude of a right triangle creates three similar right triangles. Both smaller triangles are similar to the largest triangle because they each share one acute angle and also have a right angle.

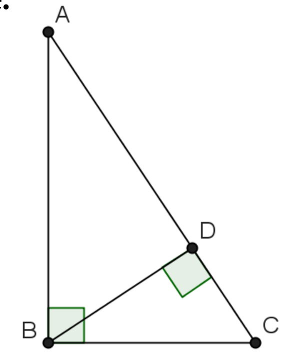
The three similar right triangles can be used to prove the Pythagorean Theorem. The Pythagorean Theorem states that in any right triangle with legs *a* and *b* and hypotenuse *c*, $c^2 = a^2 + b^2$.

Example #1. In the triangle below, B is a right angle and BD is an altitude.

a. Name three similar triangles

b. If AD=12 and BC=8, then find DC.

c. Instead, if AD=10 units and DC=5 units then find BD.



Solution:

a. ΔABC~ΔADB~ΔBDC since all have right angles and the first two both have angle A and the first and third both have angle B.

b. ΔABC~ΔBDC so $\frac{AC}{BC} = \frac{BC}{CD}$. Letting DC=*x* we get $\frac{12+x}{8} = \frac{8}{x}$.

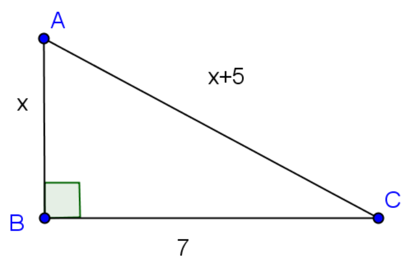
So $x^2 + 12x = 64$ *and* $x^2 + 12x - 64 = 0$ *and* $(x+16)(x-4) = 0$ *so* $x = 4$ (x = -16 makes no sense!)

c. ΔADB~ΔBDC so $\frac{AD}{BD} = \frac{BD}{CD}$. Letting BD=*x* we get $\frac{10}{x} = \frac{x}{5}$.

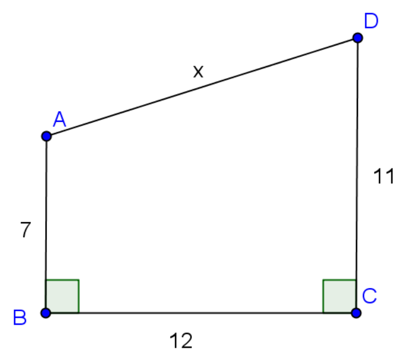
Thus $x^2 = 50$ *and* $x = \sqrt{50} = 5\sqrt{2}$

2. Example #2: Find *x* in each diagram below:

a.



b.



a. Using the Pythagorean Theorem, $x^2 + 7^2 = (x+5)^2$ so $x^2 + 49 = x^2 + 10x + 25$.

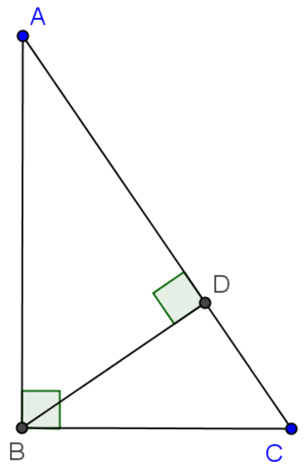
Thus $24 = 10x$ and $x = 2.4$

b. Draw a segment from A perpendicular to $\overline{CD}$; let it meet at point E. We know that $\overline{DE}$ is 11-7, or 4. And also, $\overline{AE} \perp \overline{DE}$ because ABCE is a rectangle. Thus AE=BC=12. And we can use the Pythagorean Theorem in triangle AED, giving us $12^2 + 4^2 = x^2$. So $x^2 = 160$ *and* $x = 4\sqrt{10}$.

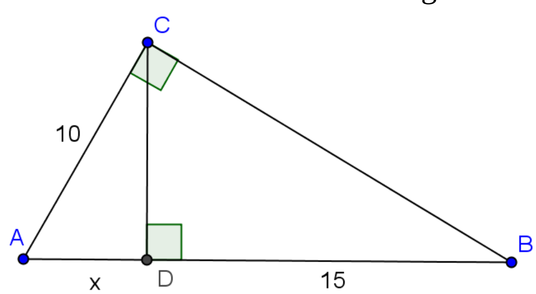
1. Use the diagram below to answer the following questions:

a. If BC=6 and AC=12, then find CD.

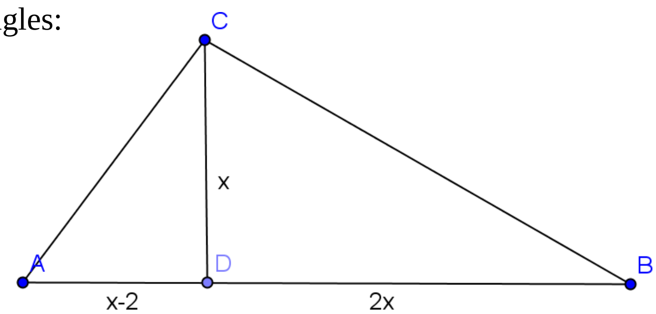
b. Instead, if BD=6 and AC=13 then find CD. (given that CD < AD)



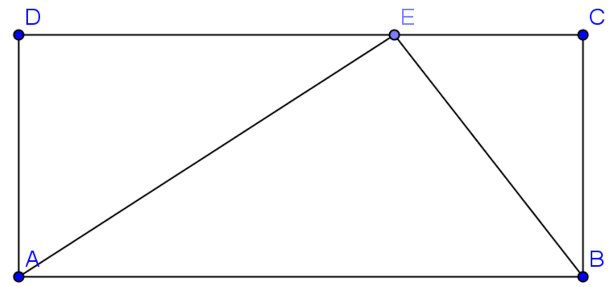
2. What is the value of *x* in the diagram below?



3. Find *x* in the diagram to the right. ACB and D are right angles:

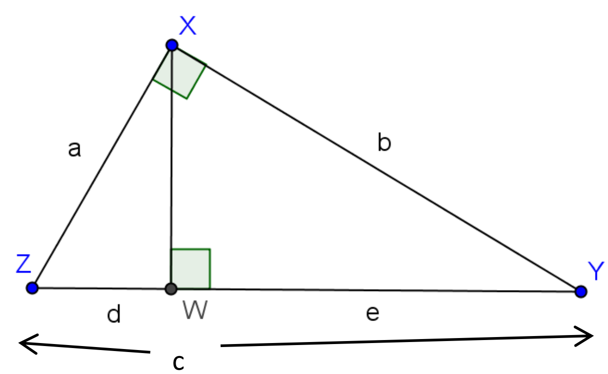


4. In the diagram below, ABCD is a rectangle and E is a right angle. If AB=12 and BC= $2\sqrt{5}$, then find EC. Note: CE<ED. Look for similar triangles!



5. In the diagram below, do the following:

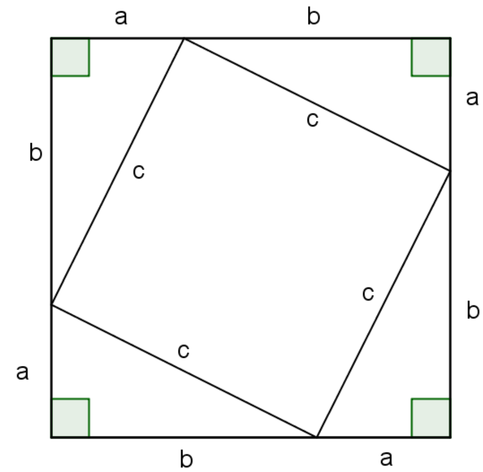
a. Show that $a^2 = cd$.



b. Show that $b^2 = ce$

c. Use your results to show that $a^2 + b^2 = c^2$

6. While we have not formally studied area yet, you should know formulas for the areas of rectangles and triangles. Answer the questions below about the diagram on the right.



a. Why must the rhombus with sides of *c* be a square?

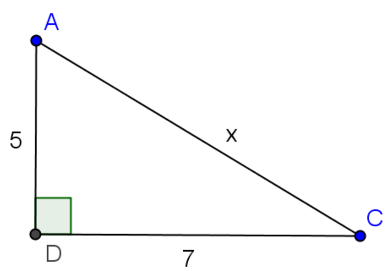
b. Find the area of the large square in terms of *a* and *b*.

c. Find the area of the large square by adding the areas of the four right triangles to the smaller square in the center.

d. How do your answers to parts *b* and *c* prove the Pythagorean Theorem?

7. Use the Pythagorean Theorem to find *x* in each part below.

a.



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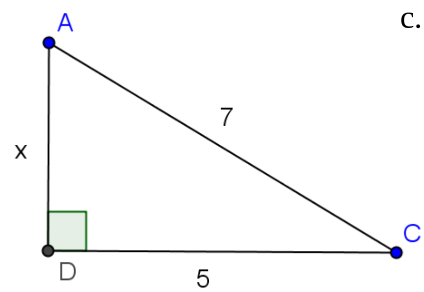
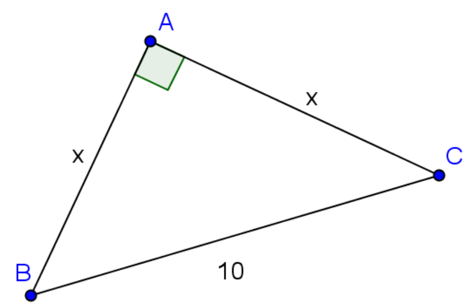
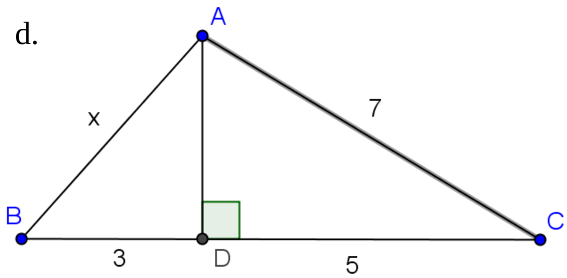


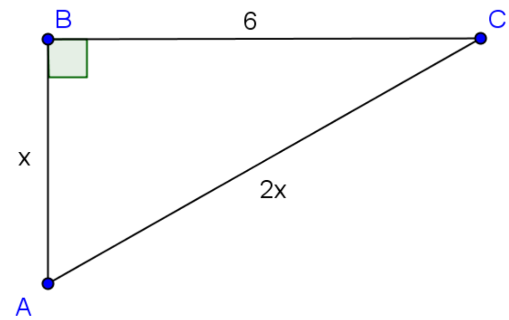
c.



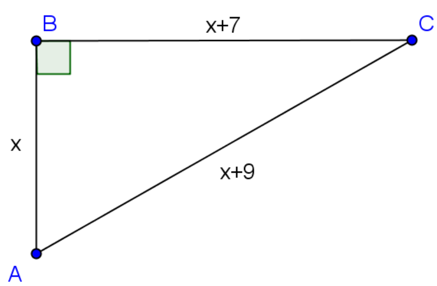
d.



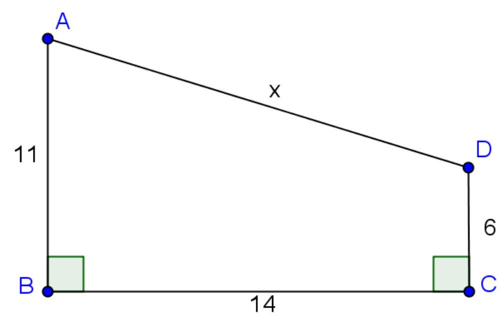
e.



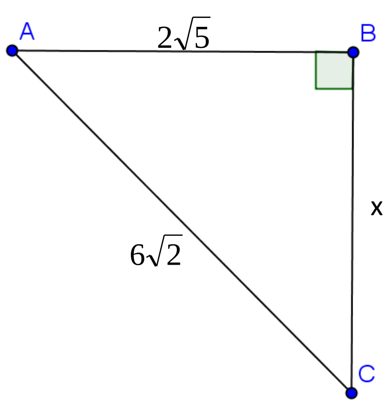
f.



g.



8. Find the value of *x* below.

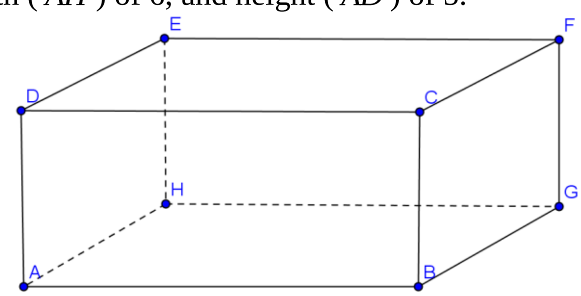


9. An isosceles triangle has a base of 12 and legs of 10. What is the length of the altitude to the base?

10. What is the converse of the Pythagorean Theorem? Do you think it is true? Explain.

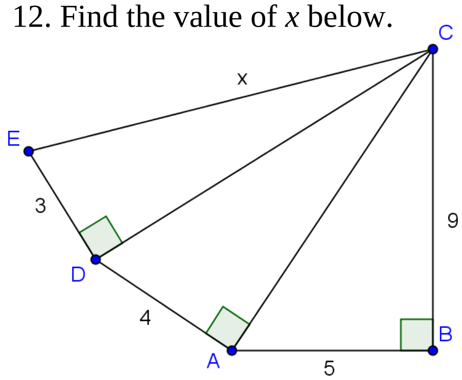
11. A box (a rectangular prism) has length ($\overline{AB}$) of 12, depth ($\overline{AH}$) of 6, and height ($\overline{AD}$) of 5.

a. What is the length of the diagonal of its bottom, $\overline{AG}$?

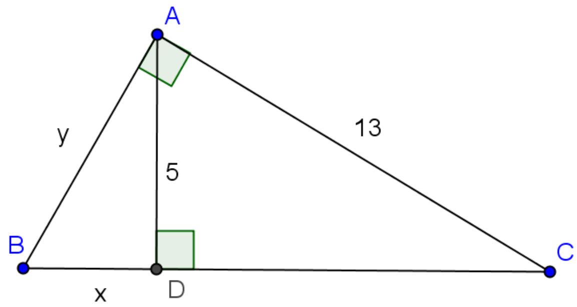


b. What is the length of the diagonal of the box $\overline{AF}$?

12. Find the value of *x* below.



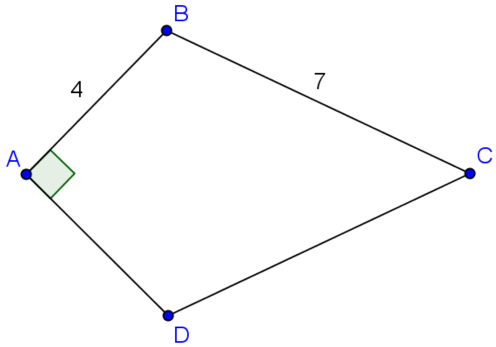
13. Find the value of *x* below, given that angles D and BAC are both right angles.



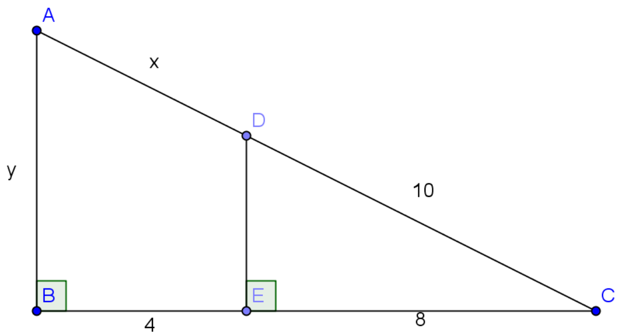
14. What is the length of an altitude of an equilateral triangle with side length of 6?

15. An isosceles trapezoid has bases of 6 and 10 and a perimeter of 24. How long are its diagonals?

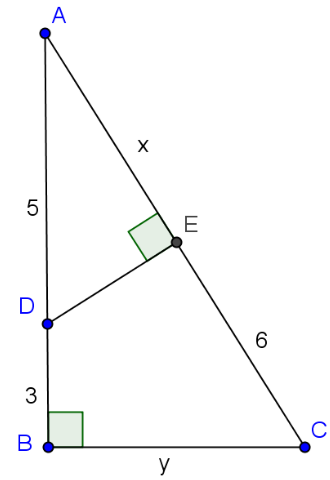
16. Two sides of the kite below are 4 and 7, and angle A measures 90°. Find the lengths of the diagonals.



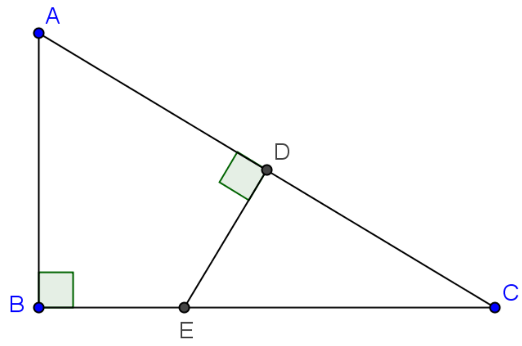
17. Find *x* and *y* in the diagram below.



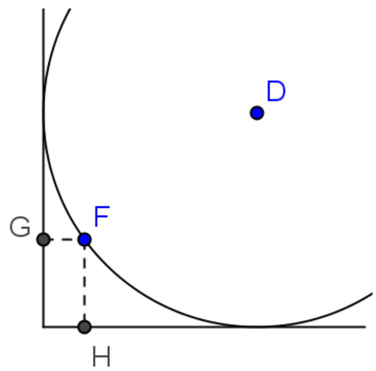
18. Find the values of *x* and *y* in the diagram below.



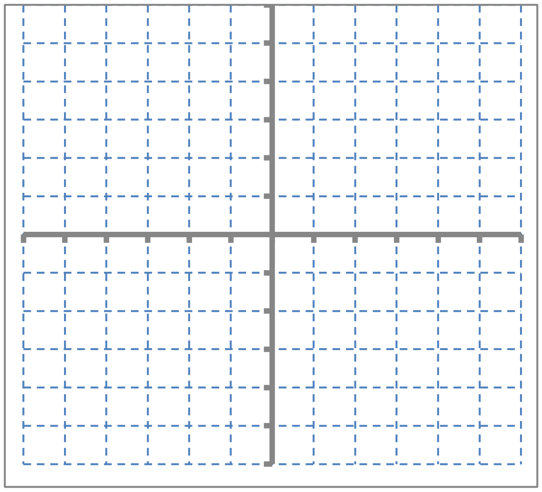
19. In the diagram below, $\overline{ED}$ is the perpendicular bisector of $\overline{AC}$. Given AB=3 and BC=4 find ED.



20. A circle is wedged into a corner (right angle), tangent to both sides. One point on the circle is 2 units from one wall (FG) and four units (FH) from the other. What is the radius of the circle?



21. Use the Pythagorean Theorem to answer the following questions about distances on the coordinate plane. Simplify all radicals. While the distance formula applies the Pythagorean Theorem, most of these will be better answering drawing triangles rather than directly plugging things into the distance formula.



a. Find the coordinates of all points on the *x*-axis five units from (0,3).

b. Find the coordinates of all points on the *x*-axis seven units from (0,3).

c. Find the coordinates of all points on line *y*=5 five units from (0,1).

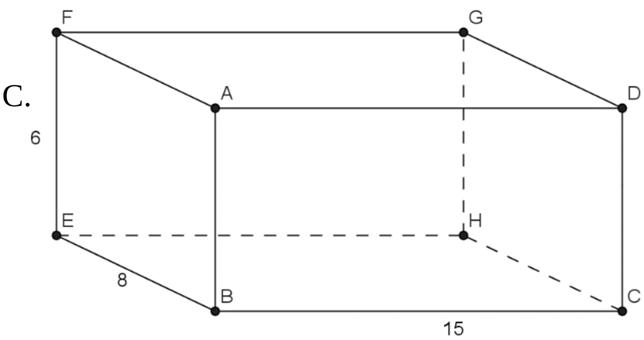
d. Find the coordinates of all points on line *y*=5 five units from (-2,2).

e. Find the coordinates of all points on line *y*=5 four units from (2,3).

f. Find the coordinates of all points on line *y*=5 four units from (3,0).

22. Four ants are at corner F of a brick below. They are trying to find the shortest way to get to the opposite corner, C. They obviously cannot walk through the brick; they must stay on its surface.

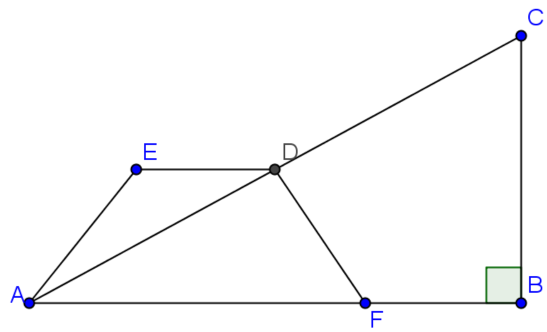
-Amelia walks to A and then diagonally across the face to C.
-Barry walks diagonally to B and then along the edge to C.
-Carmen walks across the top to D and then down the edge to C.



a. Whose walk is the shortest? Use decimals.

b. Daisy has found the shortest possible way to get from F to C on the surface of the brick. What does she do and how long is her walk?

23. AEDF is an isosceles trapezoid with ED=4, AF=10, and AE=5. D is the midpoint of $\overline{AC}$. Find BF.



Answers

1a. 3 b. 4 2. 5 3. 4 4. 2 (or 10)

6a. the larger and smaller acute angles of each triangle are complementary b. $(a+b)^2$ c. $4(0.5ab)+c^2$

d. set them equal to each other and cancel the 2ab (if you don't have a 2ab on one side, then FOIL!)

7a. $\sqrt{74}$ b. $2\sqrt{6}$ c. $5\sqrt{2}$ d. $\sqrt{33}$ e. $2\sqrt{3}$ f. 8 g. $\sqrt{221}$

8. $2\sqrt{13}$ 9. 8 10. If $a^2+b^2=c^2$ then the triangle is a right triangle. Yes, it is true

11a. $6\sqrt{5}$ b. $\sqrt{205}$ 12. $\sqrt{131}$ 13. 25/12 14. $3\sqrt{3}$ 15. $2\sqrt{19}$

16. BD= $4\sqrt{2}$ and AC= $2\sqrt{2}+\sqrt{41}$

17. x=5; y=9 18. x=4 and y=6 19. 15/8

20. 10 (draw right triangle with hypot DF…x is hypoth and legs are x-2 and x-4…)

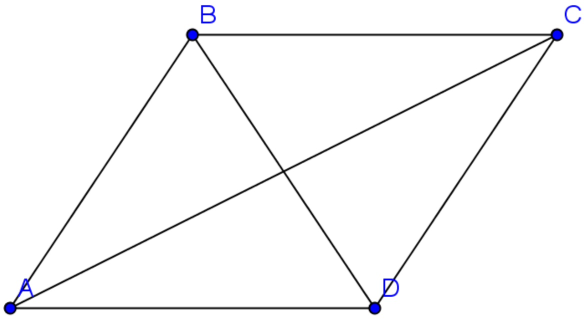
21a. (4,0), (-4,0) b. $(2\sqrt{10},0)$, $(-2\sqrt{10},0)$ c. (3,5), (-3,5) d. (-6,5), (2,5) e. $(2+2\sqrt{3},5)$, $(2-2\sqrt{3},5)$ f. none

22. Amelia≈24.16 Barry=25 and Carmen = 23 b. 20.52—"unfold" prism along AD and Daisy walks the hypotenuse of the resulting right triangle with legs of 14 and 15. Clever Daisy!

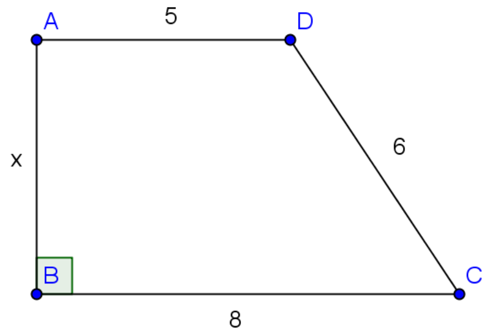
23. 4 (BC=8 since when you drop perp from D to AB, ΔABD ~ ΔABC and its sides are ½ as long…)

Unit 6 Handout #3: More Pythagorean Theorem

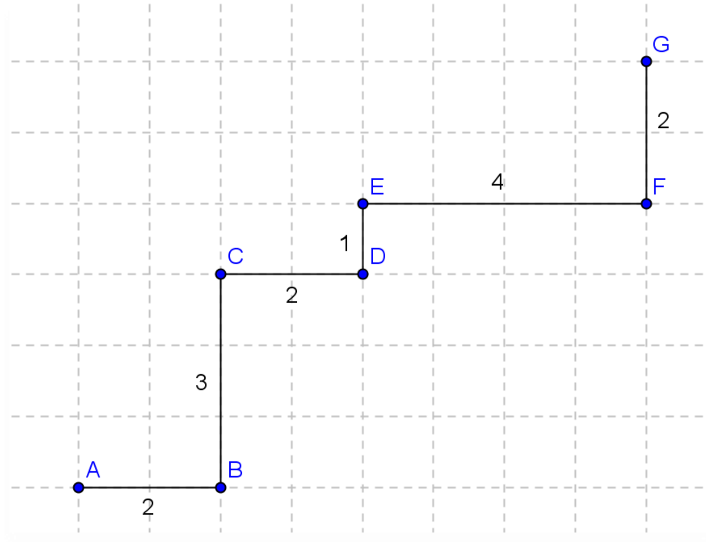
1. The diagonals of rhombus ABCD below have lengths of 10 and 14. How long is each side of the rhombus? You will need to remember a few properties of rhombi for this one!



2. Find the value of *x* in the trapezoid below:

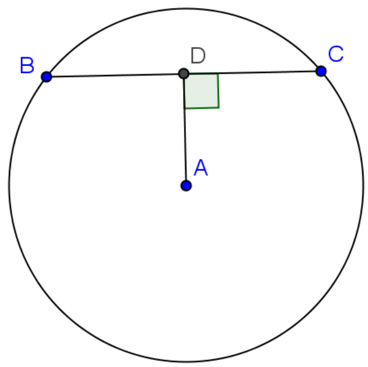


3. Tina is wandering through Manhattan, zig-zagging along the streets and avenues from point A to G below. All of her turns are 90°. She walks a total of 14 units. But how far is G from A (as the crow flies)?



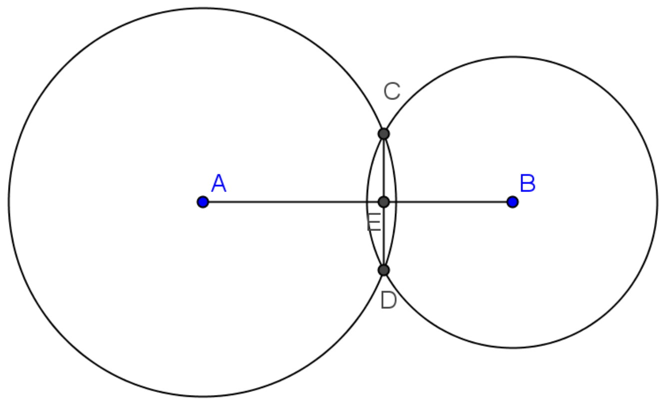
4. Circle A below has a diameter of 12 and chord $\overline{BC}$ is 10 units long.

a. Explain why the perpendicular segment from the center to a chord must bisect that chord.

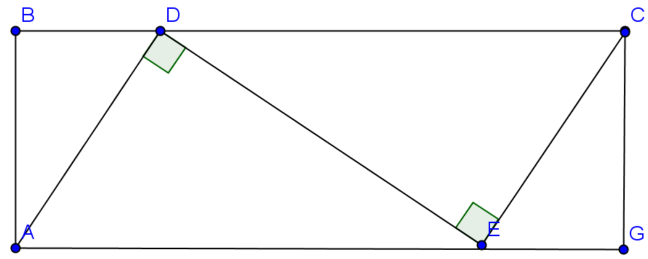


b. Find the length of $\overline{AD}$.

5. Circles with radii of 6 and 8 have a common chord $\overline{CD}$ with length 4. How far apart are their centers?



6. In rectangle ABCG, points D and E on sides $\overline{BC}$ and $\overline{AG}$ have been drawn to create right angles. The lengths of $\overline{AB}$ *and* $\overline{BD}$ are 8 and 6, respectively.



a. Find the length of $\overline{AD}$.

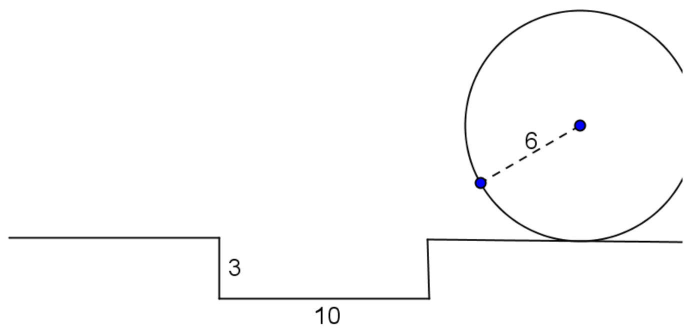
b. Why is ΔABD~ΔEDA?

c. Find the length of $\overline{DE}$.

d. What is the length of segment $\overline{AG}$?

7. A wheel with radius 6 is rolling down the road. There is a trench in the road 10 units long and 3 units deep.

a. Does the wheel hit the bottom of the trench? Justify your answer with numbers.

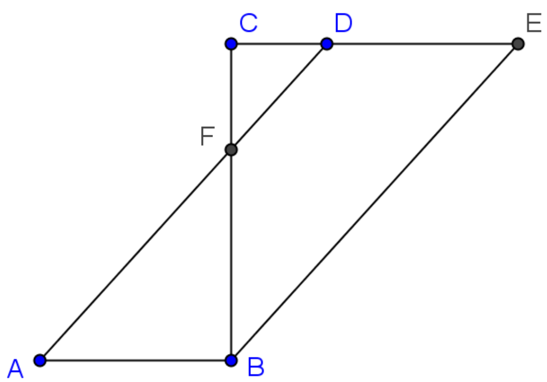


b. What radius wheel would just hit the bottom of the trench?

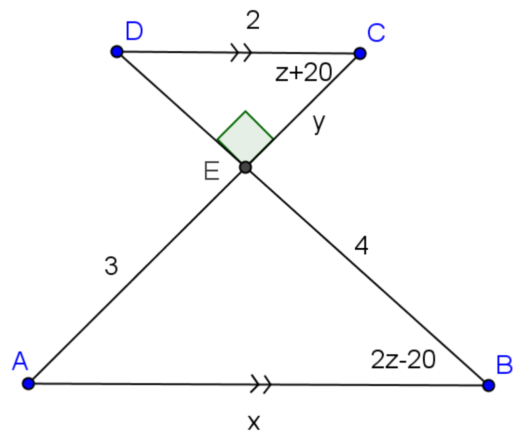
8. Verify that the following are all "Pythagorean Triples"—possible sides for right triangles. You may use a calculator.

a. 3, 4, 5 b. 5, 12, 13 c. 7, 24, 25 d. 8, 15, 17 e. 3*a*, 4*a*, 5*a*

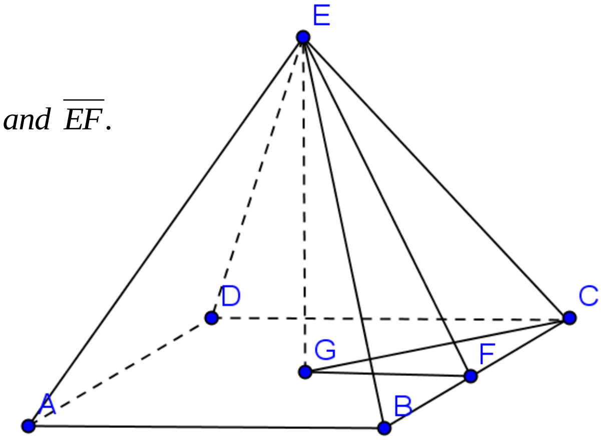
9. In the diagram below, ABED is a parallelogram; angle C is a right angle and points F and D are on segments BC and CE respectively. Given that CD=3, DE=6, and BE=15, find CF.



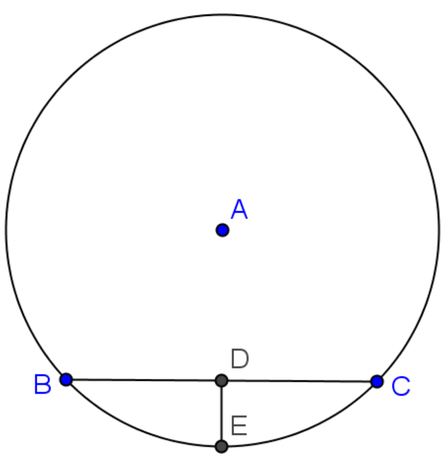
10. Find *x*, *y*, and, *z* below:



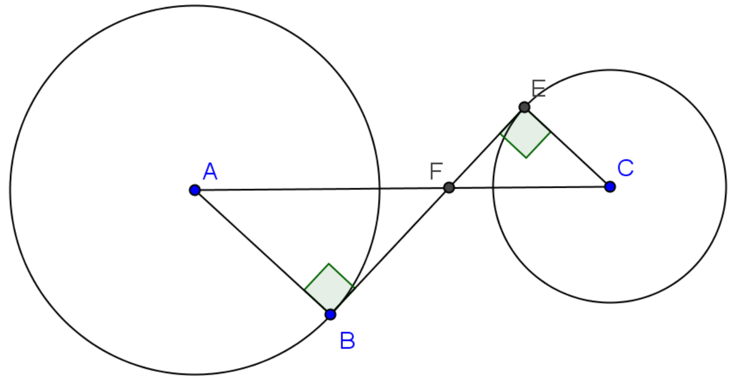
11. The pyramid below has a square base (ABCD) of side 6.
-Point G is the center of the square, and point E is directly above G;
- $\overline{EG}$ is perpendicular to the plane containing the square base, so it is (by definition) perpendicular to all lines in that plane that go through point G.
-F is the midpoint of $\overline{BC}$ so $\overline{FG} \perp \overline{BC}$.
-If $\overline{EC}$ measures 8 units, then find the lengths of $\overline{EG}$ *and* $\overline{EF}$.



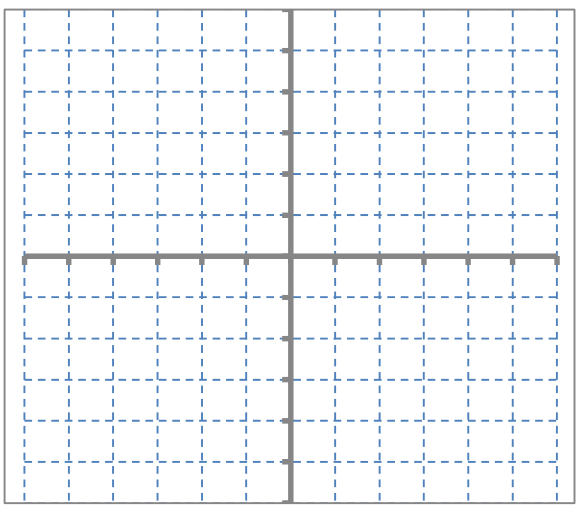
12. A bit of water is flowing through a circular draining pipe: the cross-section is shown below. The water is 3 feet deep (DE) and the width of the water's surface (BC) is 12 feet. What is the radius of the circle? Hint: call it *x* and find a right triangle where one side is known and two sides can be written in terms of *x*.



13. The radii of circles A and C below are 8 and 4 respectively. Their centers are 16 units apart. Find the length of $\overline{BE}$.

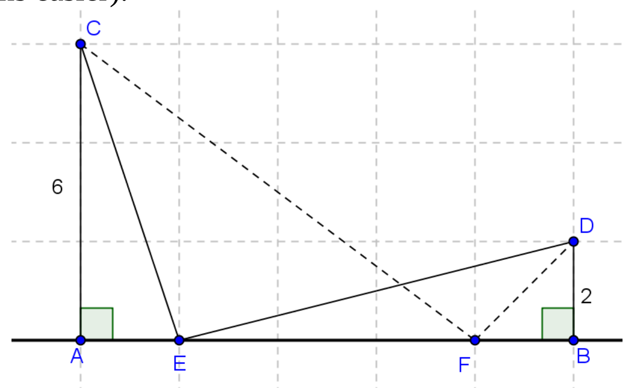


14. How far is the intersection of the lines $y = -0.5x + 3$ and $y = x - 2$ from the point (-1,0)?

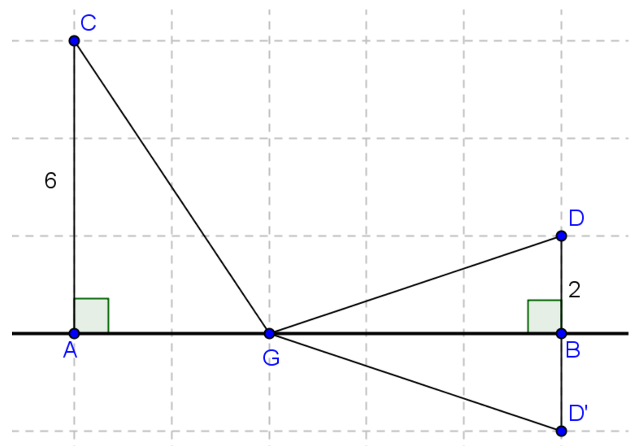


15. In the diagram on the left below, let line $\overleftrightarrow{AB}$ represent a river, where the distance between points A and B is 10 units. Andrea is located at point C. She needs to go to the river, get some water, and then get to point D. She can go to any point on the river to get the water. The diagram on the left shows two possible routes Andrea can take, where AE=FB=2.

a. Is the route from C to the river to D shorter if Andrea hits the river at E or F? Support your answer with numbers (decimals encouraged, to make comparisons easier).

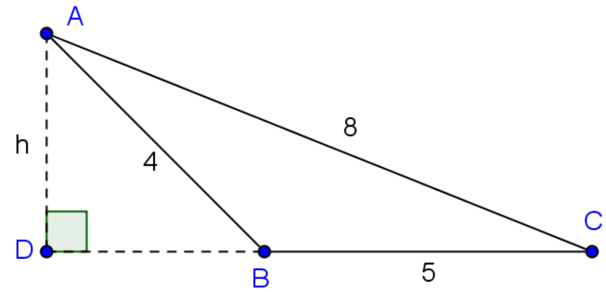


b. Andrea is trying to find an even shorter route. She reflects point D across the river to become D' (this means that DB=D'B=2 and that $\overline{DD'} \perp \overline{AB}$). Explain why $\overline{GD} \cong \overline{GD'}$.



c. Now what is the shortest route Andrea can make from C to the river to D?

16. Triangle ABC below has sides of 4, 5, and 8. You want to find *h*, the length of the altitude from A. Do so using the following steps.



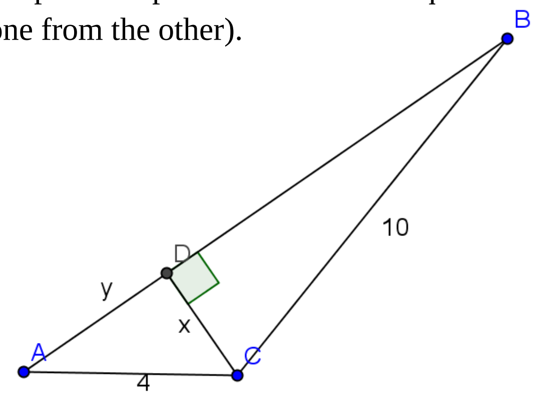
a. Call the length of segment DB *x*. Write an equation involving *x* and *h* in triangle ABD.

b. Write a different equation involving *x* and *h* in triangle ADC.

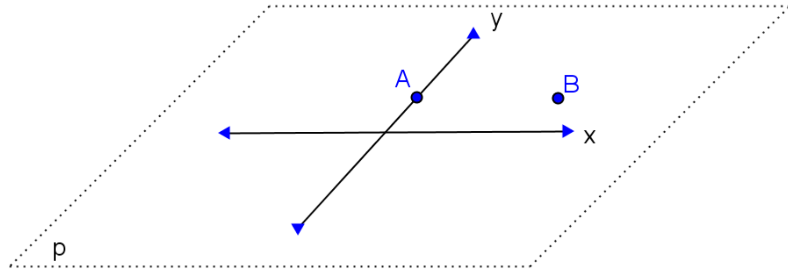
c. You can solve this system (of "second degree" equations, since both variables are squared) by substituting for h^2. Do this to find *x*.

d. Now that you know *x*, plug it into either equation (from parts *a* or *b*) to find *h*. A decimal is OK.

17. Find *x* and *y* (AD) in the diagram below, given that AB measures 12 units. Use a similar technique to the previous problem: write two equations involving *x* and *y* and solve by substitution (or by subtracting one from the other).



18. In plane *p* below, point A is located at (0,2) and point B at (8,2).

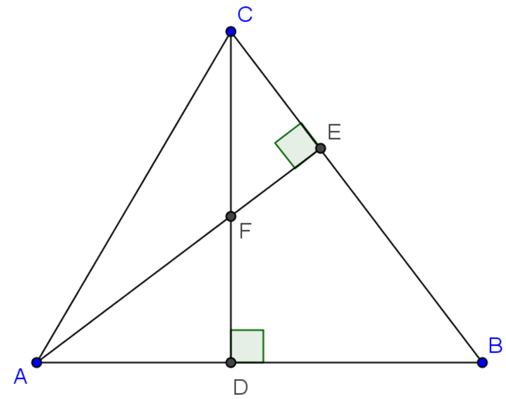


a. There are some points in plane *p* that are five units away from each point A and point B. Describe the set of such points, as specifically as possible.

b. There are some points not just in plane *p*, but in three-dimensional space, that are five units away from each point A and point B. Describe the set of such points as specifically as possible.

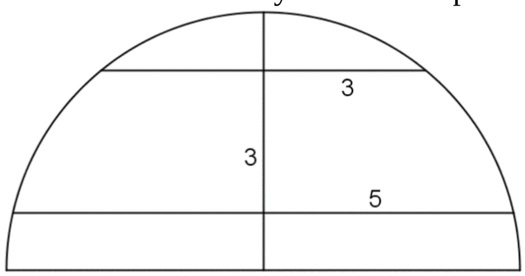
19. In scalene triangle ABC, altitudes $\overline{AE}$ *and* $\overline{CD}$ are drawn.

a. There are four similar right triangles. Name them.



b. If AD=4 and DF=CE=3, then find the lengths of all other segments. You will need both the Pythagorean Theorem and similar triangles.

20. In a circle, two chords are parallel to each other and on the same side of the diameter. Their lengths are 6 and 10 and they are 3 units apart. What is the radius of the circle?

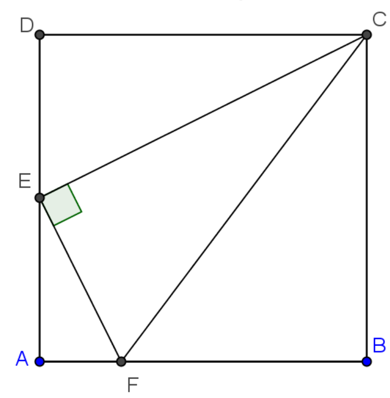


21. ABCD Is a square with side length 2 and E is the midpoint of AD. (adapted from *The Art of Problem Solving: Geometry*)

a. Why is ΔCDE~ΔEAF?

b. Find CF.

c. Explain why $\overline{EC}$ bisects angle DCF.



22. Some Pythagorean triples are listed below. Find a pattern for ones where the smallest number is odd and also where the smallest number is even. Try to verify your patterns with algebra.

Odd: 3-4-5, 5-12-13, 7-24-25; 9-40-41, 11-60-61

Even: 6-8-10; 8-15-17; 10-24-26; 12-35-37; 14-48-50

Now, find the sides of all right triangles whose side lengths are all integers and at least one side is 40 units long.

23. Find the coordinates of all points on the line $y = x + 3$ that are five units from (0,4).

Answers

1. $\sqrt{74}$ 2. $3\sqrt{3}$ 3. 10 4a. ABD and ACD are congruent by HL b. $\sqrt{11}$

5. AE= $2\sqrt{15}$ and EB= $4\sqrt{2}$ so $2\sqrt{15}+4\sqrt{2}$

6a. 10 b. Angles BAD and DEA are congruent since they are both complements of DAE c. 40/3 d. 68/3

7a. No; when the wheel sits in the trench, the center is $\sqrt{11}$ above the road surface, so the bottom of the wheel is $6-\sqrt{11}$ below the road surface, which is above the trench's bottom. b. 17/3 unit

9. 4 10. x=5; y=1.2; z=30 11. for EF use ΔEFC and get $\sqrt{55}$; for EG use EGF and get $\sqrt{46}$

12. $x^2 = 6^2 + (x-3)^2$ so x= 7.5 13. $4\sqrt{7}$ -- draw segment // to BE thru C.. then triangle is 12-x-16….

14. Meet at (10/3, 4/3) so distance is $\sqrt{\left(\frac{13}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{\sqrt{185}}{3}$

15a. through E is 14.57; through F is 12.83 b. $\Delta GBD \cong \Delta GBD'$ and CPCTC c. from C aim straight for D' since CG+GD=CG+GD' and the shortest distance is a straight line… distance is $\sqrt{164}$ =12.81

16a. $x^2 + h^2 = 16$ b. $(x+5)^2 + h^2 = 64$ c. $(x+5)^2 + (16 - x^2) = 64$ so x=2.3 d. 3.27

17. y=2.5 and x = $\sqrt{9.75} = \sqrt{39}/2$

18a. there are two points on the perpendicular bisector of AB that are five units away from both points. They are (4,5) and (4,-1). B. This is a circle of radius three with center is (4,2); it in not in plane p; it is in the plane perpendicular to segment AB. Think of it as the intersection of two spheres.

19a. ΔAEB~ΔADF~ΔCEF~ΔCDB b.AF=5; EF=2.25 CF=3.75 BD=81/16 BE=87/16 AC= $\sqrt{985}/4$

20. $\sqrt{949}/6$ 21a. angles DCE are AEF are equal since they are both complements of DEC b. AF=1/2 so CF=2.5 c. ΔCDE~ΔCEF since sides are proportional, so angles DCE and ECF are equal

22. if n is odd and at least 3: $n, \frac{n^2-1}{2}, \frac{n^2+1}{2}$ → it works since $n^2 + \left(\frac{n^2-1}{2}\right)^2 = \left(\frac{n^2+1}{2}\right)^2$

If n is even and at least 2: $n, \frac{n^2-4}{4}, \frac{n^2+4}{4}$

30,40,50 and 24,32,40 and 40,96,104 and 9,40,41 and 40,198,202 and 40,399,401

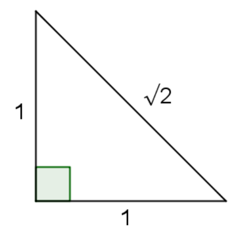
23. from (x,x+3) to (0,4)=5 so $x^2 + (x-1)^2 = 25$ and x=4 or -3 so (4,7) or (-3,0)

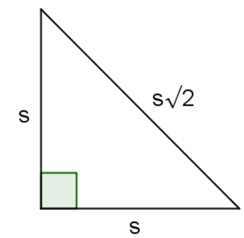
Unit 6 Handout #4: 30/60/90 and 45/45/90 triangles

Triangles with angles of 30°, 60°, and 90° or 45°, 45°, and 90° are often called "special" right triangles, and the ratios of their side lengths can be computed easily. Once you know these ratios, knowing one side of a triangle enables you to find the other two sides.

45/45/90 triangle: an isosceles right triangle:

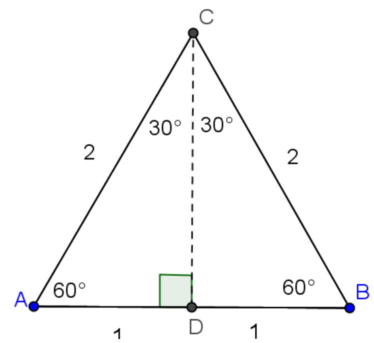
If one leg has length 1, the other leg also has length 1, and, using the Pythagorean Theorem, the hypotenuse has length $\sqrt{2}$. More generally, if the legs are length *s*, then the hypotenuse's length is $s\sqrt{2}$.



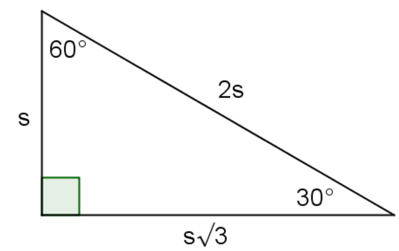


30/60/90 triangle: one half of an equilateral triangle.

Imagine drawing a median on an equilateral with side 2. The equilateral triangle is divided into two congruent triangle (by SAS) whose angles are 30°, 60°, and 90°. Using the Pythagorean Theorem on either of the two smaller triangles enables us to show that CD= $\sqrt{3}$.

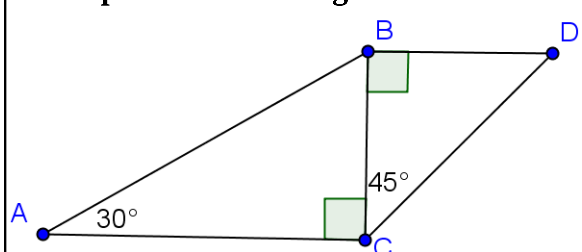


Therefore, the sides of a 30/60/90 triangle are 1, 2, and $\sqrt{3}$. Since the longest side must be opposite the largest angle, we know the hypotenuse is 2. And since the smallest side must be opposite the smallest angle, the side opposite the 30° angle is 1.



In a generic 30/60/90 triangle, the sides are *s*, $s\sqrt{3}$, and 2*s*.

Example #1: In the diagram to the left below, BC=10; find the lengths of $\overline{AB}$ *and* $\overline{CD}$.



$\overline{AB}$ first: ABC is a 30/60/90 triangle and the side opposite the right angle is twice the length of the side opposite the 30° angle. Therefore AB=20.

$\overline{CD}$: In an isosceles right triangle, the side opposite the right angle is $\sqrt{2}$ times the sides opposite the 45° angles. Therefore $\text{CD} = 10\sqrt{2}$

Example #2: In the diagram above CD=6; find the length of $\overline{AC}$.

First work in ΔBCD. Since the hypotenuse of a 45/45/90 triangle is $\sqrt{2}$ times the length of the legs, we get $BC \cdot \sqrt{2} = 6 \;\; so \;\; BC = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = 3\sqrt{2}$. Now to ΔABC. In a 30/60/90 triangle, the side opposite the 60° angle is $\sqrt{3}$ times the sides opposite the 30° angle. So $AC = \left(3\sqrt{2}\right)\sqrt{3} = 3\sqrt{6}$

1. **Isosceles Right Triangles:**

a. Show that an isosceles right triangle with legs of 1 has a hypotenuse of $\sqrt{2}$.

b. Show that an isosceles right triangle with legs of *w* has a hypotenuse of $w\sqrt{2}$.

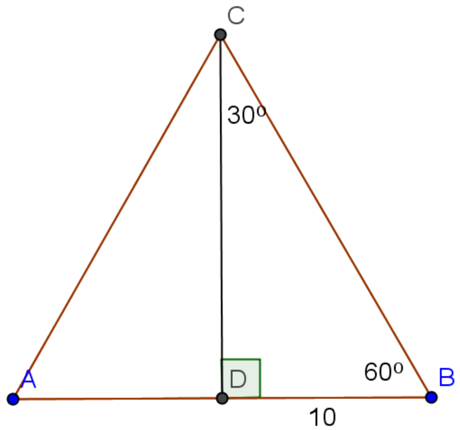
c. Show that an isosceles right triangle with a hypotenuse of *y* has legs of $\frac{y}{\sqrt{2}}$.

2. **"30/60/90" triangles.**

a. Draw an equilateral triangle with side length of 2. Then draw an altitude to any side. Explain why this divides the equilateral triangle into two congruent triangles whose angles are 30°, 60°, and 90°.

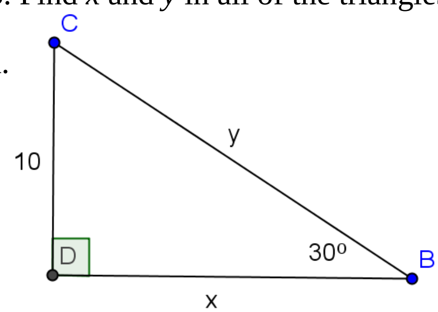
b. In either of the 30/60/90 triangles you created, show that the length of the side opposite the 30° angle is 1 and the length of the side opposite the 60° angle is $\sqrt{3}$.

c. The shortest side of a 30/60/90 triangle is 10. Find the other two sides. The diagram below may help, as you can think of it as half of an equilateral triangle—so the hypotenuse should be obvious!

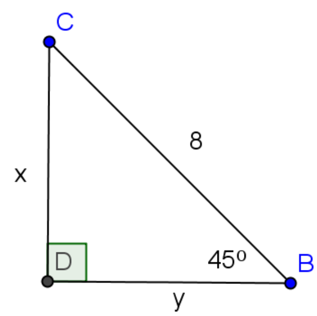


3. Find *x* and *y* in all of the triangles below:

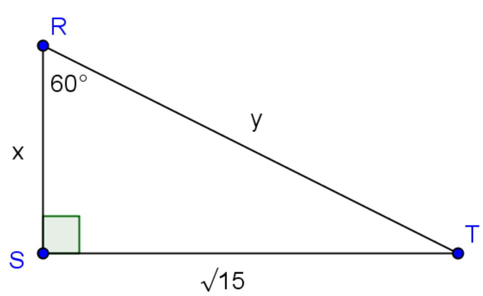
a.



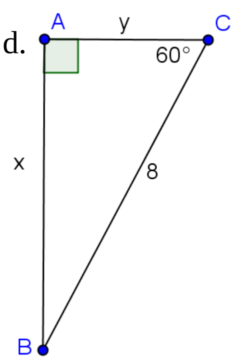
b.



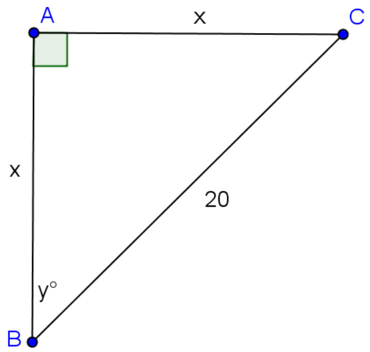
c.



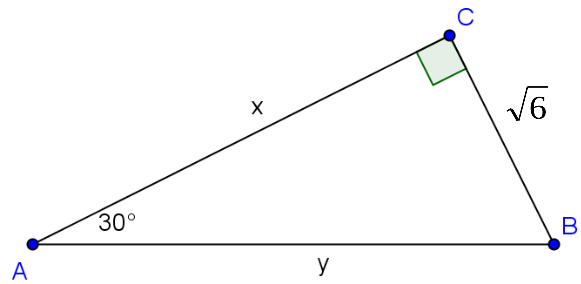
d.



e.

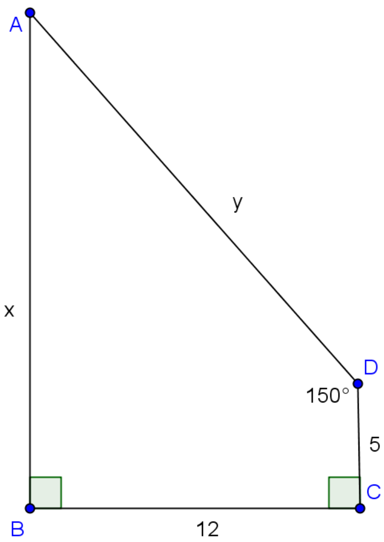


f.

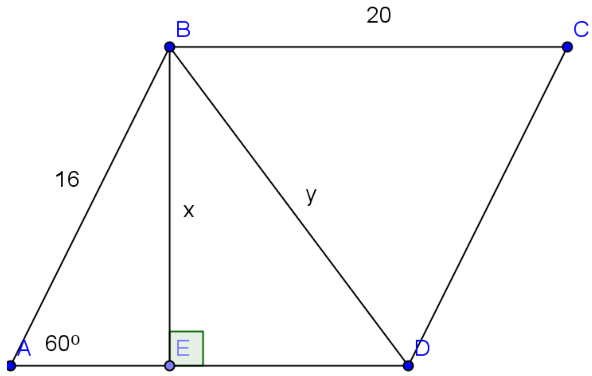


4. Find *x* and *y* in each diagram below.

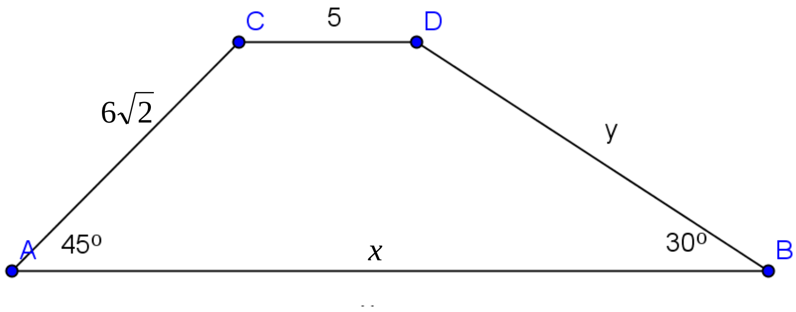
a.



b. ABCD is a parallelogram



5. Find x and y in the trapezoid below:



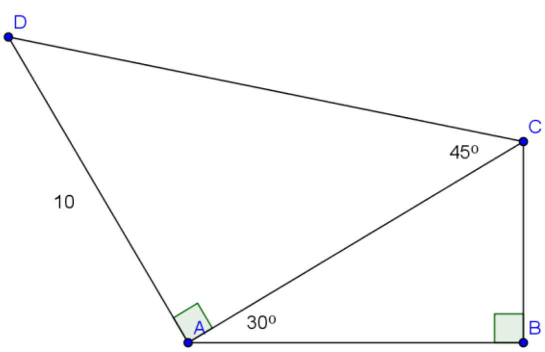
6. What is the length of the diagonal of a square with…

a. A side of 6?

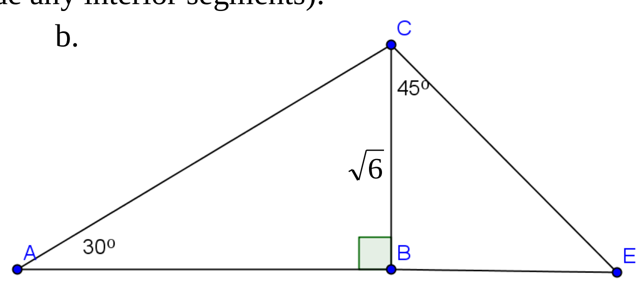
b. An area of 10?

7. Find the perimeter of the figures below (do not include any interior segments):

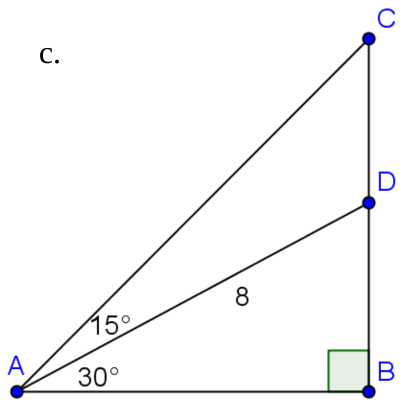
a.



b.

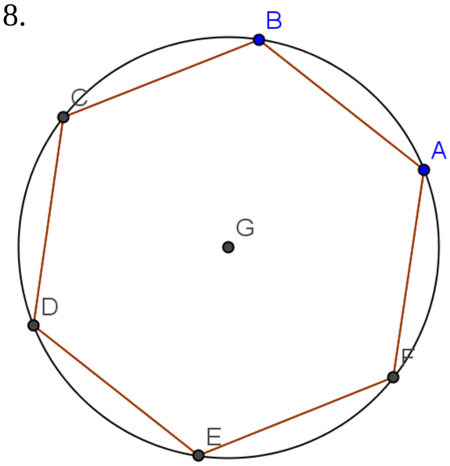


c.



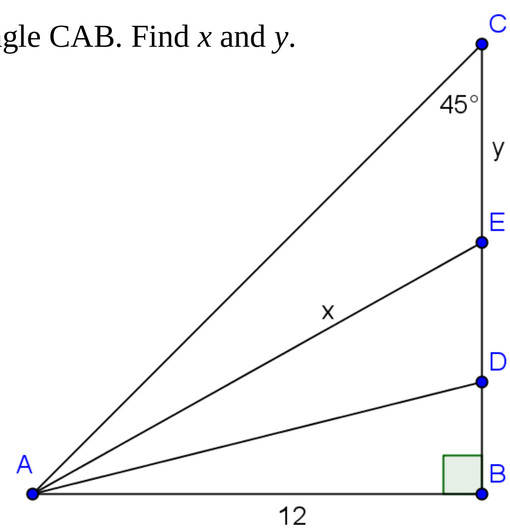
8. In the diagram below, a regular hexagon is inscribed in a circle of radius 8.

a. What is the perimeter of the hexagon?



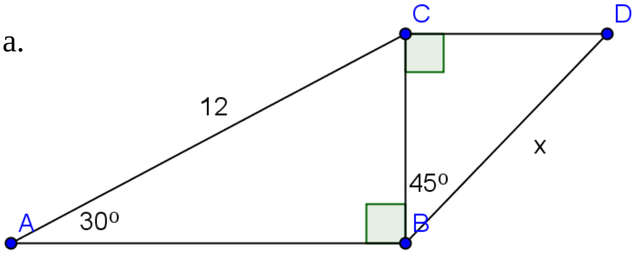
b. Find the length of $\overline{AE}$.

9. In the triangle on the right, segments $\overline{AE}$ and $\overline{AD}$ trisect angle CAB. Find *x* and *y*.

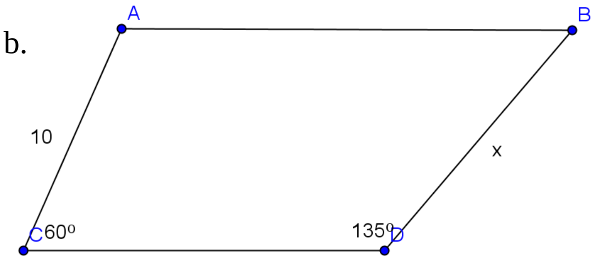


10. Find the value of *x* in the figures below. In all diagrams, $\overline{AB} // \overline{CD}$..

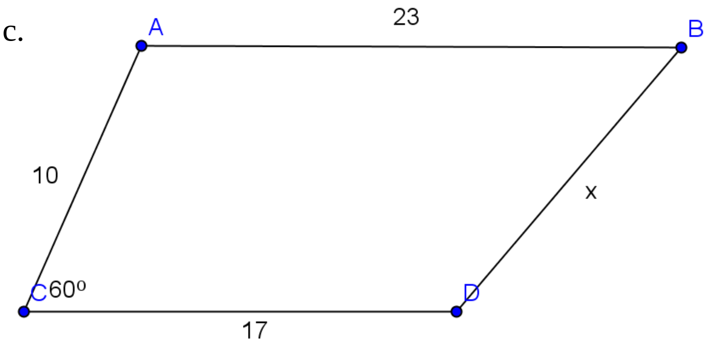
a.



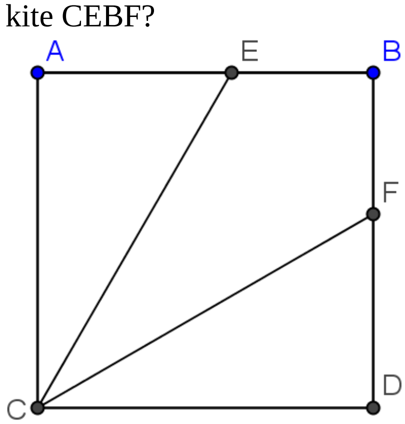
b.



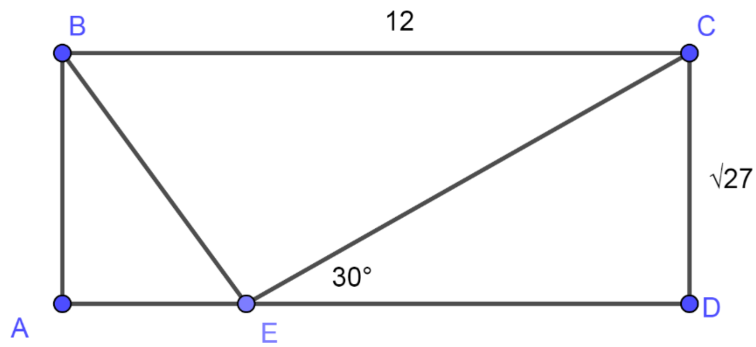
c.



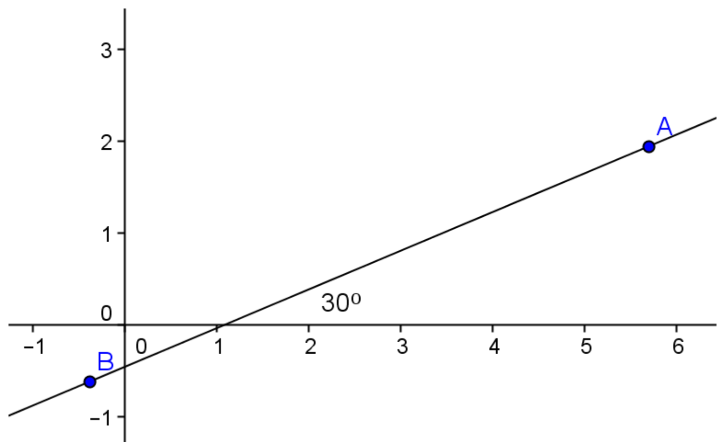
11. Square ABCD has a side length of 4. Angle C is trisected by $\overline{CE}$ and $\overline{CF}$. What is the perimeter of kite CEBF?



12. Rectangle ABCD has length $\overline{AB}$ 12 and width $\sqrt{27}$. Find the length of $\overline{BE}$.



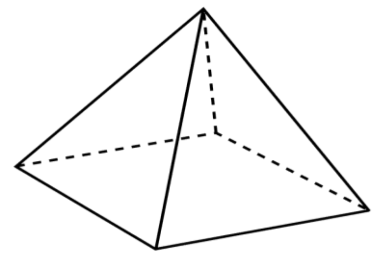
13. Line $\overleftrightarrow{AB}$ in the diagram below makes a 30° angle with the *x*-axis. What is its slope?



14. A cube has a side length of 2. How long is its diagonal (from one corner through the center to the opposite corner)?

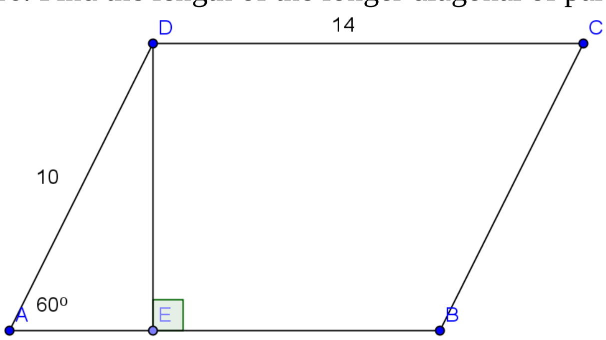
15. A pyramid has a square base and four equilateral triangles as faces (see diagram). The length of each of the eight edges is 10 units.

a. How long is the altitude of one of the triangles on the face?

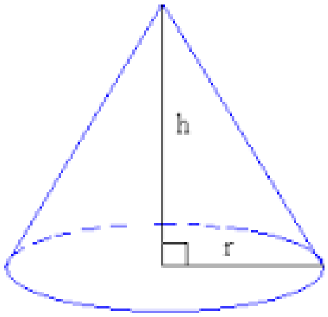


b. How tall is the pyramid, measured from the center of the base to the point at the top? (draw a right triangle!)

16. Find the length of the longer diagonal of parallelogram ABCD:

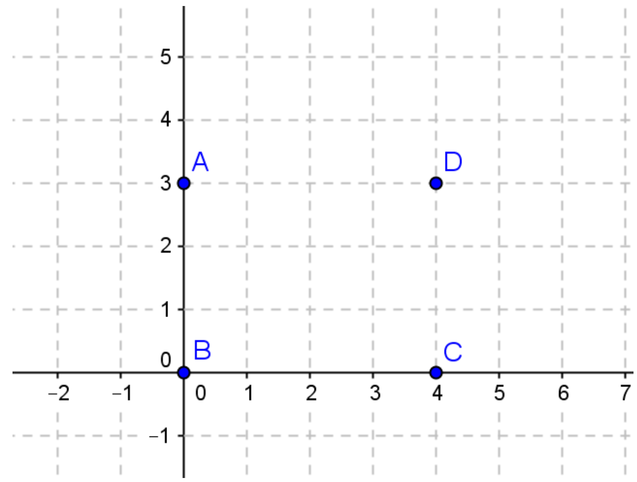


17. A cone has a circular base with radius of 6 and height of 10. What is the "slant height": the distance from the tip along the edge to any point on the edge of the circular base?



18. Do the following given the points plotted below: Hint: draw triangles!

a. What are the coordinates of the point when C is rotated 60° counter-clockwise around B?

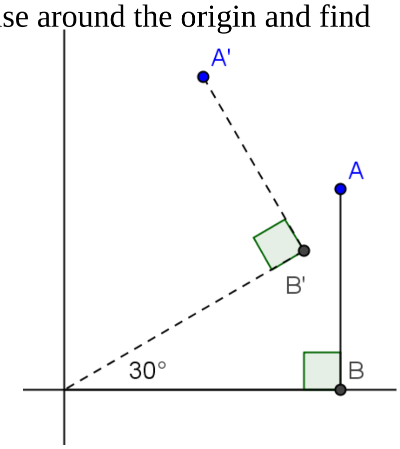


b. What are the coordinates of the point when A is rotated 45° clockwise around D?

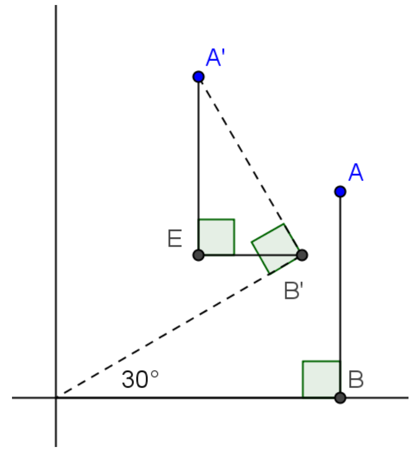
c. What are the coordinates of the point when A is rotated 30° counter-clockwise around B?

19. Point A's coordinates are (8,6). You want to rotate it 30° counter-clockwise around the origin and find the coordinates of A' that results. Here's how:

a. First find the coordinates of point B and rotate B 30° counter-clockwise around the origin to find the coordinates of B'.



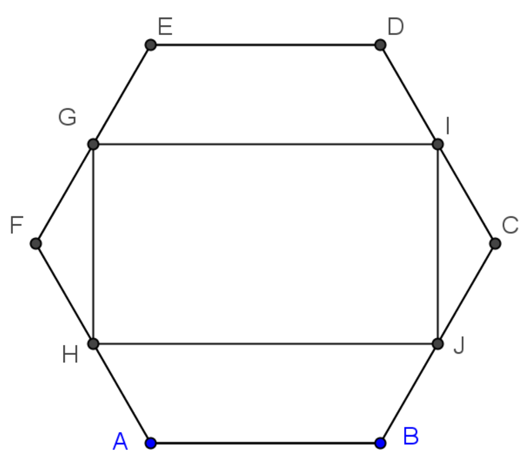
b. Now, use triangle B'EA' in the next diagram (where B'E is horizontal and A'E is vertical) to find distances B'E and A'E.



c. Now find the coordinates of A'.

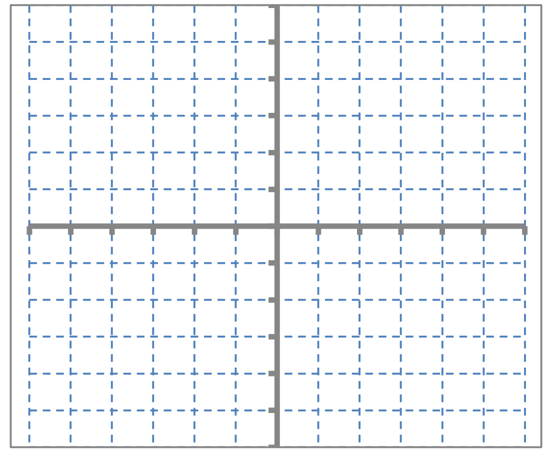
d. One way to check: the distance from the origin to A' should be the same as the distance from the origin to A. Is it?

20. ABCDEF is a regular hexagon with side length of 6. Points G, H, I, and J are the midpoints of four sides. They are connected to form a rectangle. What are the dimensions of this rectangle?



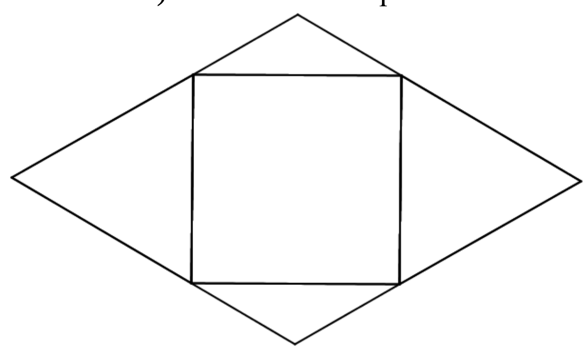
21. Plot points A(-4,1), B(2,3), C(1,-4), and D(4,-3) on the grid at the right.

a. Is angle BAC equal to 60°, greater than 60°, or less than 60°? Explain. Hint: look at the side lengths.

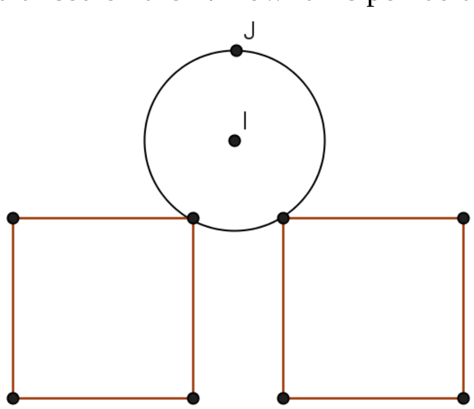


b. What is the measure of angle DAB? Why?

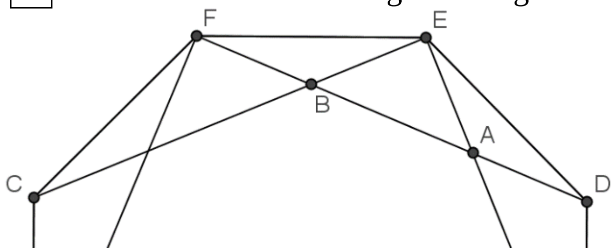
22. A rhombus with side six has a square inscribed in its center, creating equilateral triangles on the left and the right and isosceles triangles on the top and bottom. What is the length of the square's side? Leave your answer as a simplified radical (but it does not have to be rationalized). Hint: call the square's side $2x$ and write an equation that you can solve for x.



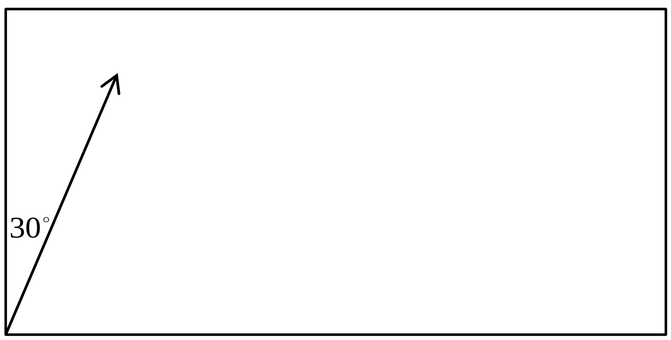
23. Two squares with side 4 are set 2 units apart; their bases are on the same line. A circle with radius 2 is balanced on them. How far is point J above the line through the bottoms of the squares?

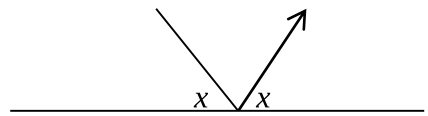


24. CDEF are vertices of a regular octagon. Find the ratio of the length of $\overline{AB}$ to the length of $\overline{DF}$.



25. The rectangular box below has a height of 10 units. A ball leaves one corner at a 30° angle and bounces off the top, then the bottom, then the top again, then the bottom again, before hitting the upper-right corner. Note: it bounces "cleanly" meaning the acute angle it makes "coming in" is the same angle that it makes "going out". See the diagram on the right for an illustration.





a. How wide is the box?

b. How far did the ball travel?

Answers

1a. $1^2 + 1^2 = h^2$ so $h = \sqrt{2}$ b. $w^2 + w^2 = h^2$ so $h^2 = 2w^2$ *and* $h = w\sqrt{2}$

1c. $x^2 + x^2 = y^2$ *so* $x^2 = \frac{y^2}{2}$ *and* $x = \frac{y}{\sqrt{2}}$

2a. 2 triangles are congruent by HL so they are both 30/60/90 b. base is bisected so base halves are each 1 and $1^2 + x^2 = 2^2$ *so* $x = \sqrt{3}$

2c. hypotenuse is 20 (twice 10) so $x^2 + 10^2 = 20^2$ *and* $x = \sqrt{300} = 10\sqrt{3}$

3a. x=$10\sqrt{3}$ y=20 b. $x = y = \frac{8}{\sqrt{2}}$ $(= 4\sqrt{2})$ c. $x = \sqrt{5}$ $y = 2\sqrt{5}$

d. $x = 4\sqrt{3}$ $y = 4$ e. $x = 10\sqrt{2}$ y=45 f. $x = 3\sqrt{2}$ $y = 2\sqrt{6}$

4a. $x = 5 + 12\sqrt{3}$ $(NOT\ 17\sqrt{3})$ y=24 b. $x = 8\sqrt{3}$ $y = \sqrt{336} = 4\sqrt{21}$

5. $x = 11 + 6\sqrt{3}$ y=12 6a. $6\sqrt{2}$ b. $\sqrt{20} = 2\sqrt{5}$

7a. $15 + 5\sqrt{3} + 10\sqrt{2}$ b. $3\sqrt{6} + 2\sqrt{3} + 3\sqrt{2}$ c. $8\sqrt{3} + 4\sqrt{6}$ 8a. 48 b. $8\sqrt{3}$

9. $x = 24/\sqrt{3}$ $(or\ 8\sqrt{3})$ $y = 12 - 12/\sqrt{3}$ $(or\ 12 - 4\sqrt{3})$ 10a. $6\sqrt{2}$ b. $5\sqrt{6}$ c. 14 11. $8 + 8/\sqrt{3}$

12. 6 13. $1/\sqrt{3}$ 14. $2\sqrt{3}$

15a. $5\sqrt{3}$ b. $5\sqrt{2}$ 16. $\sqrt{436} = 2\sqrt{109}$ 17. $\sqrt{136} = 2\sqrt{34}$

18a. $(2, 2\sqrt{3})$ b. $(4 - 2\sqrt{2}, 3 + 2\sqrt{2})$ c. $(-1.5, 1.5\sqrt{3})$

19a. $(4\sqrt{3}, 4)$ b. B'E=3; A'E=$3\sqrt{3}$ c. $(4\sqrt{3} - 3,\ 4 + 3\sqrt{3})$

d. $(4\sqrt{3} - 3)^2 + (4 + 3\sqrt{3})^2 = 48 + 9 - 24\sqrt{3} + 16 + 27 + 24\sqrt{3} = 100$; so 10 units from origin, as is A

20. $3\sqrt{3}$ by 9

21a. $\angle BAC = \angle ABC$ and both are more than angle ACB so greater than 60°
b. 45° because triangle ABD is an isosceles right triangle.

22. $2x + 2x/\sqrt{3}$=6 so $x = \frac{6}{2 + 2/\sqrt{3}} = \frac{3\sqrt{3}}{1 + \sqrt{3}}$ 23. $6 + \sqrt{3}$

24. Since angles FEC and DEA are each 22.5°, angle BEA is a right angle. And BF=BE=EA=AD so triangle BEA is 45/45/90. Thus the ratio is $\frac{\sqrt{2}}{2 + \sqrt{2}}$

25a. $50/\sqrt{3}$; b.$100/\sqrt{3}$ → you can "unfold" it and make it like one big triangle…

Unit 6 Handout #5: Trigonometric Ratios

In a right triangle, we can define ratios of sides as a trigonometric function. Define the following:

-The sine of an acute angle is the ratio of the side opposite that angle to the hypotenuse, or

$\sin A = \dfrac{opposite}{hypotenuse}$.

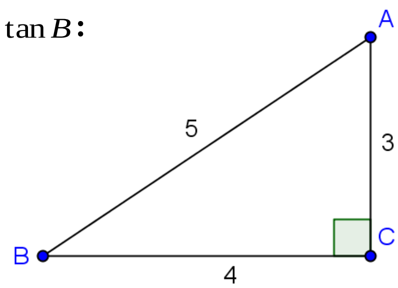
-The cosine of an acute angle is the ratio of the side adjacent to that angle to the hypotenuse, or

$\cos A = \dfrac{adjacent}{hypotenuse}$.

-The tangent of an acute angle is the ratio of the side opposite that angle to the side adjacent to that angle, or $\tan A = \dfrac{opposite}{adjacent}$.

Millions of students have used the acronym SOH-CAH-TOA to remember this!

Example #1: In the triangle below, find the values of $\sin A$ *and* $\tan B$:

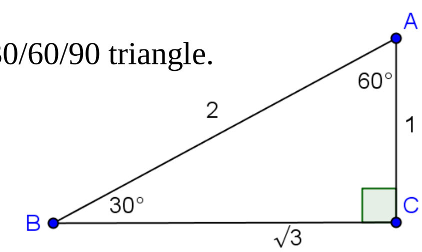


- $\sin A$ is the ratio of the length of the side opposite angle A to the hypotenuse. Thus $\sin A = \dfrac{BC}{AB} = \dfrac{4}{5}$

- $\tan B$ is the ratio of the length of the side opposite angle B to the side adjacent to angle B. Thus $\tan B = \dfrac{AC}{BC} = \dfrac{3}{4}$

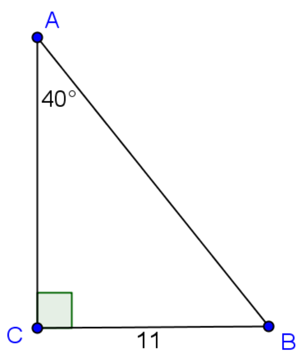
Example #2: Find the sine, cosine, and tangent of a 30° angle.

For this, we can draw any right triangle with a 30° angle, so any 30/60/90 triangle.



So $\sin 30^\circ = \dfrac{AC}{AB} = \dfrac{1}{2}$; $\cos 30^\circ = \dfrac{BC}{AB} = \dfrac{\sqrt{3}}{2}$; $\tan 30^\circ = BC = \dfrac{1}{\sqrt{3}}$

Example #3: A calculator or trigonometric table shows that $\sin 40^\circ \approx 0.64$. Use that to find AB and AC in the triangle below.



-Since the sine involves the opposite and hypotenuse, we will find AB first.

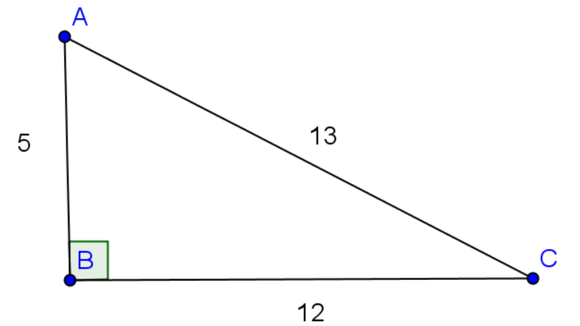
Let x be the length of AB. $\sin 40^\circ = \dfrac{BC}{AB} = \dfrac{11}{x}$. $0.64 = \dfrac{11}{x}$ and

$0.64x = 11 \;\; so \;\; x = 11/0.64 \approx 17.2$

-Now we can use the Pythagorean Theorem to find AC.

$(AC)^2 + 11^2 = 17.2^2$ and AC≈13.2

1. In the triangle below, find the following:



a. $\sin A$

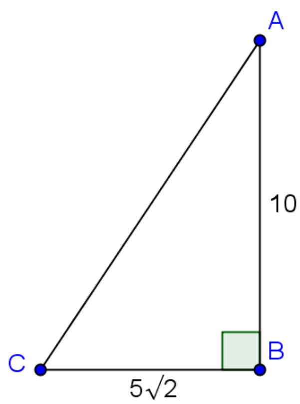
b. $\cos A$

c. $\tan A$

d. $\cos C$

e. $\sin C$

2. In the triangle below, find the following:



a. $\sin A$

b. $\cos A$

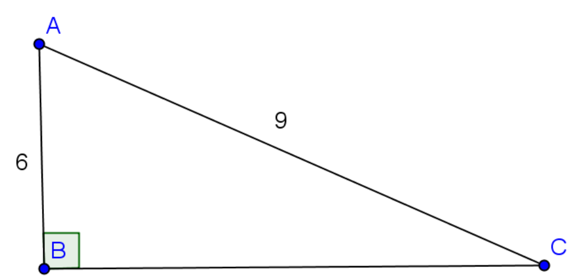
c. $\tan C$

3. In the triangle below, find the following:

a. $\sin C$

b. $\cos C$

c. $\tan A$



4. Draw a 30/60/90 triangle and a 45/45/90 triangle and find the following. No calculators!

a. $\sin 30^\circ$

b. $\cos 30^\circ$

c. $\tan 30^\circ$

d. $\sin 45^\circ$

e. $\cos 45^\circ$

f. $\tan 45^\circ$

g. $\sin 60^\circ$

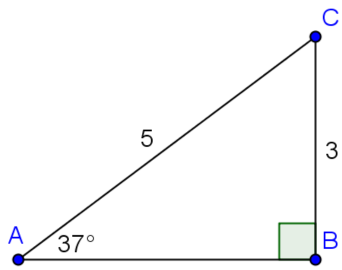
h. $\cos 60^\circ$

i. $\tan 60^\circ$

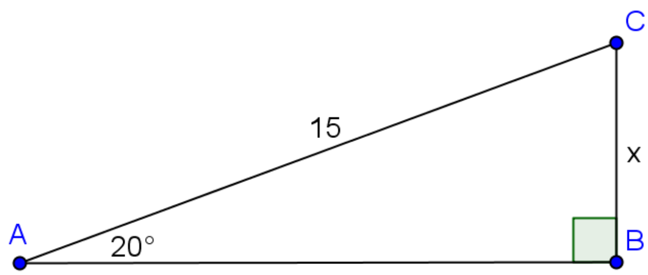
5. Based on your answers to question #4 above, does it appear that $\sin 30^\circ + \sin 30^\circ = \sin 60^\circ$? What does this tell you about the truth of the statement $\sin A + \sin B = \sin(A+B)$?

6. In ΔABC, $\angle B = 90^\circ$, AB=7, and AC=9. Find $\sin A$ *and* $\tan C$

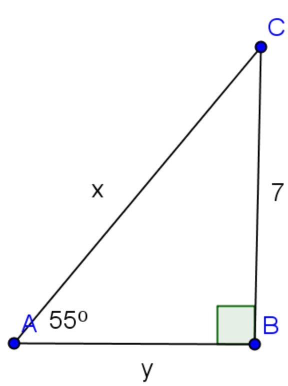
7. Given that the triangle below is approximately correct (the angle rounds to 37°, but is not ***exactly*** 37°), approximate the values of $\sin 37^\circ$, $\cos 37^\circ$, and $\tan 37^\circ$. How about $\tan 53^\circ$?



8. Given that $\sin 20^\circ \approx 0.34$, find *x* in the diagram below (use your calculator!). Then use the Pythagorean Theorem to find the length of side $\overline{AB}$.



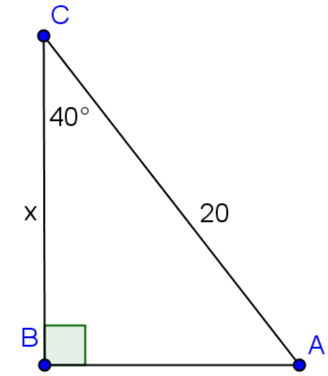
9. Given that $\sin 55^\circ \approx 0.82$, find *x* and *y* in the diagram below .



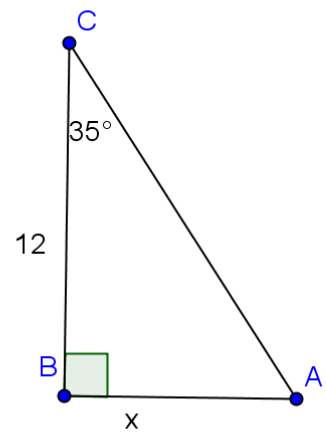
10. This table below shows approximations for sine, cosine, and tangent of some angles. Use it to find approximate values of *x* in all of the questions below.

Angle	Sine	Cosine	Tangent
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391

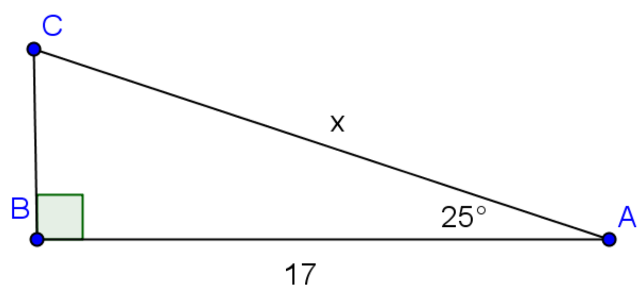
a.



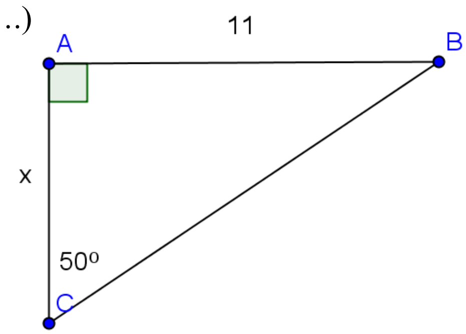
b.



c.

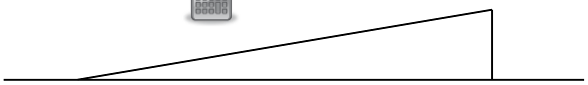


d. (hmmm….)



Note: for the remainder of the semester, any problem or part of problem that requires trigonometric buttons on your calculator will have  *next to it. Problems without this symbol can be done without calculators—consider special triangles!*

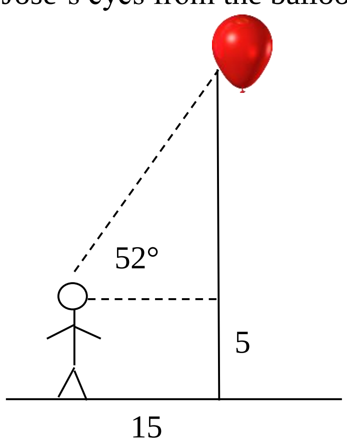
11. A ramp meets the ground with an angle of 10°. How long must it be (the slanted part) for it to gain 4 feet in elevation? 



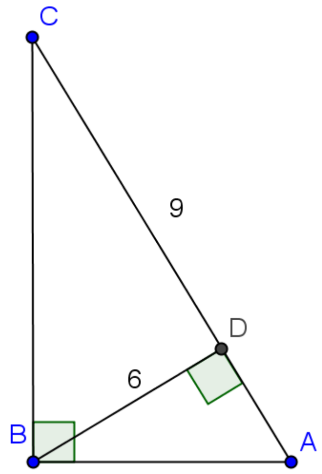
12. Jose is standing up, looking at a balloon. From his eye level, 5 feet above the ground, the angle of elevation is 52°. His feet are 15 feet from the spot on the ground directly below the balloon.

a. How high above the ground is the balloon? 

b. How far are Jose's eyes from the balloon? 



13. Earlier in this unit, we used similar triangles to find all segments in a diagram such as the one below. It may be easier to use trigonometry instead, as we will then not need to draw three triangles separately.



a. What is $\tan C$?

b. Name an angle congruent to angle C.

c. Given that its tangent must be the same as $\tan C$, you know the length of one other segment. Which one?

d. Now find the lengths of the remaining segments.

answers

1a. 12/13 b. 5/13 c. 12/5 d. 12/13 e. 5/13 2a. $\frac{5\sqrt{2}}{5\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ b. $\frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

2c. $\frac{10}{5\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$ 3a. 2/3 b. $3\sqrt{5}/9$ c. $\sqrt{5}/2$

4a. ½ b. $\sqrt{3}/2$ c. $1/\sqrt{3}$ d. $1/\sqrt{2}$ e. $1/\sqrt{2}$ f. 1 g. $\sqrt{3}/2$ h. ½ i. $\sqrt{3}$

5. it appears not to be true

6. $\sin A = \frac{4\sqrt{2}}{9}$ $\tan C = \frac{7}{4\sqrt{2}}$ 7. $\sin 37^\circ \approx 0.6$; $\cos 37^\circ \approx 0.8$, $\tan 37^\circ \approx 0.75$, $\tan 53^\circ \approx 4/3$

8. x=5.1 and AB≈14.11 9. x≈8.54 y≈4.89 10a. 15.32 b. 8.4 c. 18.76 d. 9.23 11. 23.04
12a. 24.2 b. 24.4 13a. 2/3 b. ABD c. AD=x and x/6 = 6/9 since both are tanC so x=4

13d. using the Pythag Theorem AB=$2\sqrt{13}$ and $BC = 3\sqrt{13}$

Unit 6 Handout #6: Trigonometry: Finding Angles from Sides

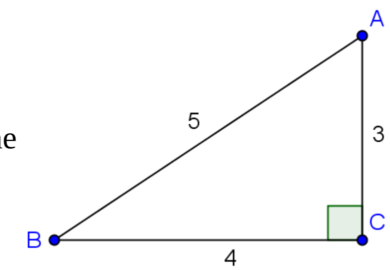
In the prior section we used the trigonometric ratios of an angle to find an unknown side or two. We can also use trigonometry to find an angle if we know two sides. To do this, we use the "inverse trigonometric functions".

The inverse sine function is $\sin^{-1}$; its input is a ratio and its output is an angle. For example, $\sin^{-1}(1/2) = 30^\circ$ because the angle whose sine is ½ is 30°, as we know from our special triangles!

Key ideas:

1. ***Use the "regular" trig functions when you know the angle and want to find the ratio. Use the inverse trig functions when you know the ratio and want to find the angle.***
2. When you use the inverse sine function, think of it as saying "the angle whose sine is __". Obviously you can use the same logic with inverse cosine and inverse tangent.
3. On your calculator, use the 2^{nd} button then the sine button.
4. The notation can be confusing. With exponents, -1 denotes the reciprocal. In this context it does not!

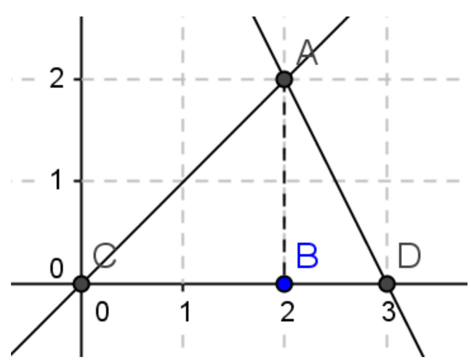
Example #1: Find the smallest angle in a 3-4-5 triangle.



We know the smallest angle is opposite the shortest side, so angle B must be the smallest angle. Thus $\sin B = \frac{3}{5}$. Since we want to find the angle, we use the inverse sine function: $B = \sin^{-1}(3/5)$. Again, this means B is ***the angle whose sine is 3/5***. The calculator shows that B≈36.9°.

Note: you can also get the answer using $B = \cos^{-1}(4/5)$ or $B = \tan^{-1}(3/4)$

Example #2: Find the measure of the acute angle at which the lines $y = x$ *and* $y = 6 - 2x$ intersect.



A little algebra shows that they intersect at A(2,2) and one line crosses the *x*-axis at C(0,0) and the other at D(3,0).

We want to find the measure of angle CAD. Draw a perpendicular from A to point B on the *x*-axis (2,0) so we have right triangles to work with.

Break it into two parts.

The measure of angle BAD; $\tan(BAD) = BD / AB = 1/2$

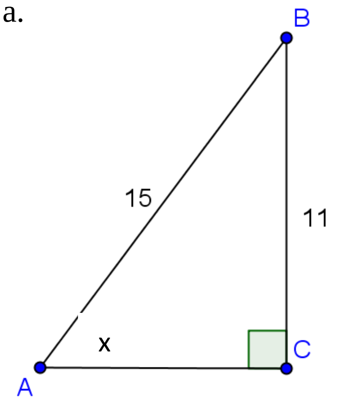
so BAD= $\tan^{-1}(1/2) \approx 26.6°$

The measure of angle BAC; $\tan(BAC) = BC / AB = 1$ so BAC= $\tan^{-1}(1) = 45°$

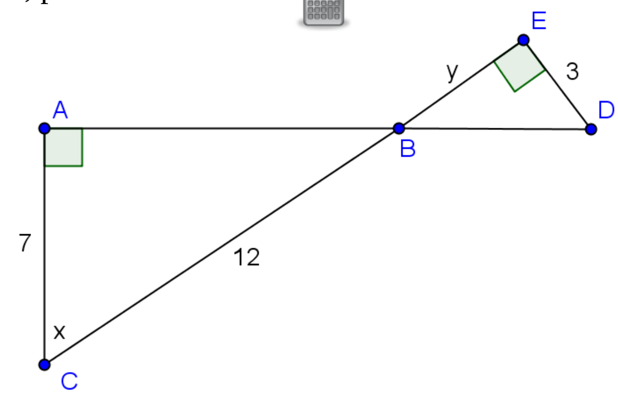
Therefore angle CAD, the angle at which the lines meet, is about 26.6+45 = 71.6°

1. Find the values of *x* and *y* in the diagrams below. In part *b*, point B is on $\overline{CE}$.

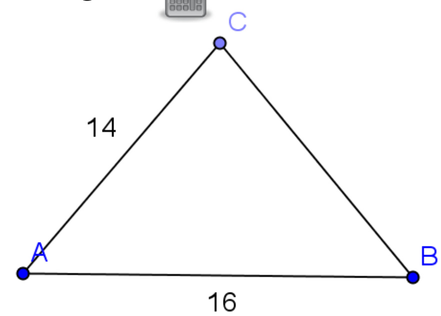
a.



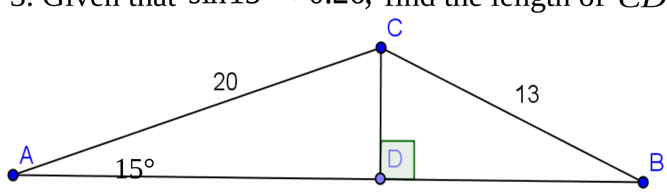
b.



2. Given that angles A and B are congruent in the diagram below, find $\sin A$. Note: angle C is NOT a right angle, so you'll need to find one somewhere else! Then find the measure of angle A.

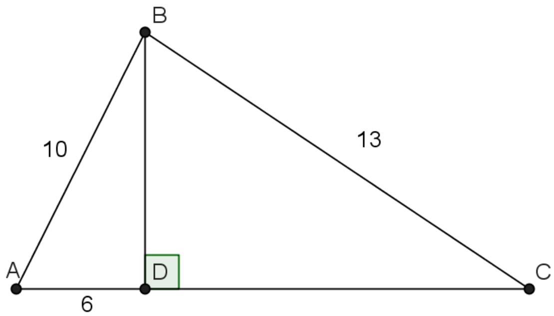


3. Given that $\sin 15^\circ \approx 0.26$, find the length of $\overline{CD}$. Then find the measure of angle B.

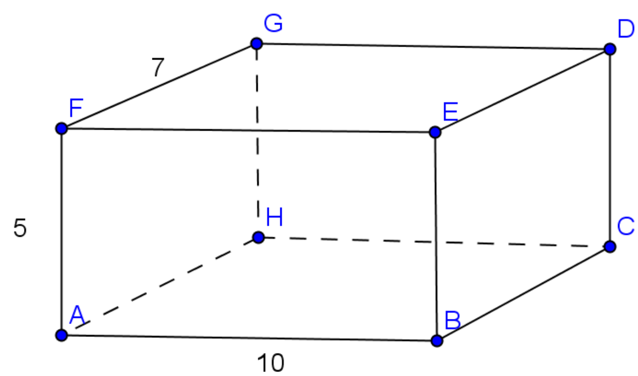


4. Find the measure of the largest angle in the triangle whose sides are 10, 10, and 12. 

5. In the diagram below, ABC is not a right angle; find its measure. 



6. Find the following in the rectangular prism below. 

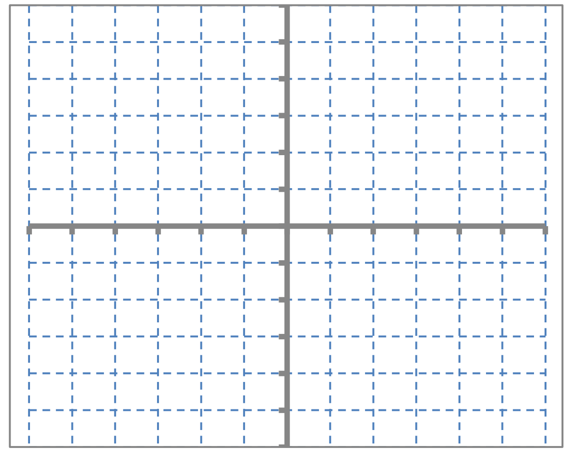


a. The measure of angle AFH.

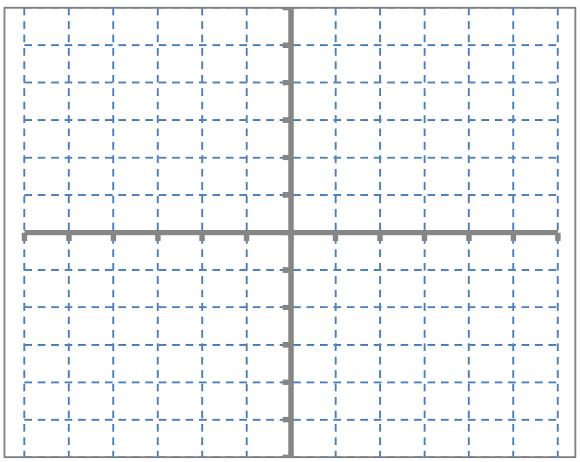
b. The measure of angle BHE.

c. The measure of angle ADC.

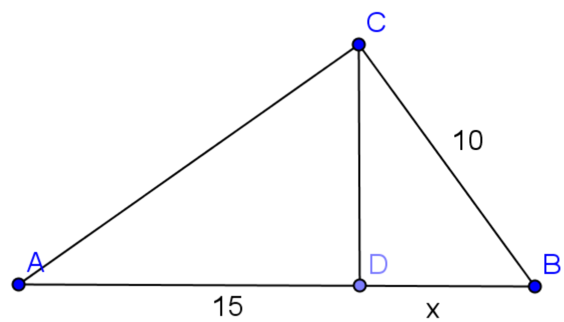
7. Graph the lines $y = 0.5x + 2$ *and* $y = -3x + 9$ below. A triangle is formed by these two lines and the *x*-axis. Find the measure of the three angles of the triangle. 



8. At what acute angle do the lines $y = x - 2$ *and* $x + 3y = 6$ meet? Find some right triangles! 



9. Angles C and CDB are right angles in the diagram below. We are trying to find the value of *x*. Earlier in this unit, we used similar triangles. Now let's use trigonometry instead. It may be easier since we do not need to draw the three similar triangles separately.



a. In triangle ABC, write $\sin A$ in terms of *x*.

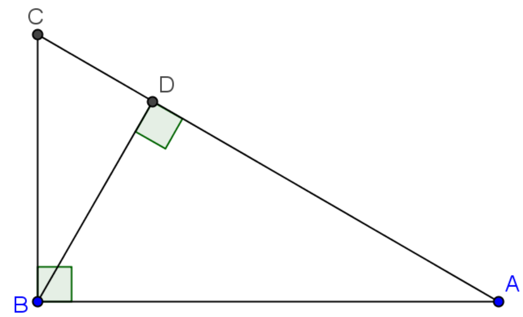
b. Find another angle that is congruent to A.

c. Write the sine of that angle in terms of *x*.

d. Given that those two angles are congruent, their sines must be equal. Set them equal and solve for *x*.

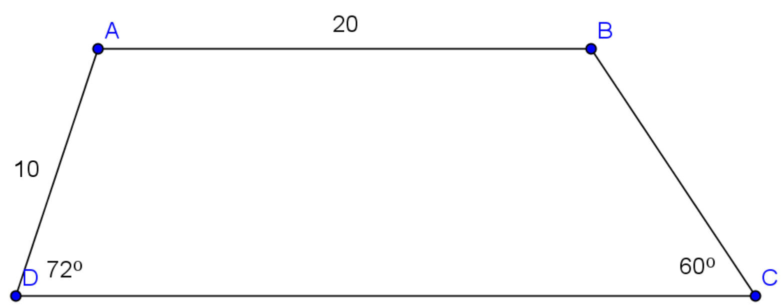
10. In the diagram below, ABC is a right triangle and $\overline{BD}$ is the altitude to the hypotenuse. We solved these problems earlier using similar triangles, now try to solve them with trigonometry as in problem #9 above.

a. If BC=6 and AC=12, then find the length of $\overline{CD}$.

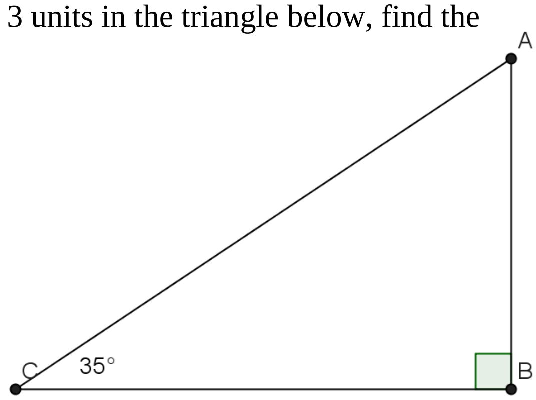


b. Instead, if BD=6 and AC=13 then find the length of $\overline{CD}$. (given that CD < AD)

11. In the trapezoid below, find *BC* and *CD*. You'll need your calculator for $\sin 72^\circ$, but you should be able handle the 60° angle without it!



12. Given that the length of $\overline{BC}$ exceeds the length of $\overline{AB}$ by 3 units in the triangle below, find the length of $\overline{AC}$. 



Answers

1a. x=42.8° b. x=54.3° y=4.18

2. draw altitude from C is perp bisector of base so altitude = $\sqrt{132} = 2\sqrt{33}$ so $\sin A = \frac{\sqrt{33}}{7}$ and A=55.2°

3. CD=5.18 so using Pythag, BD=11.92 b=23.5°

4. 73.74° 5. 88.89° 6a. 54.46° b. 22.27° c. 67.73° 7. 26.6°, 71.6°, and 81.8° 8. 63.4°

9a. $\frac{10}{x+15}$ b. BCD c. x/10 d. x=5 (by factoring)

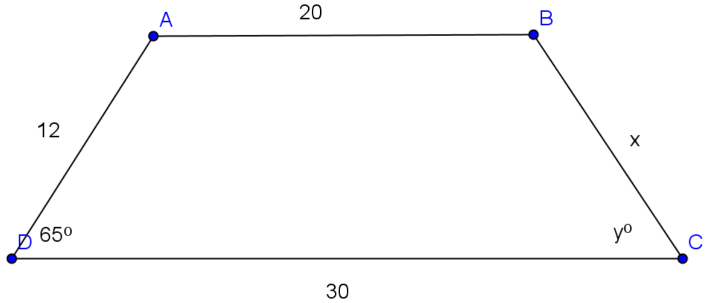
10a. $\cos C = 6/12$ in triangle ABC; in triangle BCD, $\cos C = x/6$ so x=3 b. CD=4

11. drop vertical from A to CD to meet at E; AE=9.5 and DE=3.12;
Drop vertical from B to CD to meet at F; BF=9.5 so BC=$19/\sqrt{3}$ (about 10.97); CF is about 5.48
So CD = 3.12 + 20 + 5.48 = about 28.6

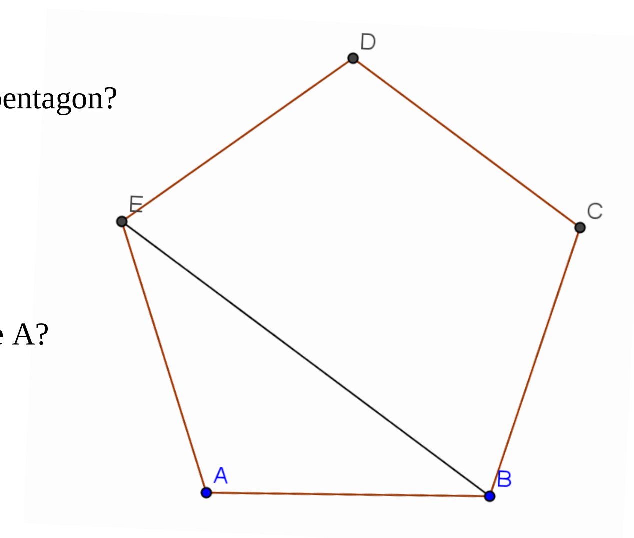
12. 12.216

Unit 6 Handout #7: Trigonometry Problem-Solving

1. Find *x* and *y* in the trapezoid below. 



2. The regular pentagon has side length of 2.

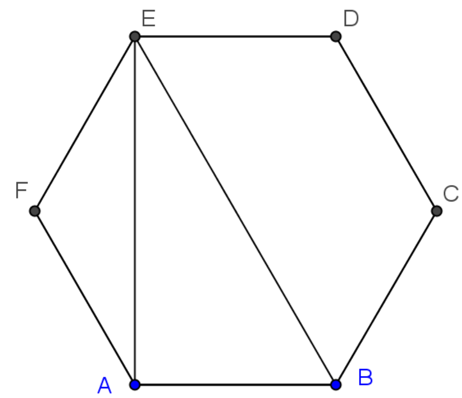


a. What is the measure of each internal angle of a regular pentagon?

b. Why must the perpendicular from A to $\overline{BE}$ bisect angle A?

c. How long is diagonal $\overline{BE}$? 

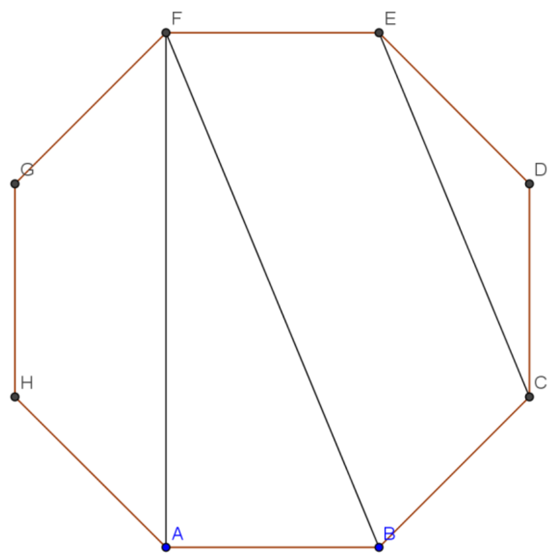
3. Given the regular hexagon with side of 8.



a. Why must $\angle BAE$ be a right angle?

b. Find the lengths of $\overline{AE}$ and $\overline{BE}$.

4. The regular octagon below has side length of 4.



a. What is the measure of each interior angle?

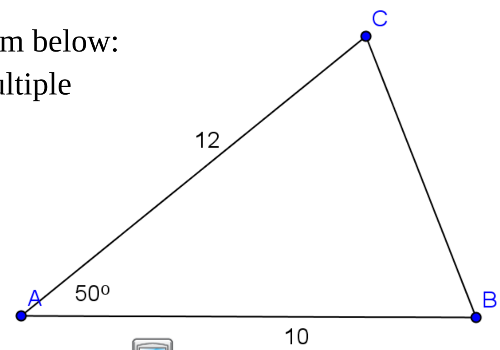
b. Find the length of $\overline{AF}$ by dividing it into three pieces.

c. Find the length of $\overline{BF}$. 

d. Find the length of $\overline{CE}$. 

5. The base of an isosceles triangle is 6 and the angle opposite it is 30°. Find the length of its legs.

6. Answer the following questions about triangle $\overline{AB}$ in the diagram below:



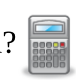
a. Is there only one possible length for side $\overline{BC}$, or are there multiple possible lengths? Briefly explain why.

b. Find the length of $\overline{BC}$. Hint: you may want to draw an additional line. 

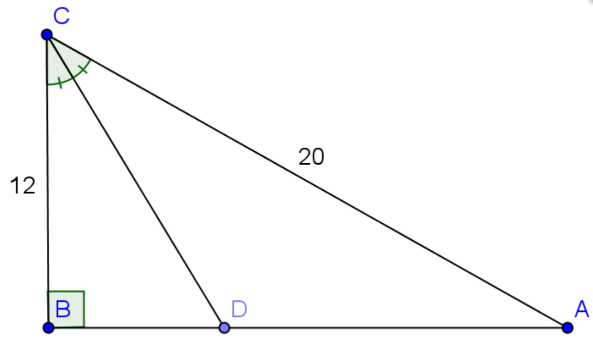
c. Find the measure of angle B. 

7. The ladder on a fire truck is mounted 7 feet above ground level. The ladder itself is 70 feet long.

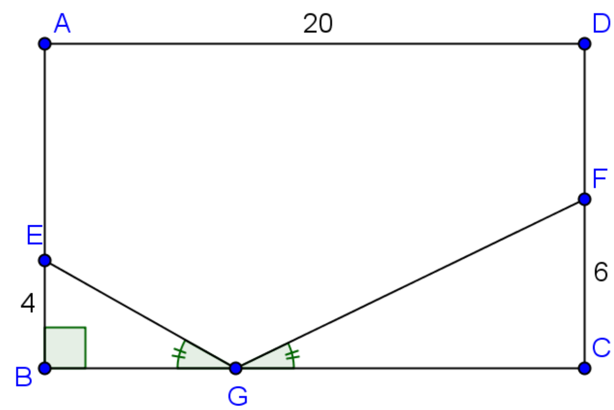
a. The maximum angle the ladder can be elevated to is 70° (almost vertical!). What is the highest it can reach? 

b. Instead, if the top of the ladder is 45 feet above ground level, at what angle is it elevated? 

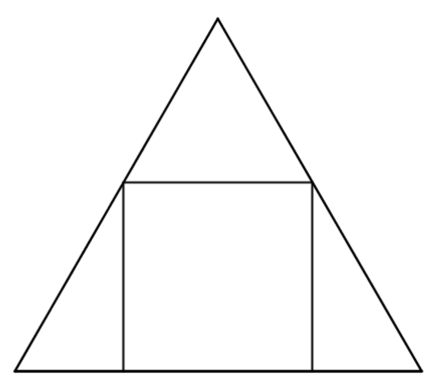
8. In triangle ABC below, $\overline{CD}$ bisects angle BCA. What is the length of segment $\overline{CD}$? Hint #1: there are two different right triangles you want to work with. Hint #2: why is it important that $\overline{CD}$ is the angle bisector, instead of some random segment?



9. In rectangular box ABCD, a ray leaves point E and reflects off $\overline{BC}$ at point G, arriving at point F. The reflection is clean, in that the “angle in” (BGE) is equal to the “angle out” (CGF). What is the measure of these angles?

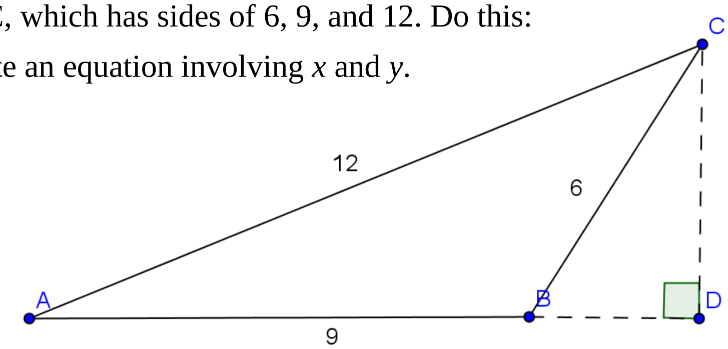


10. A square fits inside an equilateral triangle, where one side of the square has the same midpoint as the base of the triangle. If the triangle’s perimeter is 18, find the perimeter of the square. A decimal approximation is fine. It is tempting to assume the upper vertices of the square are the midpoints of the triangle’s sides, but looks may be deceiving!



11. You want to find the angles of triangle ABC, which has sides of 6, 9, and 12. Do this:

a. Call CD *x* and DB *y*. In triangle BCD, write an equation involving *x* and *y*.



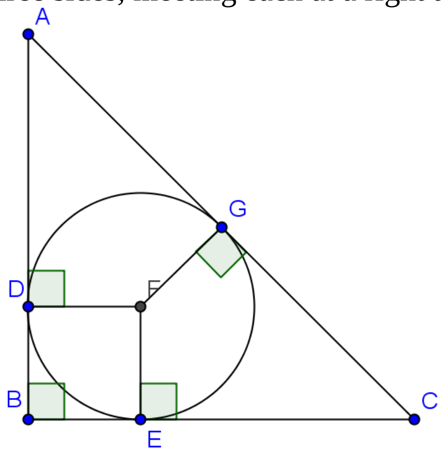
b. Write another equation involving *x* and *y* in triangle ADC.

c. Use algebra to solve this system of two equations for *y* (you may want to substitute for x^2 or subtract one equation from the other).

d. Find the measures of angles A and CBD. 

e. What are the measures of angles ABC and ACB? 

12. The isosceles right triangle below has legs of length one. A circle fits inside of it and is tangent to all three sides, meeting each at a right angle. What is the radius of the circle? A decimal answer is fine.



answers

1. x=11.94, y=65.6° 2a. 108° b. cuts ABE into 2 congruent triangles by HL postulate c. 3.24

3a. FAE=30° b. AE=$8\sqrt{3}$ and EB=16 4a. 135° b. $4+\dfrac{8}{\sqrt{2}}=4+4\sqrt{2}$ c. 10.45 d. 7.39

5. 11.59 6a. one; by SAS b. draw altitude from C… get 9.47 c. 76°

7a. 72.78 feet b. about 33° 8. BCA = 53.1° so BCD=26.6° so CD=13.42

9. 26.6°; by similar triangles BG=8 and GC=12…

10. 2.785 per side; so perim= 11.14 (let 2x be square side; at either triangle on the bottom, $\tan 60^\circ = \dfrac{2x}{3-x}$

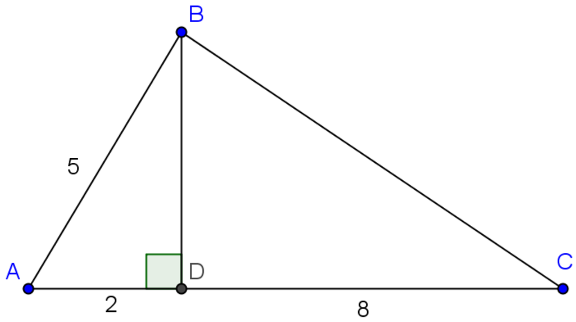
11a. $x^2+y^2=36$ b. $x^2+(y+9)^2=144$ c. y=1.5 d. A=29° and CBD=75.5°

e. so ABC=104.5° and C=180-104.5-29=46.5°

12. About 0.29: draw CF and let the radius be x. CF bisects ECG (by HL). In CFE, $\tan 22.5^\circ = \dfrac{x}{1-x}$

Unit 6 Handout #8: Review Problems

1. In the diagram below, find the length of segment BC. Then determine whether if B is a right angle. Justify your conclusion.

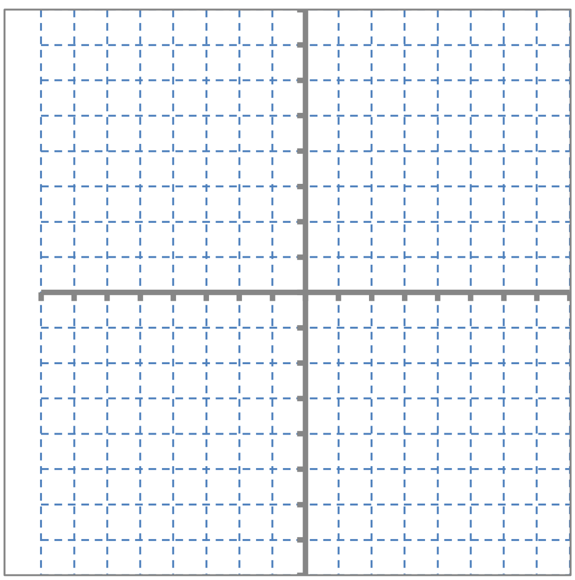


2. Three sides of a triangle are formed by the lines $y = 0.5x - 4,\ x = -2,\ and\ y = 2 - x.$

   a. What is the perimeter of the triangle?

   b. The angle located at which vertex is the largest? Explain.

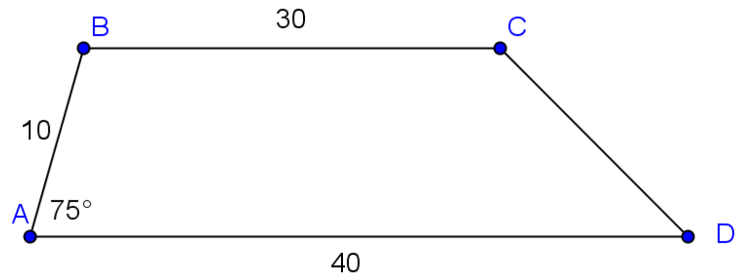
   c. Use trigonometry to find the measure of the largest angle. 



3. Find the following without using a trig table or your calculator.

a. $\tan 60^\circ$ b. $\sin 30^\circ$ c. $\cos 45^\circ$

4. In trapezoid ABCD below, find the following;

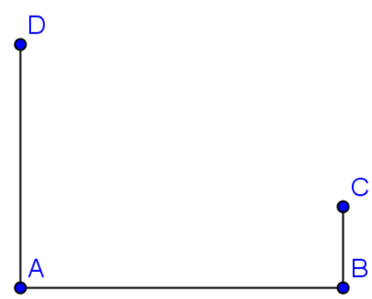


a. The measure of angle D. 

b. The length of $\overline{CD}$. 

c. The length of diagonal $\overline{BD}$. 

5. Find AB below given than CD=12, AD=10, and BC=4.

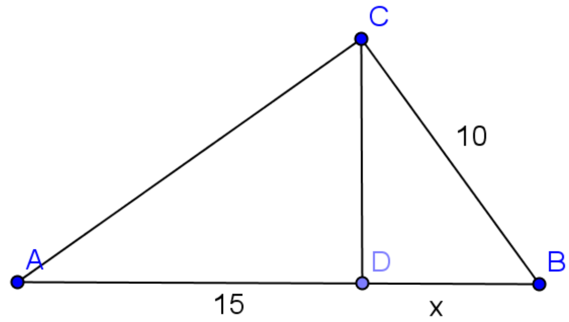


6. How long is the diagonal of a cube if the diagonal of one face of the cube measures 6 units?

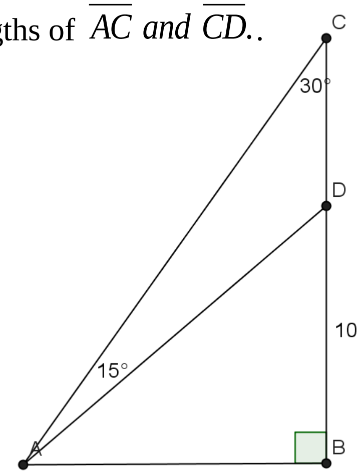
7. Angles C and CDB are right angles in the diagram below.

a. What is the value of *x*? Hint: angle A is congruent to…

b. What is the measure of angle A?



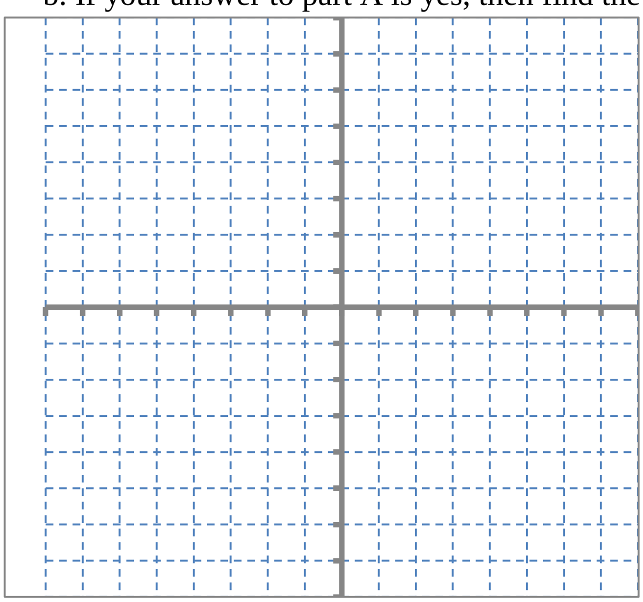
8. In the diagram on the right below, $\overline{BD}$ measures 10 units. Find the lengths of $\overline{AC}$ *and* $\overline{CD}$..



9. Given triangle ABC whose vertices are A (0,0), B (2,8), and C(6,7) do the following:

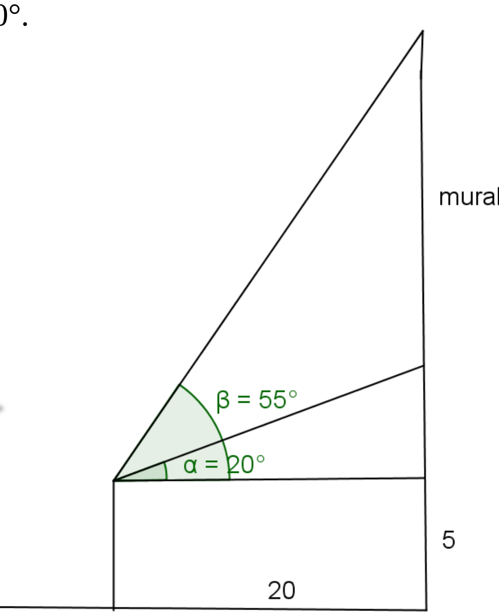
a. Is ABC a right triangle? Explain. (There are at least two good ways to do this!)

b. If your answer to part A is yes, then find the measure of angle A.



10. Ivana is standing up, looking at a mural on a wall twenty feet away. From her eye level, 5 feet above the ground, the angle of elevation to the top of the mural is 55°. From her eye level to the bottom of the mural is an angle of elevation of 20°. Note; the 55° includes the 20°.

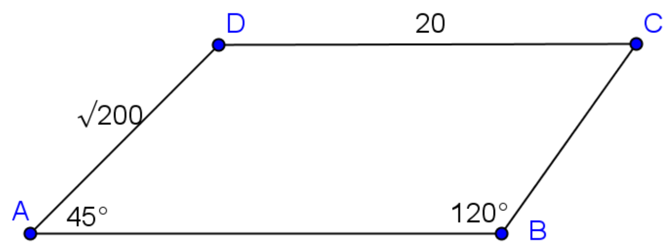
a. How high above the ground is the top of the mural? 



b. How high is the mural itself (from its top to its bottom)? 

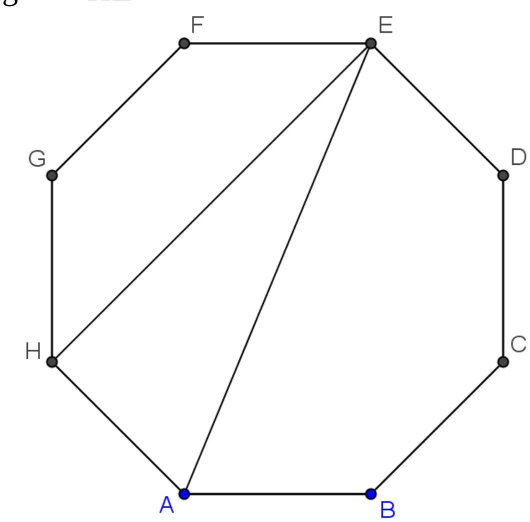
11. In the diagram below, $\overline{AB} \parallel \overline{CD}$.

a. Is ABCD a parallelogram? Explain briefly.

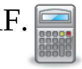


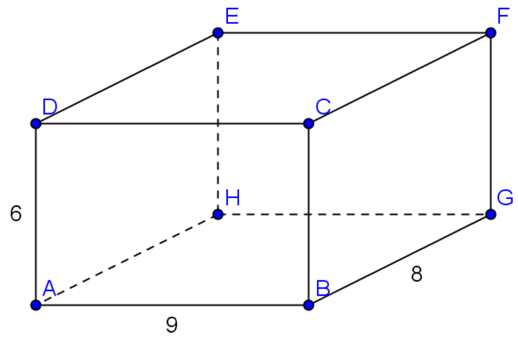
b. Find the perimeter of ABCD.

12. The regular octagon has side length of 6. Find the length of segment $\overline{HE}$.

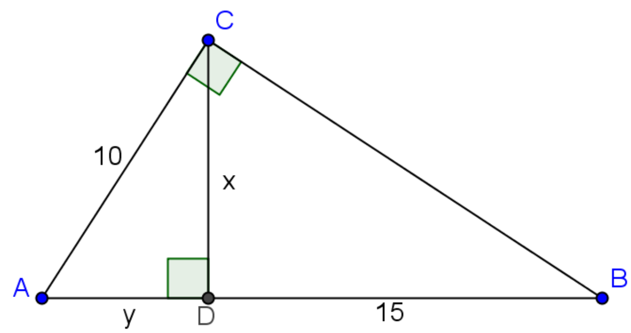


13. Find the length of segment $\overline{AF}$ in the rectangular prism below. Then find the measure of angle GAF.

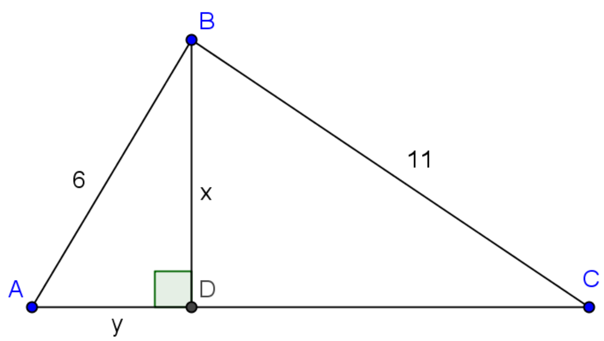




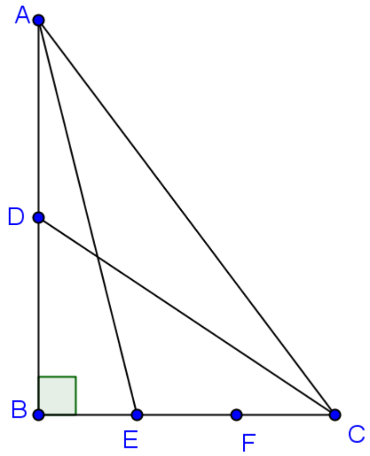
14. Find *x* and *y* in the triangle below.



15. In the triangle below, side AC measures 15 units. B is NOT a right angle. Find the values of *x* and *y*. Hint: you will need to use the Pythagorean Theorem in the two right triangles and solve the system of equations! You should be able to get an exact answer for *y* (it will be a fraction). Then you can use your calculator to get a decimal approximation for *x*.



16. In triangle ABC below, D is the midpoint of AB and E and F trisect BC. If AE=20 and CD=15, then find AC.



Answers

1. $\sqrt{85}$; no Pythag Thm does not hold

2a. $9+6\sqrt{2}+3\sqrt{5}$ b. the one at (4,-2) b/c it is opposite the largest side c. 71.6°

3a. $\sqrt{3}$ b. ½ c. 1/ $\sqrt{2}$ 4a. 52.5° b. 12.2 c. 38.6 5. $6\sqrt{3}$

6. $3\sqrt{6}$ 7a. 5 b. 30° 8. AC=20; CD=$10\sqrt{3}-10$ 9a. yes; slopes or Pythag Them b. 26.6°

10a. 33.6 feet b. 21.3 11a. no; adjacent angles are not supplementary b. $50+10/\sqrt{3}+10\sqrt{2}$

12. $6+6\sqrt{2}$ 13. $\sqrt{181}$; 26.5°

14. angle B=angle DCA so 10/(y+20)=y/10 and y=5 so $x=5\sqrt{3}$ 15. y=14/3 and x≈3.77

16. Let AD=x and BE=y.. then $(2x)^2+y^2=400$ and $x^2+(3y)^2=225$ therefore

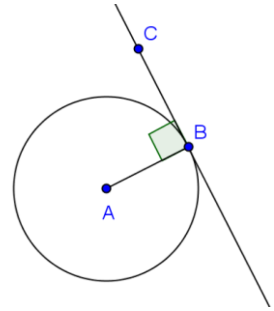
$(AC)^2=4x^2+9y^2$ and $AC=\dfrac{60}{\sqrt{7}}$

Unit 7 Handout #1: Tangents to Circles

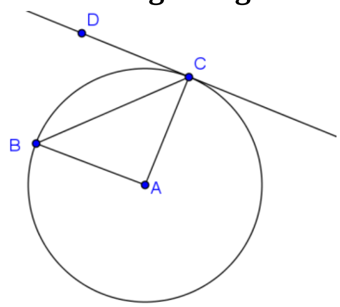
A ***circle*** is defined as the set of points equidistant from a fixed point. The fixed point, of course, is the ***center***. And the distance from any point on the circle to the center is the ***radius*** of the circle.

A ***tangent line*** touches a circle at a single point. The tangent line is perpendicular to the radius at the point of tangency. In the diagram at the right, $\overline{BC}$ is tangent to circle A at point B.

Given that the tangent line creates a right angle with the radius, many problems involving tangent lines may involve right triangles, so expect to use the Pythagorean Theorem and trigonometry!

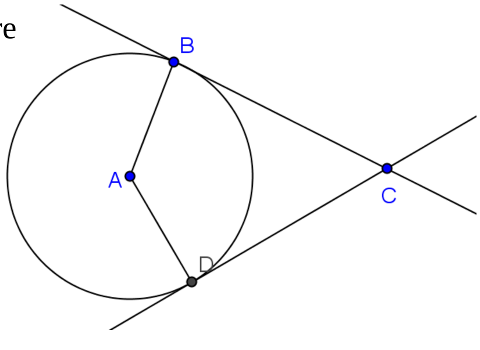


Example #1: In the diagram below, $\overline{CD}$ is tangent to circle A. $\overline{BC}$ bisects $\angle ACD$. Explain why $\angle A$ must be a right angle.



Since the tangent line is perpendicular to the radius, $\angle ACD$ is a right angle, so $\angle ACB$ measures half of that, or 45°. Because $\overline{AC}$ and $\overline{AB}$ are radii, they are congruent. Therefore, $\angle ACB \cong \angle ABC$. Since both are 45° and the angles in any triangle sum to 180°, angle A must measure 90°.

Example #2: Tangent lines $\overline{CB}$ *and* $\overline{CD}$ are draw from point C to circle A, whose radius is 6. They intersect at a 50° angle. Explain why $\overline{CA}$ bisects $\angle BCD$ and find the length of $\overline{BC}$ (using your calculator). 



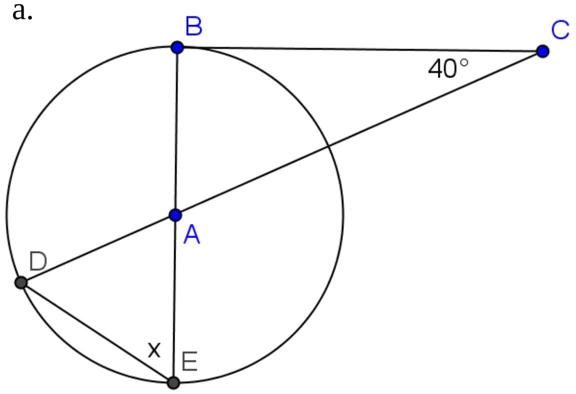
Draw $\overline{AC}$. It must be the case that $\Delta ABC \cong \Delta ADC$ by HL. They are both right triangles (since tangents are perpendicular to the radii at the point of tangency). They share side $\overline{AC}$ and legs $\overline{AB}$ *and* $\overline{AD}$ must be congruent since they are both radii. Therefore angles BCA and DCA are congruent (by CPCTC), so $\overline{CA}$ bisects $\angle BCD$.

Since $\overline{CA}$ bisects $\angle BCD$, angle DCA must measure half of angle DCB, so it measures 25°. In triangle ACD, $\tan 25^\circ = \frac{AD}{DC} = \frac{6}{DC}$.

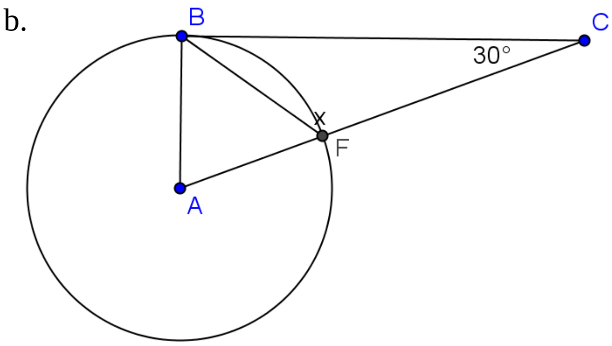
Therefore DC≈12.87.

1. Find *x* in each diagram below. In both cases, $\overline{BC}$ is tangent to circle A.

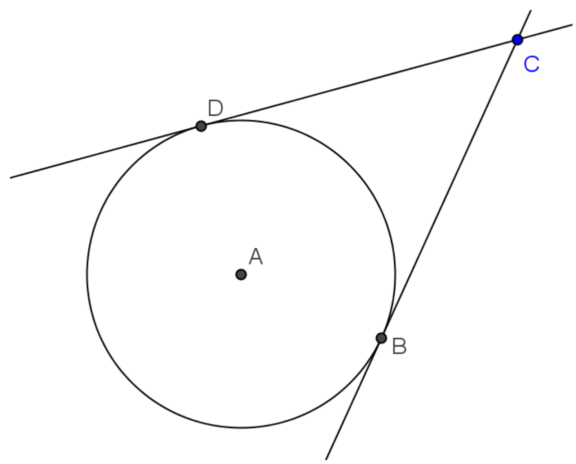
a.



b.



2. Given that the radius of circle A is 6, do the following:



a. Explain why tangent segments $\overline{BC}$ *and* $\overline{CD}$ are congruent.

b. What shape is ABCD? Why?

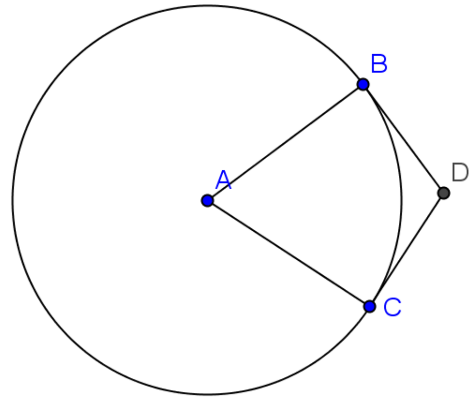
c. Given AC=10, find the length of $\overline{CD}$.

d. Why must $\overline{AC}$ be the perpendicular bisector of $\overline{DB}$?

e. Find the length of $\overline{DB}$.

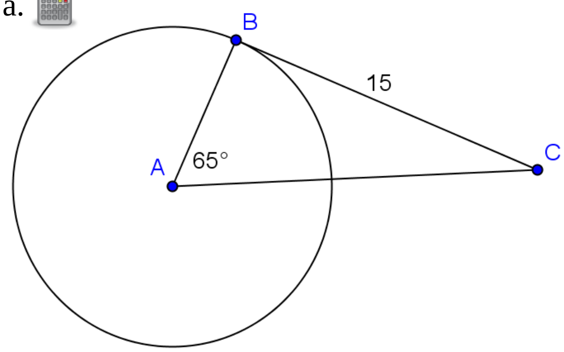
f. Find the measure of angle DAB. 

3. $\overline{DB}$ and $\overline{CD}$ are tangents to circle A with radius 12. BAC measures 60°. Find the lengths of $\overline{DB}$, $\overline{AD}$, and $\overline{BC}$.

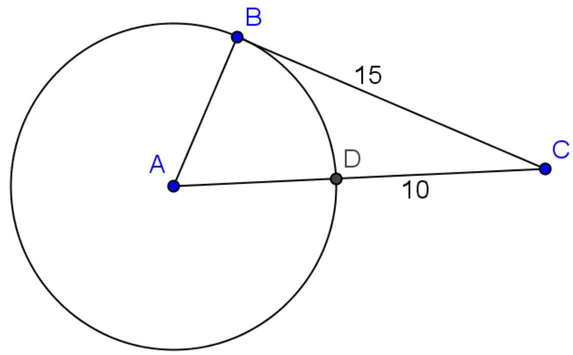


4. Find the radius of the circle in each part below, given that $\overline{BC}$ is tangent to circle A.

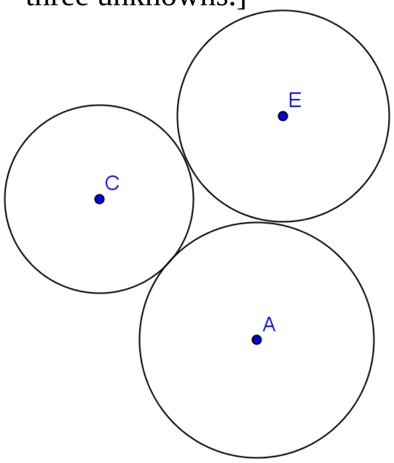
a. 



b.

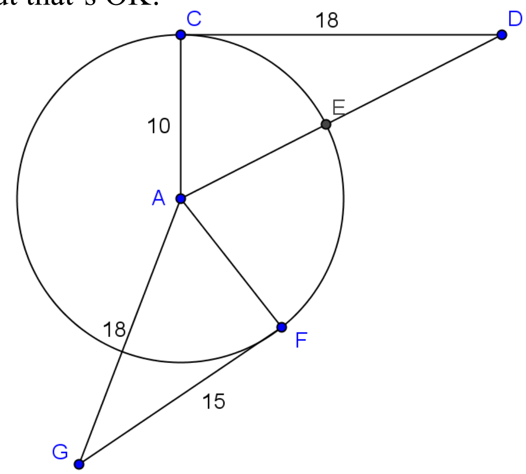


5. Each circle below is tangent to the other two circles. Given that AC=13, EC=12, and AE=15, find the radii of the circles. [There are several ways to approach this; one involves writing three equations with three unknowns.]



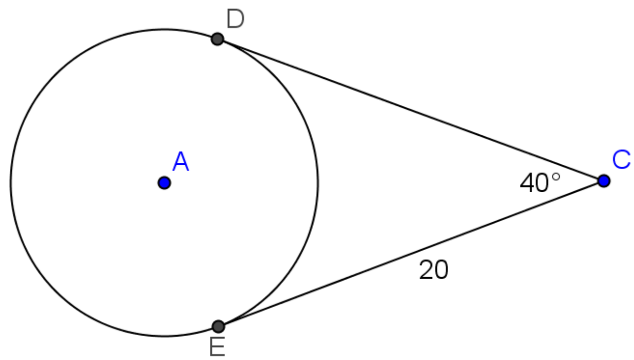
6. Answer the following questions about the diagram below. Note: $\overline{CD}$ is tangent to circle A.

a. Find the length of $\overline{DE}$. Note: your answer will look funny, but that's OK!



b. Is $\overline{FG}$ tangent to circle A? Explain.

7. The tangent segment $\overline{CE}$ from point C to circle A is 20 units long and; it meets tangent line $\overline{CD}$ at a 40° angle.



a. Explain why segment $\overline{CD}$ must also be 20 units long.

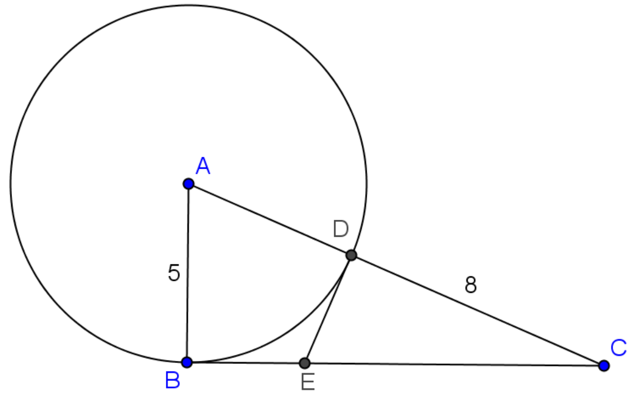
b. Must $\overline{AC}$ bisect $\angle DCE$?

c. What is the radius of the circle? 

8. A chord of length 10 of a circle of radius 12 is how far from the center? (measured on the perpendicular, of course!)

9. In the diagram below, segments $\overline{BC}$ *and* $\overline{DE}$ are tangent to circle A.

a. Explain why ΔABC~ΔEDC.

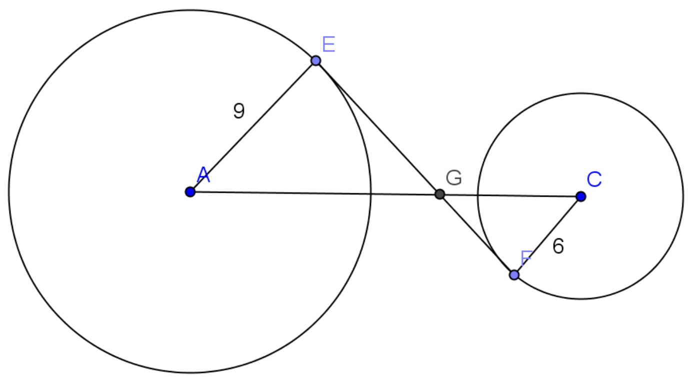


b. Find the length of $\overline{DE}$.

c. What is angle C?

10. The centers of circles A and C are 20 units apart. Segment $\overline{EF}$ is tangent to both circles.

a. Explain why the two triangles must be similar.



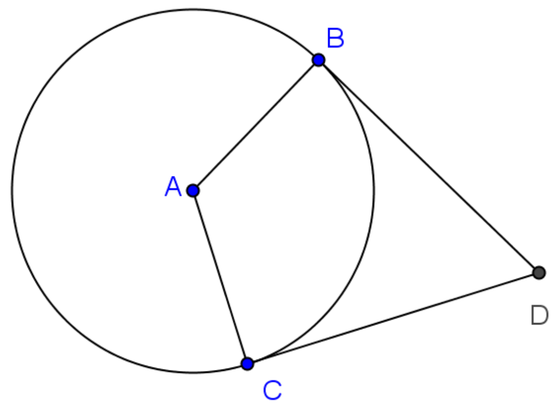
b. Use similar triangles to find the length of $\overline{AG}$.

c. Find the length of segment $\overline{EF}$.

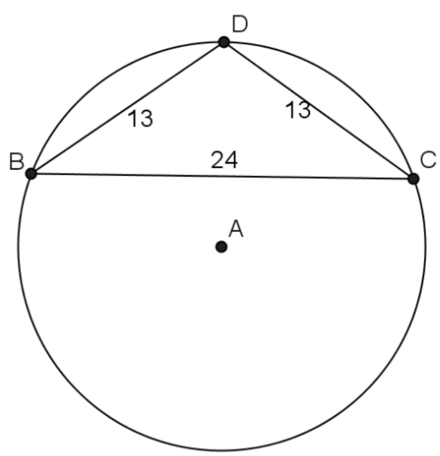
d. What is the measure of angle EGA?

11. A circle with radius 5 has a center (0,0). What is the equation of the tangent line through the point (4,3), and where does it intersect the *x*-axis?

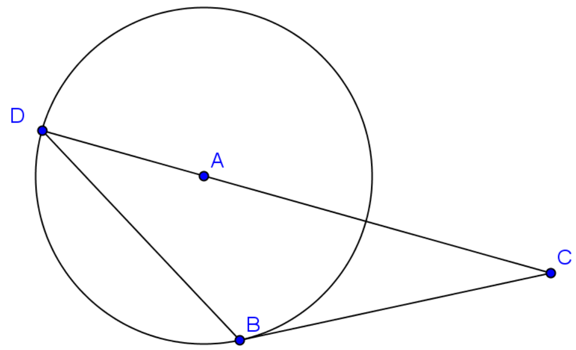
12. In the diagram below, $\overline{DB}$ *and* $\overline{DC}$ are tangent to circle A. Their coordinates are A(0,0), B(5,5), and C(1,-7). Use algebra to find the exact coordinates of D.



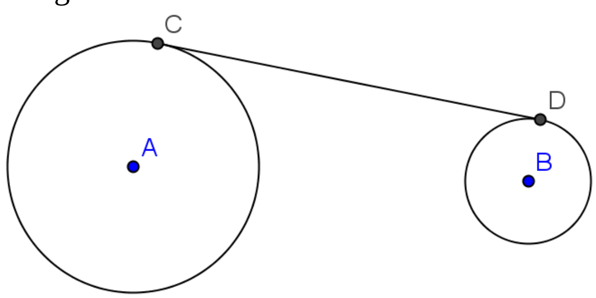
13. Find the radius of the circle below.



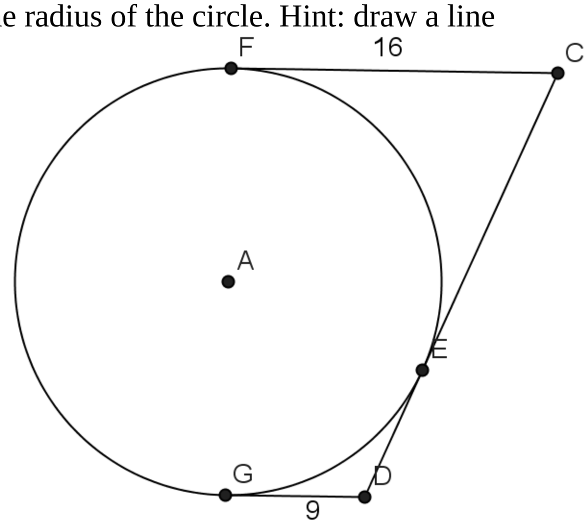
14. In the diagram below, segment $\overline{BC}$ is 12 units long and is tangent to circle A. $\overline{CD}$ is 20 units long. Find the radius of the circle. Hint: draw a segment!



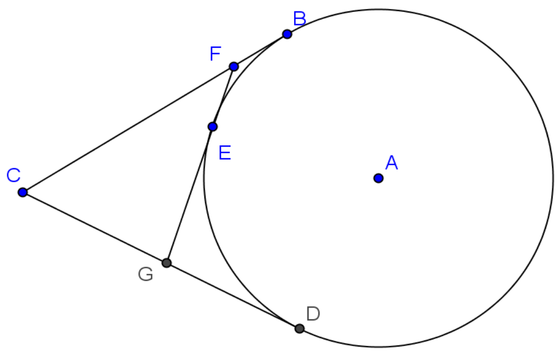
15. $\overline{CD}$ is tangent to circles A and B below. Its length is 16. The radii of the circles are 6 and 4. Find the length of $\overline{AB}$.

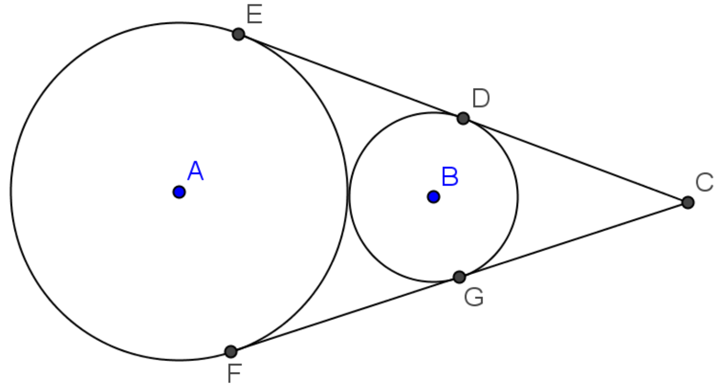


16. Given that tangent segments $\overline{DG}$ *and* $\overline{CF}$ are parallel, find the radius of the circle. Hint: draw a line segment somewhere!



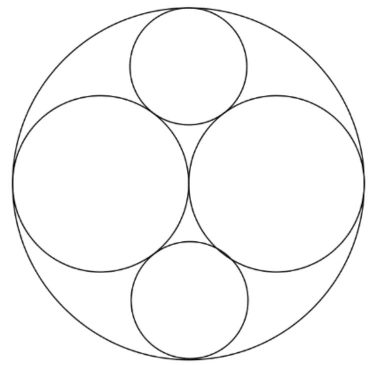
17. In the diagram on the left below, segments $\overline{CD}, \overline{CB}$, *and* $\overline{FG}$ are tangent to circle A. Given that the length of $\overline{CD}$ is 10 units, find the perimeter of triangle CGF.





18. In the diagram on the right above $\overline{CE}$ *and* $\overline{CF}$ are tangent to both circles below if ED=DC=4 then find the radius of the smaller circle.

19. The medium circles have radius of one. What is radius of smallest circles?

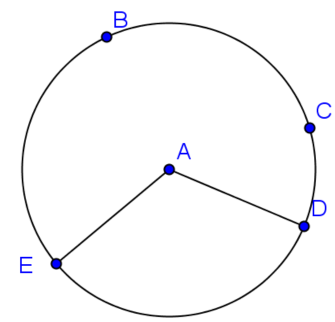


Answers

1a. 65° b. 120° 2a. draw the radii to B and D and AC; ADC is congruent to ABC by HL; then CPCTC
2b. kite since two pairs of cong adj sides and no opposite sides parallel c. 8
2d. A & C are both equid from B & D so are both on the perp bisector e. 9.6 (using similar Δs) f. 106.3°
3. BD= $4\sqrt{3}$; AD= $8\sqrt{3}$; and BC=12
4a. 6.99 b. 25/4 5. 8, 7, and 5 6a. $2\sqrt{106}-10$ b. no; Pythagorean Them does not hold
7a. HL b. yes by CPCTC c. 7.28 8. $\sqrt{119}$ 9a.both are right triangles with angle C b. 10/3 c. 22.6°
10a. right triangles with same vertical angle b. 12 c. $5\sqrt{7}$ using Pythag twice d. 48.6°
11. $y=-\frac{4}{3}x+\frac{25}{3}$ hits x-axis at (6.25,0) 12. (15,-5) 13. 16.9
14. draw AB and get 6.4 15. Draw line thru B parallel to AC and get $2\sqrt{65}$
16. draw line thru D parallel to diameter FG- its length is 24 so the radius is 12
17. 20 since EG=GD so CG+CG=10; CB=CD=10 and EF=FB so CF+FE=10 also
18. $\sqrt{2}$ – let x be small radius, then larger one is 2x and AB is 3x. Thus AC is 6x; using Pythag in AFC yields $8^2 + (2x)^2 = (6x)^2$
19. 2/3. From the center of the largest circle draw a segment to the center of one of the medium circles; its length is 1. The draw a radius of the largest circle through the center of the top small circle. Look at the right triangle formed by the center of the largest circle, the center of the medium circle and the center of the smallest circle. If the radius of the small circle is x then this right triangle has legs of 1 and 2-x and a hypotenuse of 1+x.

Unit 7 Handout #2: Arc Measures and Central Angles

An ***arc*** is a section of a circle joining two points on the circle. A ***minor arc*** is the smaller of the two arcs whose endpoints are the two given points. For example, at the circle on the right, there are two arcs whose endpoints are B and D. Minor arc BD contains point C but not point E. Major arc DB contains point E but not point C



A ***central angle*** of a circle is the angle created by the intersection of two radii at the circle's center. Angle EAD in the circle to the right is a central angle.

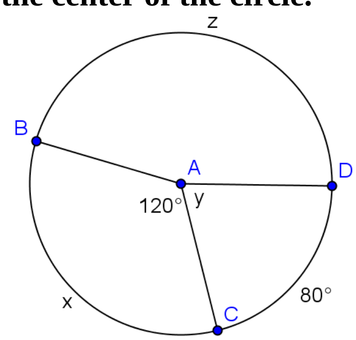
Arcs can be measured in two ways, in degrees and in length. In this section, we will only measure them in degrees. By definition, the arc that represents the entire circle measures 360°, as does the central angle. Thus the ***measure of a central angle is equal to the measure of the arc it intercepts***. For example, if $\angle EAD$ measures 100°, then so does minor arc ED.

Example #1: Find the values of *x*, *y*, and *z* in the diagram below. A is the center of the circle.

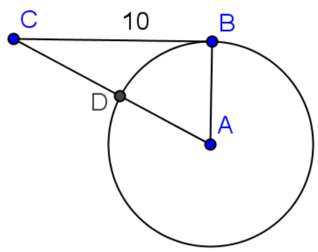
The measure of the arc is equal to the measure of the central angle that intercepts it.
Thus *x* measures 120° and *y* measures 80°.

Angle BAD must measure (360°-120°-80°)=160°.

So *z* is also equal to 160°.



Example #2: Given that minor arc BD below measures 60° and $\overline{BC}$ is tangent to circle A, find the length of line segment $\overline{DC}$.

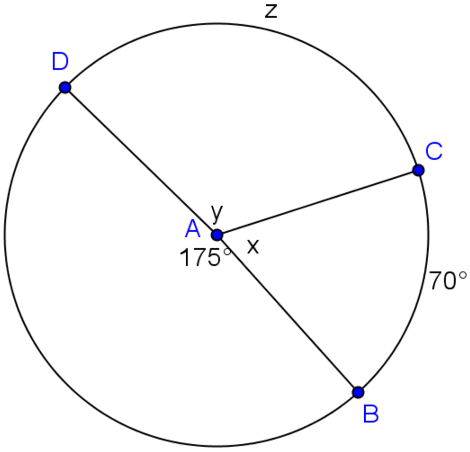


Since minor arc BD measures 60°, so does angle BAD. Thus ΔABC is a 30/60/90 triangle. And using trigonometry or special triangles, we know that $\overline{AB} = 10/\sqrt{3}$ and $\overline{AC} = 20/\sqrt{3}$. Since $\overline{AB}$ and $\overline{AD}$ are both radii, they are congruent.

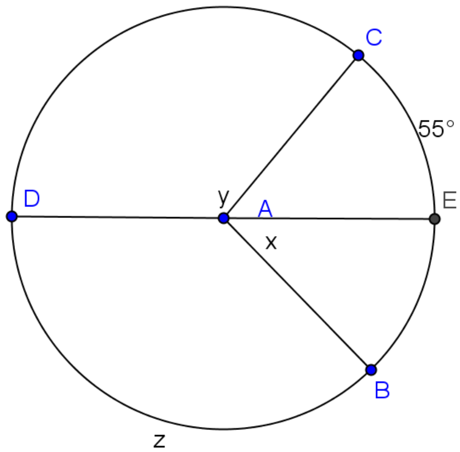
Therefore $DC = AC - AD = 20/\sqrt{3} - 10/\sqrt{3} = 10/\sqrt{3}$

1. Find *x*, *y*, and *z* in the diagrams below. Point A is the center of each circle.

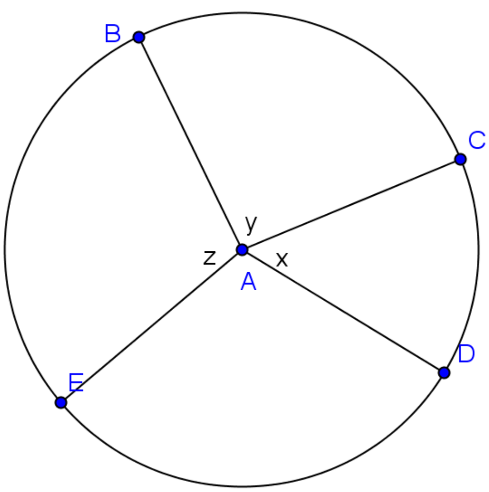
a.



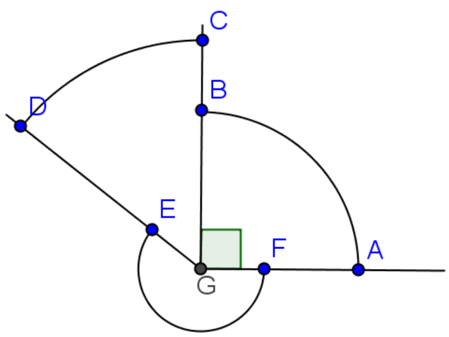
b. Diameter $\overline{DE}$ bisects angle CAB.



c. Central angle EAC measures 160°, minor arc BE measures 100°; minor arc BD measures 150°

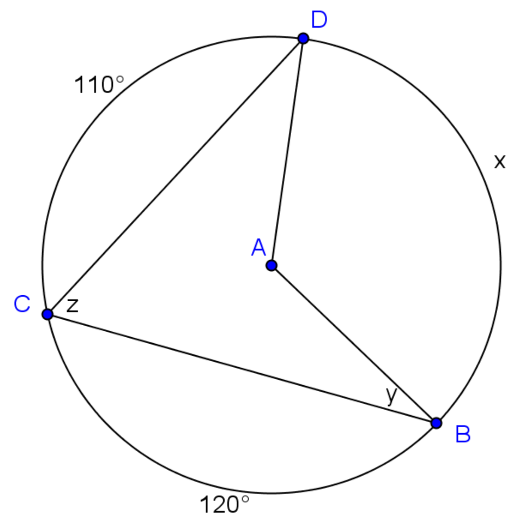


2. AB, CD, and EF are all arcs centered at G. Rank them from large to small (when measured in degrees)

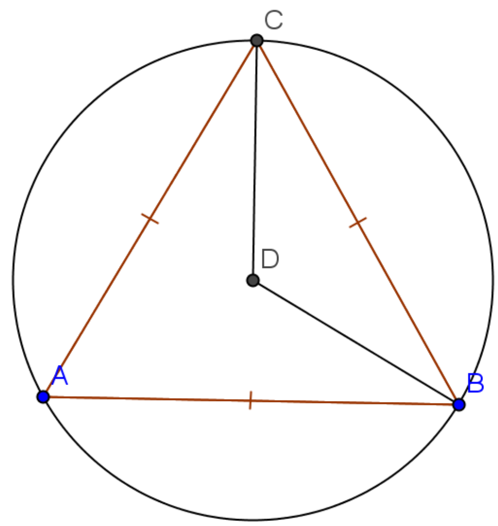


3. Two points on a circle create a major arc and a minor arc. If the measure of the major arc is 20 degrees less than 3 times the measure of the minor arc, then what is the measure of the major arc?

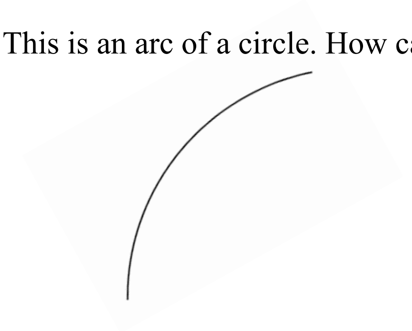
4. Find *x*, *y*, and *z* in the diagram below. Hint: draw $\overline{AC}$.



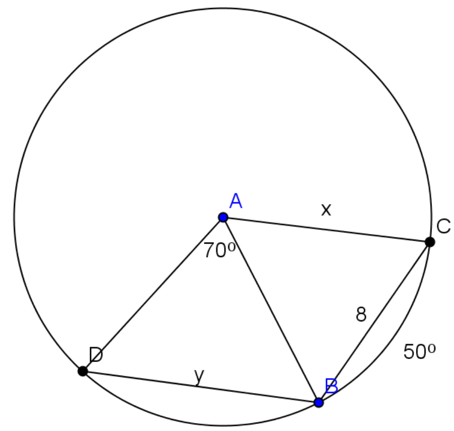
5. In the diagram below, equilateral triangle ABC has side 10. Find the measure of angle D and the radius of the circle. Remember your special triangles!



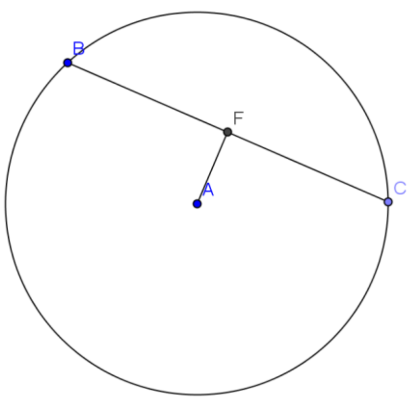
6. This is an arc of a circle. How can you find the circle's center and radius?



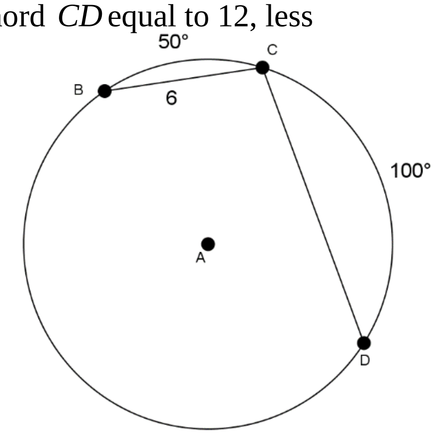
7. In the diagram on the right, find *x* and *y*.



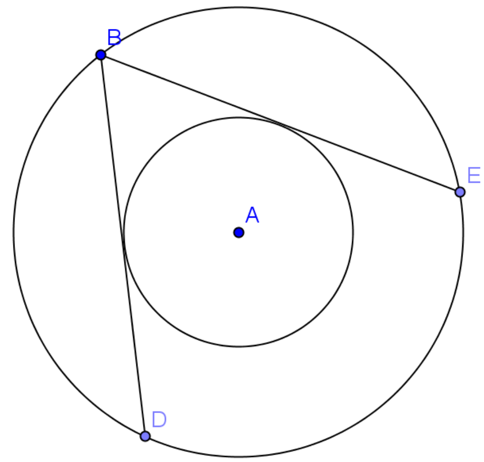
8. Minor arc BC measures 120° and chord $\overline{BC}$ measures 12. What is the radius of circle A?



9. In circle A below, it would be tempting to conclude that chord $\overline{CD}$ is 12 units long, since it cuts off an arc that is twice the measure of the arc cut off by chord $\overline{BC}$. Is the length of chord $\overline{CD}$ equal to 12, less than 12, or more than 12? Explain.



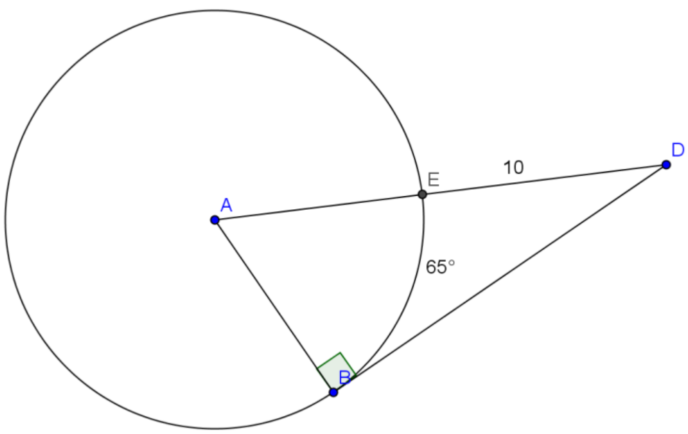
10. In the diagram below, the two circles are concentric. One has radius *x* and the other has radius 2*x*. Chords $\overline{BE}$ *and* $\overline{BD}$ are tangent to the smaller circle.



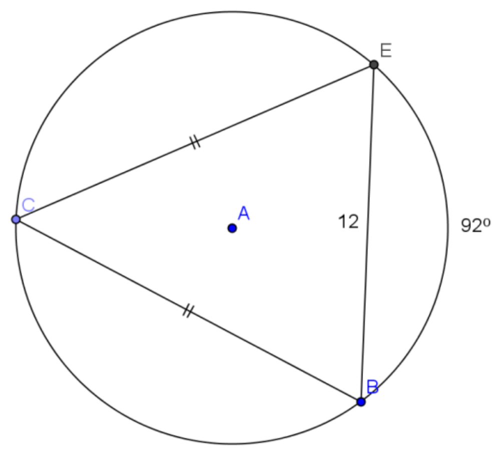
a. What is the measure of angle DBE?

b. What is the length of chord $\overline{BE}$ (in terms of *x*)?

11. Find the length of tangent segment $\overline{BD}$ given that $\overline{ED}$ is 10 and minor arc BE measures 65°. Hint: find the radius of the circle first. 

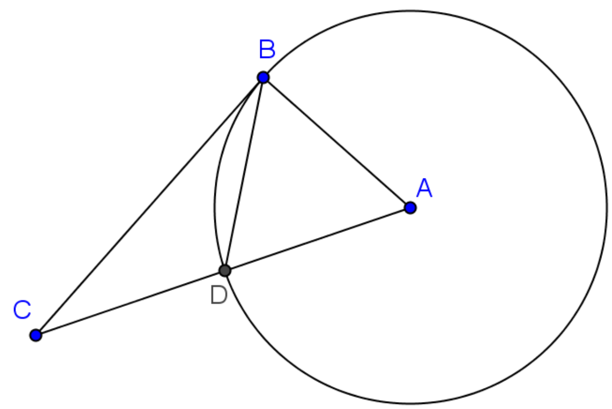


12. Find the length of $\overline{CE}$ below. 

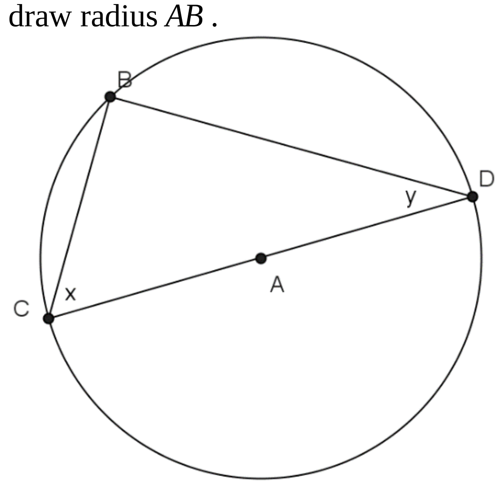


13. In a given circle, a chord of length 10 cuts off a minor arc of 60°; what is the measure of the arc cut off by a chord of length 8? 

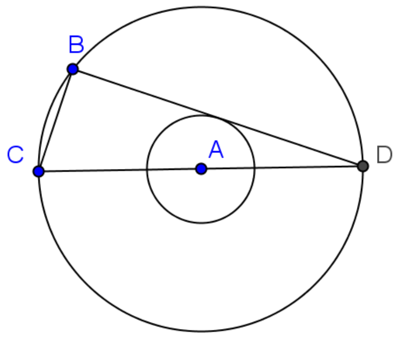
14. Segment $\overline{BC}$ is tangent to circle A. Let the measure of angle BAD be *x*. What is the measure of angle CBD (in terms of *x*)?



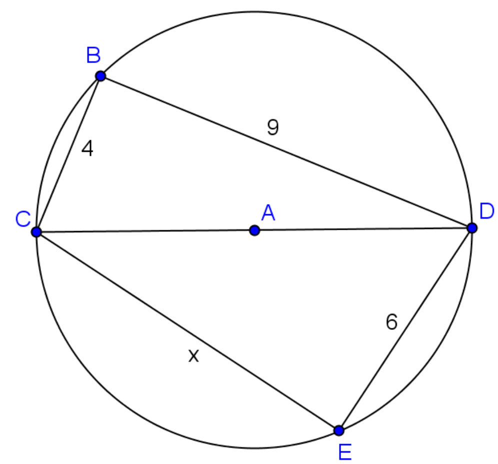
15. ***Inscribed angles and intercepted arcs, part I.*** Inscribed angle B intercepts a semi-circle, an arc measuring 180°. Show that angle B must measure 90°. It may help to draw radius $\overline{AB}$.



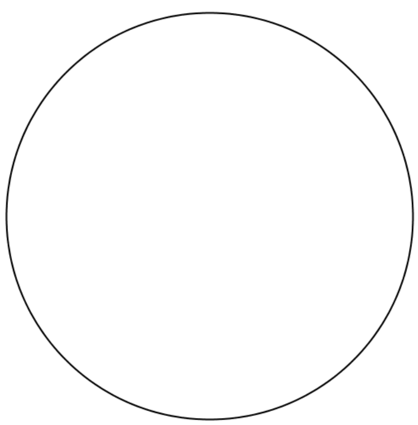
16. The two circles below are concentric; both have their centers at point A. The radius of the large circle is three times the radius of the small circle. $\overline{BC}$ measures 8 units and $\overline{BD}$ is tangent to the small circle. What is the radius of the large circle?



17. $\overline{CD}$ is a diameter of circle A. Find *x*, and justify your answer

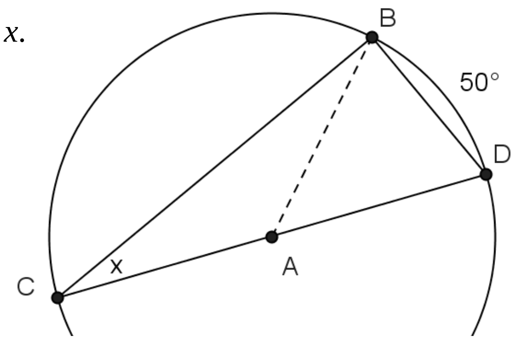


18. Inscribe any right triangle in the circle below. Then use what you know about inscribed angles to explain why the midpoint of the hypotenuse of any right triangle must be equidistant from the triangle's three vertices.



19. ***Inscribed angles and intercepted arcs, part II.***

a. Assume minor arc BD measures 50°. Find the measure of angle *x*.

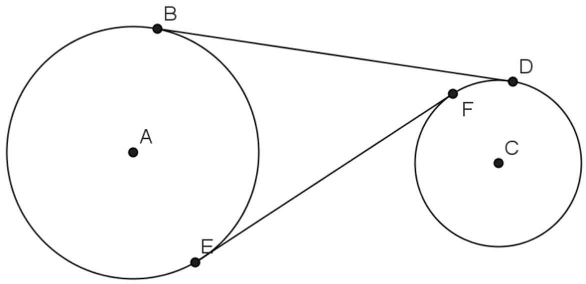


b. Now instead of measuring 50°, it measures *y*°. Find *x* in terms of *y*.

Conjecture: How does measure of an inscribed angle seem to relate to the measure of the arc that it intercepts?

Is there anything about the diagrams above that makes you not completely sure that this relationship always holds?

20. Circles A and C have radii of 8 and 5 and their centers are 20 units apart. Find the lengths of the common external tangent $\overline{BD}$ and the common internal tangent $\overline{EF}$.



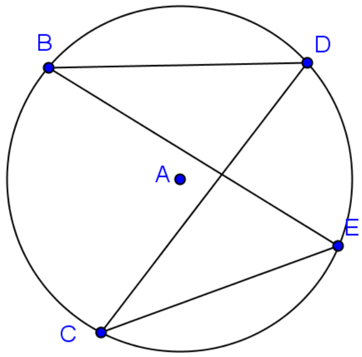
Answers

1a. x=70°; y=z=115° b. x=55°; y=z=125° c. d. x=50°; y=z=100° 2. EF>AB>CD 3. 265°
4. x=130°; y=30°; z=65° 5. D=120° and the radius is $10/\sqrt{3}$
6. pick 3 points on it and draw 2 segments connecting them; draw perp bisectors of the segments and they will meet at the center of the circle.
7. x=9.46 and y=10.86 8. $12/\sqrt{3} = 4\sqrt{3}$ 9. Less; draw two chords each intersecting ½ of arc CD; they are each 6 and they form a triangle where the longest side is less than the sum of the other 2
10. 60° (twice $\sin^{-1}(0.5)$) b. $2x\sqrt{3}$
11. Cos(65)=r/(r+10) so r=7.32 and BD=15.7 12. 15.36 13. 47.2° 14. x/2
15. ABD=y so BAD=180-2y; ABC=x so BAC=180-2x so 180-2x+180-2y=180 and 2x+2y=180 so x+y=90; thus two smaller angles are complements and angle B must be a right angle
16. 12 17. Angles B & E are right since they intercept semi-circles… so in ΔBCD, CD=$\sqrt{97}$, so x=$\sqrt{61}$
18. the hypotenuse must be the diameter of the circle. So its midpoint is the circle's center. By definition, this is equidistant from all points on the circle, which include the triangle's vertices!
19. BAD=50 so CAB=130 and ACB=ABC=25; y/2 20. BD=$\sqrt{391}$ and EF=$\sqrt{231}$

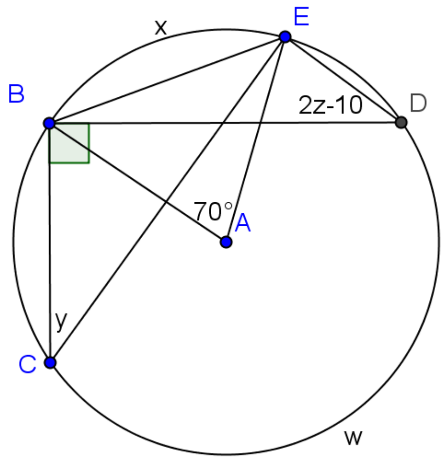
Unit 7 Handout #3: Inscribed Angles

An ***inscribed angle*** is formed by two chords meeting on a circle. Angles B and C in the diagram below are inscribed angles. Both of these angles intercept minor arc DE.

The ***measure of an inscribed angle*** is exactly one-half of the measure of the arc it intercepts. Thus if minor arc DE measures 60°, then angles B and C both measure 30°.



Example #1: Find the values of *x, y, z,* and *w* in the circle A below.



Since the measure of the inscribed arc is equal to the measure of the central angle that intercepts it, *x* must be 70°.

The measure of the inscribed angle is one half the measure of the intercepted arc, so *y* measures one half of 70, or 35°.

For the same reason, 2*z*-10 must also be 35 so *z*=22.5°

Finally, right angle CBD intercepts arc *w*, so *w* must measure 180°.

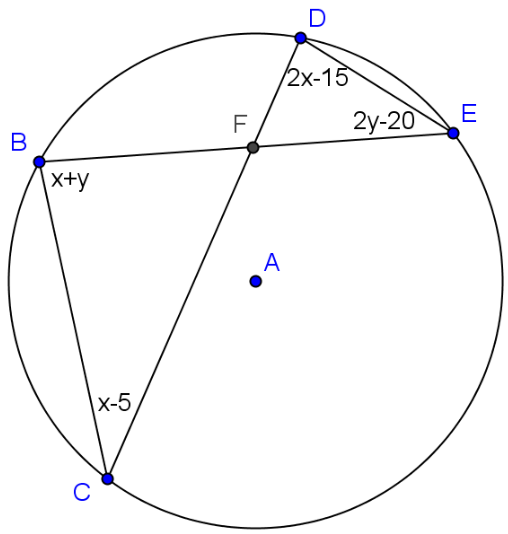
Example #2: Find the measure of minor arcs BD and CE in the diagram below. Then explain why ΔDFE~ΔBFC.

Since angles C and E both intercept minor arc BD, they must be congruent. Thus $x-5=2y-20$ *or* $2y-x=15$.

Since angles B and D both intercept minor arc CE, they must also be congruent. So $x+y=2x-15$ *or* $x-y=15$.

We can solve this system by elimination, adding the two equations together to eliminate *x*. This yields $y=30$. Substituting this *y* into either equation gives us a value of 45 for *x*.

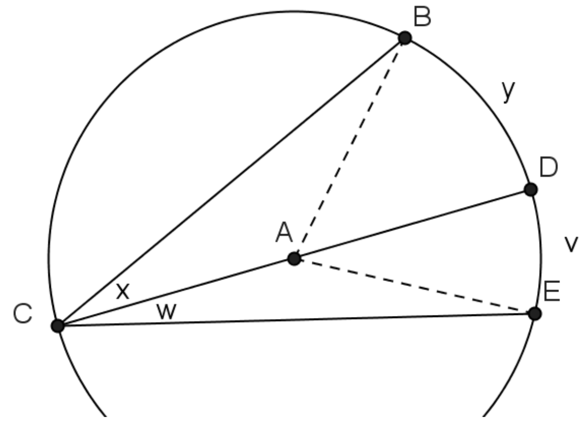
Thus minor arc BD is twice the measure of angle C, or 80°. And minor arc CE is twice is measure of angle D, or 150°.



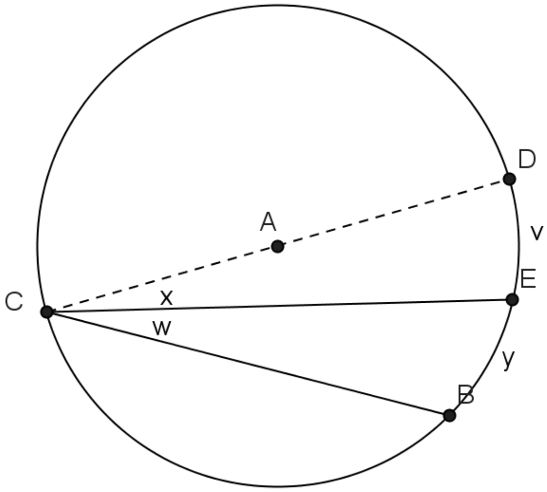
Because inscribed angles that intercept the same arc both measure half of the arc, they must be congruent. Thus $\angle B \cong \angle D$ *and* $\angle C \cong \angle E$ so ΔDFE~ΔBFC by AA (one could also use the fact that vertical angles BFC and DFE are congruent.

1. ***Inscribed angles and intercepted arcs, part III.*** What if one ray of the inscribed angle is not a diameter?

Your goal is to find the measure of angle C (*x*+*w*) in terms of the arc it intercepts, *y*+*v*. Do something similar to what you did in the questions near the end of the last handout!



2. ***Inscribed angles and intercepted arcs, part IV.*** What if both rays of the inscribed angle are on the same side of a diameter? Use the prior results to show that angle *w* is one half of arc *y*.



3. Minor arcs DE and BF measure 150° and 30° respectively

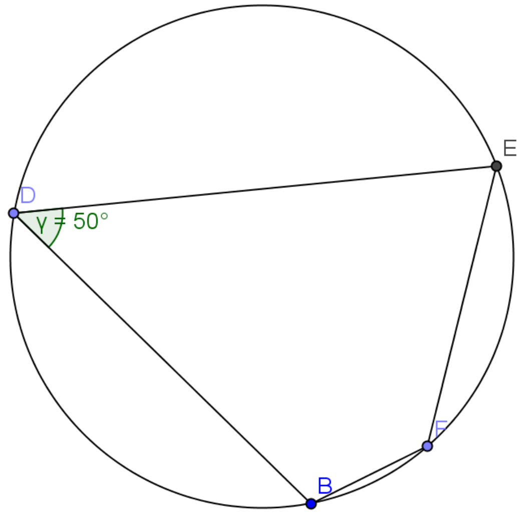
a. What is the measure of minor arc BE?

b. What is the measure of major arc BE?

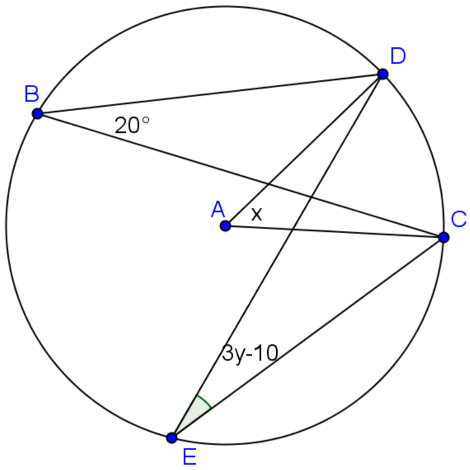
c. What is the measure of angle F?

d. What is the measure of angle B?

e. What is the measure of angle E?

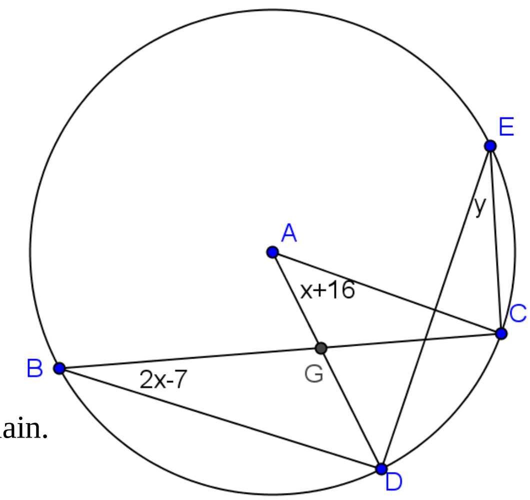


4. Find the values of *x* and *y* in the diagram below. The center of the circle is A.



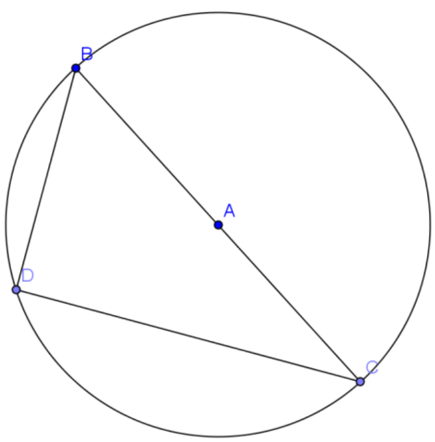
5. Answer the following questions about circle A below.

a. What are the values of *x* and *y*?



b. It appears that $\angle BDA$ is larger than $\angle BCA$. Must it be? Explain.

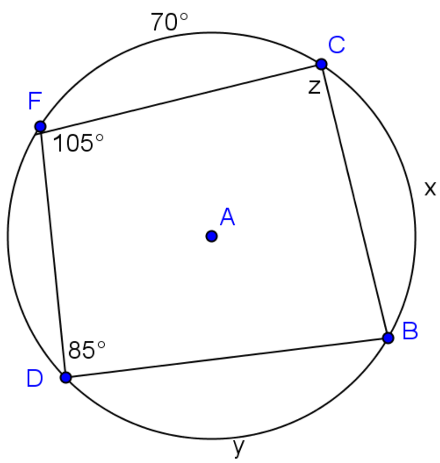
6. Circle A below has radius of 10 and minor arc BD measures 76°. Find the measure of the angles of triangle BCD as well as the length of $\overline{CD}$.



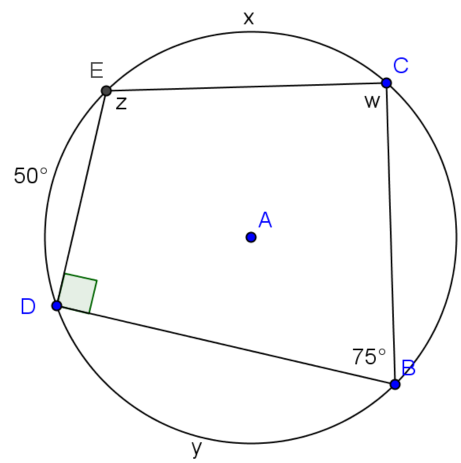
7. Isosceles triangle ABC is inscribed in a circle (meaning, of course, that points A, B, and C are all on the circle). Angle A intercepts an arc of 100°. Find the three possible measures of the arc that angle B intercepts.

8. Find all variables in the diagrams below. Think supplementary!

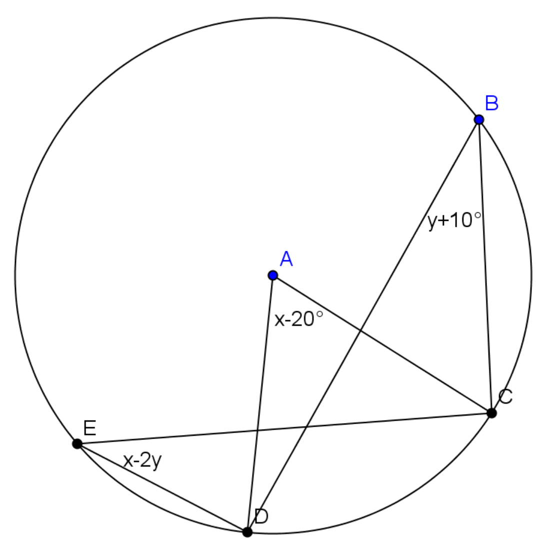
a.



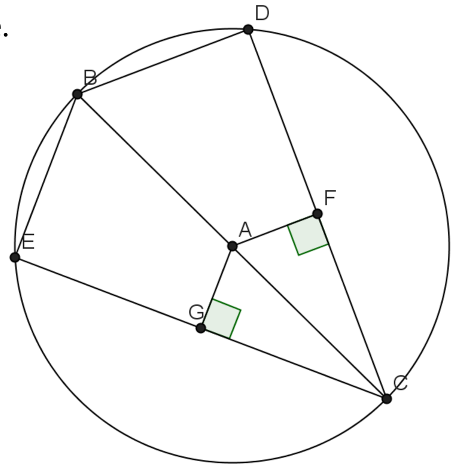
b.



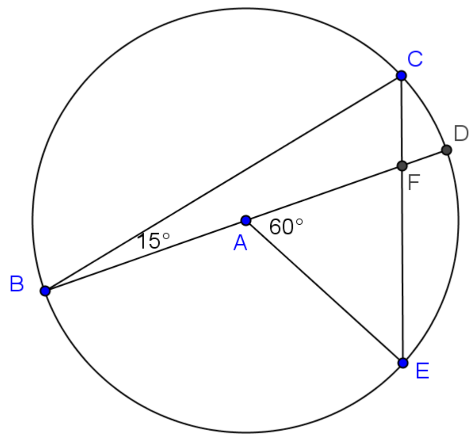
9. Find the values of *x* and *y* in the diagram below.



10. In circle A below, $\overline{CF} \cong \overline{CG}$. Explain why CEBD must be a kite.

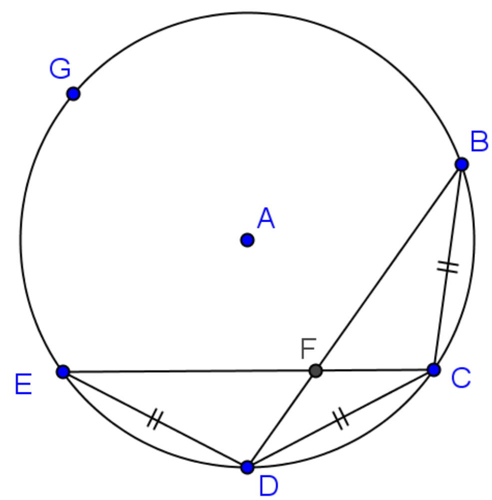


11. In circle A below with diameter $\overline{BD}$, find the measures of minor arc BC and angles BCE and AEF.



12. Given angle CDE measures130° in circle A below.

a. Must ΔEDC be congruent to ΔDCB? Explain.

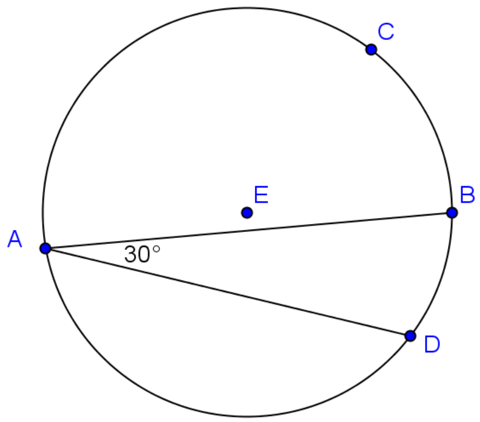


b. Find the measure of arc EGB.

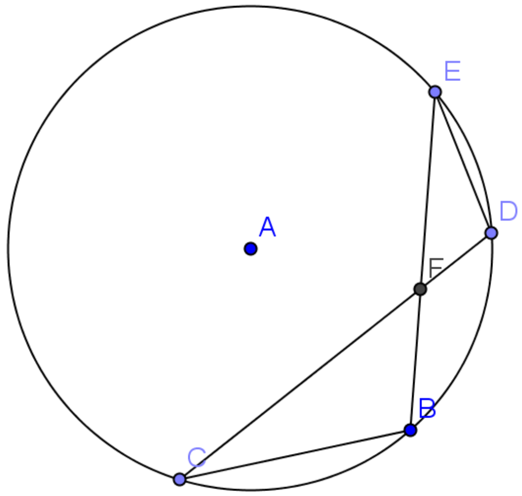
c. Find the measure of angle DFC.

d. Find the measure of angle FAB.

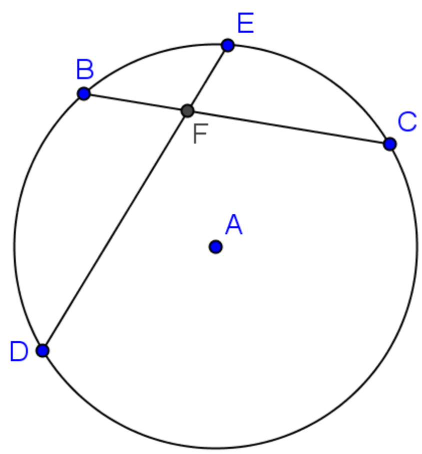
13. Given arc ACB=190° and major arc CD=260°, find minor arc AC.



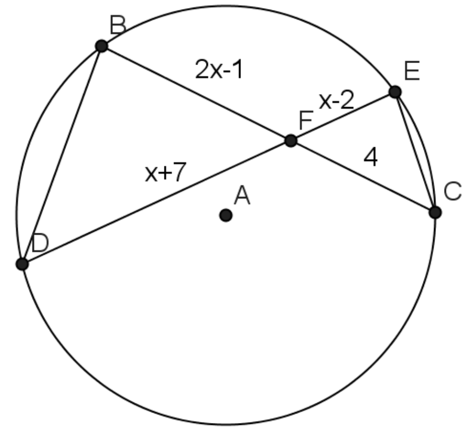
14. Similar triangles? It appears that the two triangles in the diagram below are similar. Can you explain why?



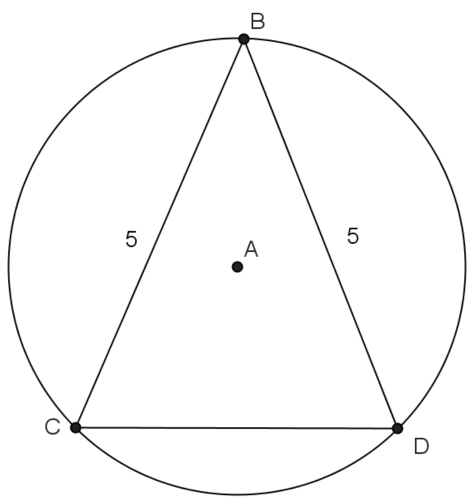
15. In the diagram below, explain why $DF \cdot FE = BF \cdot FC$.



16. Find the value of *x*.



17. Given that the circle's radius is 3, find the length of $\overline{CD}$..



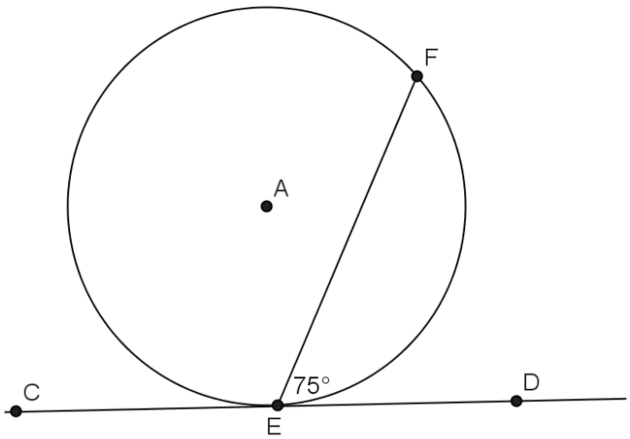
Answers

1. x+w = ½(y+v) so inscribed angle is ½ of intercepted arc 2. (x+w)= ½ (v+y) and x=(1/2)v so w=(1/2)y
3a. 100° b. 260° c. 130° d. 110° e. 70° 4. x=40 and y=10 (since 3y-10=20)
5a. x=10 and y=13 b. yes since A>B and AGC=BGD and ΔACG & ΔBGD must have angles sum to 180
6. C=38°; B=52° CD=15.76 7. 100 or 160 (if triangle is 50-50-80); 130 (if triangle is 50-65-65)
8a. x=100, y=110, z=95 b. x=100; y=130, w=90, and z=105 9. x=100; y=30
10. ΔCAG and CAF are congr by HL so AG=AF; D&E are right angles since they intercept a ½ circle so ΔCAG~ΔCEB and this BE=BD; and by HL or pythag them CE=CD so 2 pairs of cong adj sides
11. BC=150°; BCE=60° ; AEF=45° (draw BE so AEB=30 and BEC=75)
12a. yes; draw radii to B, C, D, and E and get congruent triangles so angle EDC=angle DCB and triangles are congruent by SAS b. 210° c. 130° d. 75° 13. 150°
14. Angles CFB and DFE are congruent because they are vertical. And angles B and D are congruent because they intercept the same arc (as do angles C and E).
15. ΔEFC~ΔBFD because angles DBA and CED are congruent b/c intercept the same arc…
16. $\frac{4}{x-2}=\frac{x+7}{2x-1}$ *so* $x=5$ (x=-2 makes no sense!)
17. Draw line thru AB.. hits CD at E. Let AE= x and DE=y. In ΔBED we have $(x+3)^2 + y^2 = 25$
And in AED we have $x^2 + y^2 = 9$ so 6x+9=16 and x=7/6 so y= $5\sqrt{11}/6$ and CD= $5\sqrt{11}/3$

Unit 7 Handout #4: More Relationships among Arcs, Angles, and Segments

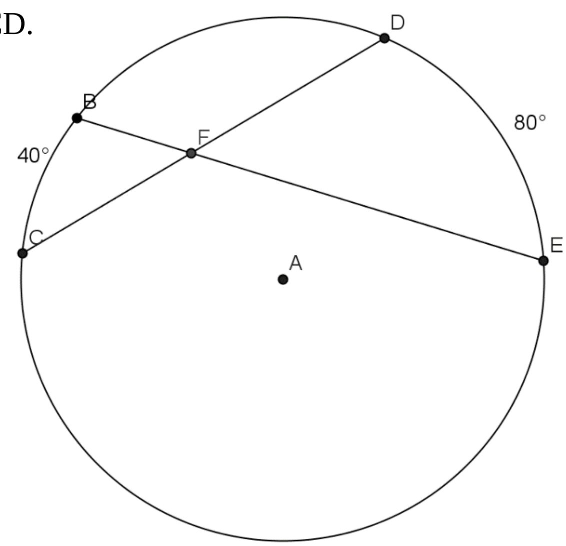
1. ***Angles formed by a tangent and a secant.*** $\overline{CD}$ is tangent to circle A at point E. Angle DEF measures 75°. Find the measure of minor arc EF.

Instead of 75°, assume $\angle DEF$ measured x° (where x<90). Now what is the measure of minor arc EF?



2. ***Angles formed by intersecting chords.*** Look at the diagram below. Your goal is to show how an angle formed by intersecting chords (angle DFE) relates to the measures of the arcs it intercepts (BC and DE).

a. Draw chord $\overline{CE}$ and find the measure of angles CEB and ECD.

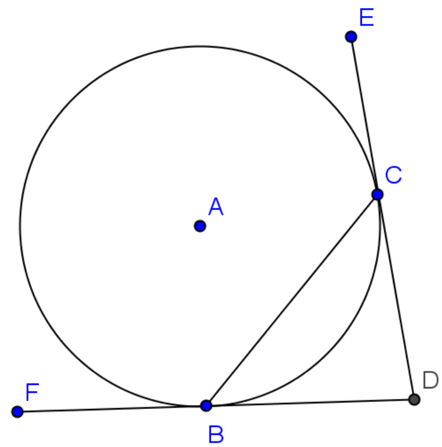


b. Use them to find the measure of angle DFE.

c. Try the same exercise where minor arcs BC and DE measure a and b instead of 40° and 80°.

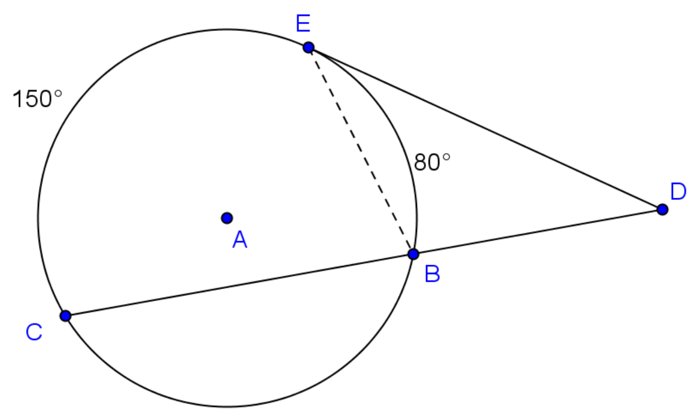
3. In the diagram below, minor arc BC measures 100° and $\overline{DF}$ *and* $\overline{DE}$ are tangent to circle A.

a. Find the measure of angles BCD and D.

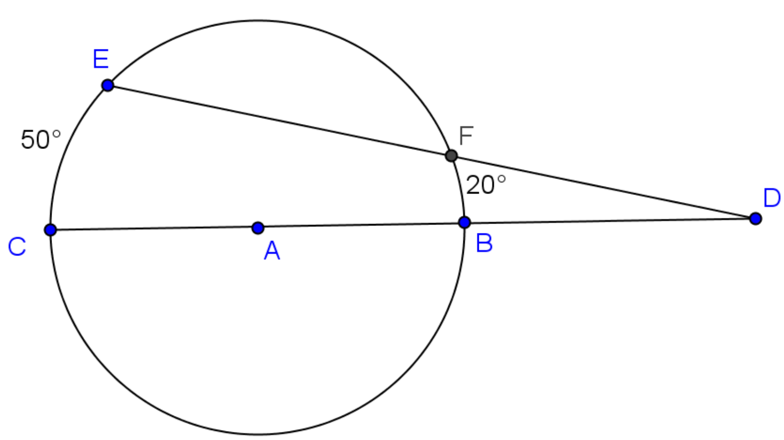


b. Does $\overline{AD}$ bisect minor arc BC? Explain.

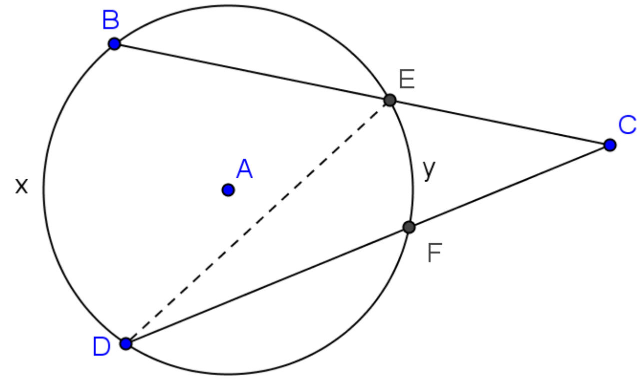
4. ***Angles outside the circle and arcs they intercept I.*** $\overline{DE}$ is tangent to circle A. Find the measure of angle D below.



5. ***Angles outside the circle and arcs they intercept II.*** Find the measure of angle D in the diagram below. There are many ways to proceed; you'll probably need to draw at least one extra segment.



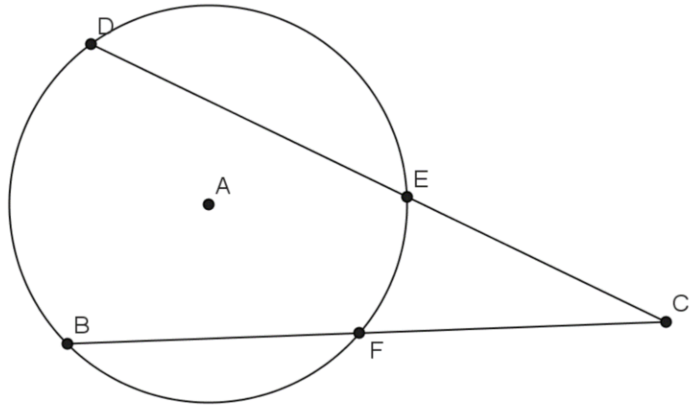
6. ***Angles outside the circle and arcs they intercept III.*** The previous two examples used lines through the external point that were either tangent lines or a diameter. This example is more general, showing that the measure of angle C is $\frac{x-y}{2}$.



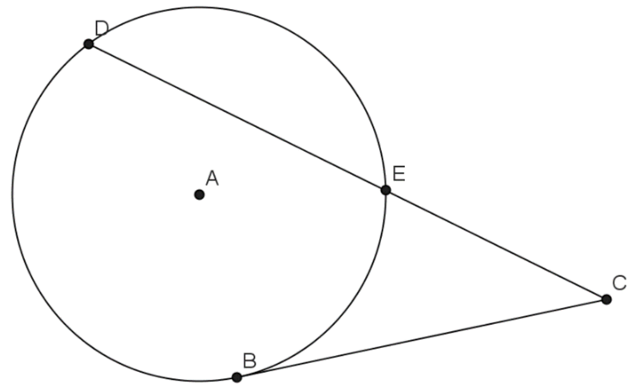
a. Find the measure of angles CDE and BED in terms of *x* and *y*.

b. Now look at triangle CDE and find the measure of angle C in terms of *x* and *y*. You may use the external angle, but do not need to.

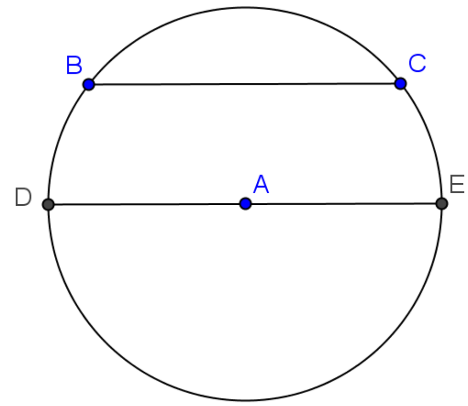
7. Explain why $CE \cdot CD = BC \cdot CF$. Look for similar triangles….



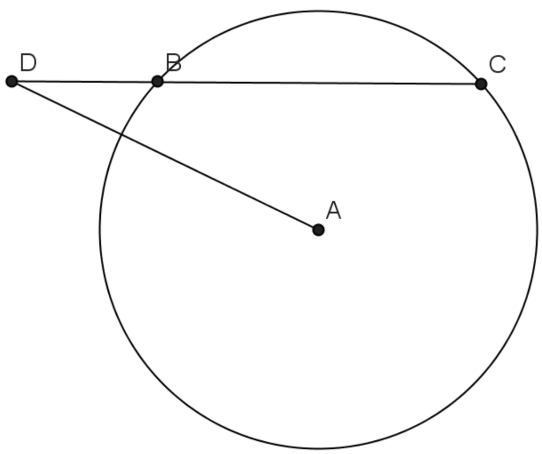
8. Given tangent segment $\overline{BC}$, explain why $BC^2 = CD \cdot CE$



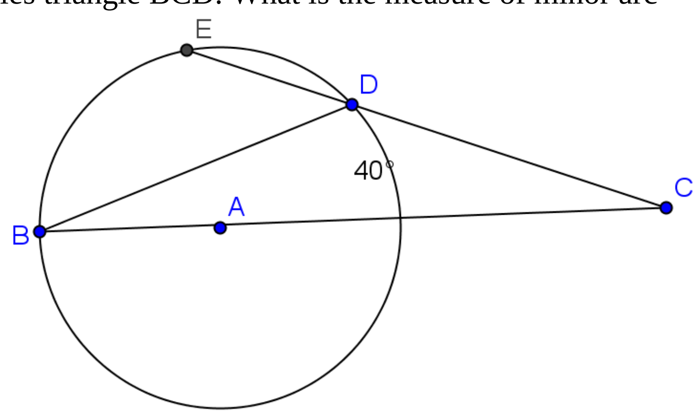
9. In circle A below, chords $\overline{BC}$ *and* $\overline{DE}$ are parallel. Must minor arcs BD and CE have the same measure? Explain.



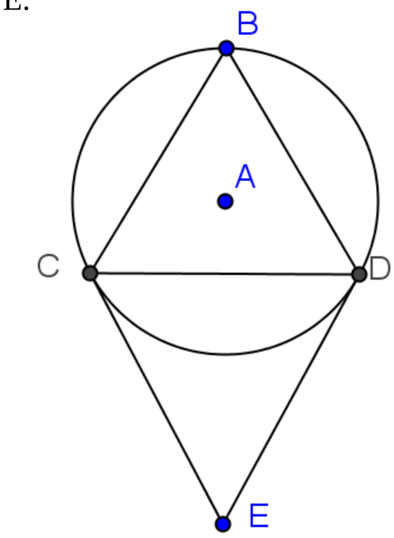
10. In the diagram below, DB=4, BC=10, and AD=15. Find the radius of the circle.



11. The center of circle A is on base $\overline{BC}$ of isosceles triangle BCD. What is the measure of minor arc DE?



12. BCD is an isosceles triangle with BC=DB. CE and DE are tangent segments to circle A. The measure of angle BCD is twice the measure of angle E. Find the measure of angle E.



Answers

1a. draw diameter thru A and E; small arc measures 30° so EF measures 150°; 2x°

2a. CEB=20° and ECD=40° b. DFE is an external angle to triangle CFE so its measure is the sum of the remote interior angles, or 60°. c. (a+b)/2 → the average of the intercepted arcs!

3a. BCD=50°; D=80° b. yes, since DA bisects angle BAC

4. 35° since CBE=75° and BED=40° 5. 15°; either draw AF and AE or draw CF or BE

6a. CDE=y/2; BED=x/2 b. $\angle BED = \angle EDC + \angle C$ so (x/2)=(y/2)+C and C=(x-y)/2

7. draw DF and BE; ΔDCF~ΔBCE so CE/CF = BC/DC

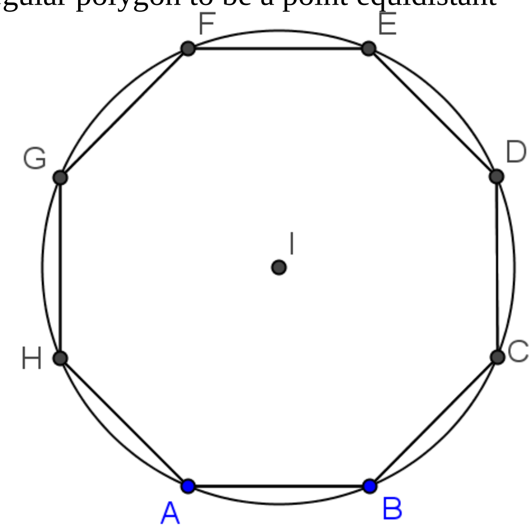
8. angles EBC and EDB are congruent (intersect same arc), so ΔBCE~ΔDCB so BC/CE=DC/CB

9. yes; draw CD and see that angles BCD and CDE are congruent (alt inter).. so intercepted arcs are equal

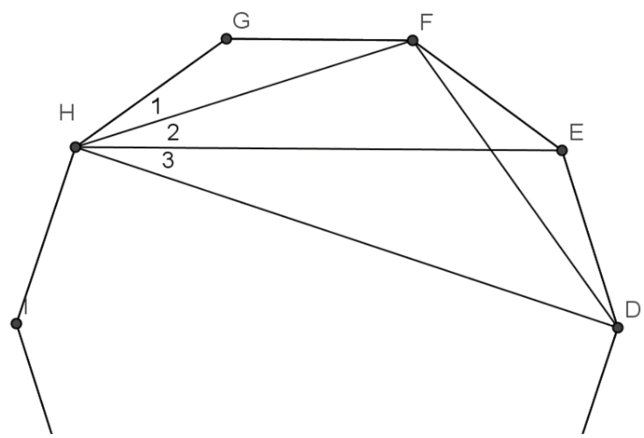
10. 13 – use Pythag twice or 4(14)=(15-x)(15+x) 11. 60° 12. 180/7

Unit 7 Handout #5: Polygons and Circles

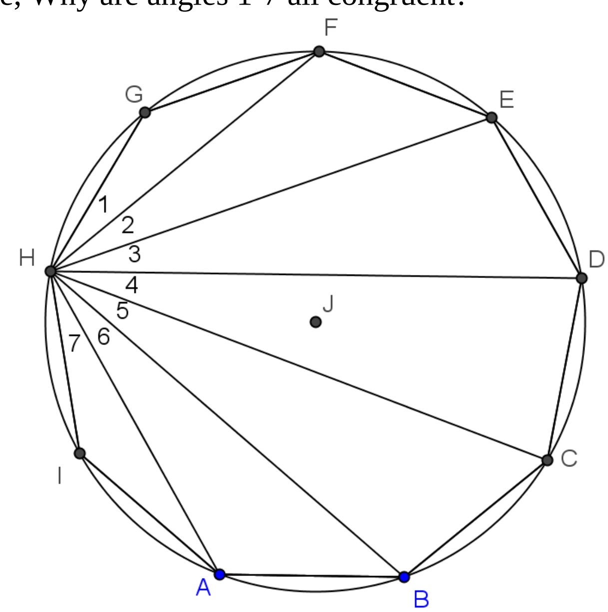
1. A regular octagon is inscribed in a circle, creating 8 minor arcs around the circle. Explain why those arcs must all be congruent. You may consider the center of a regular polygon to be a point equidistant from all vertices.



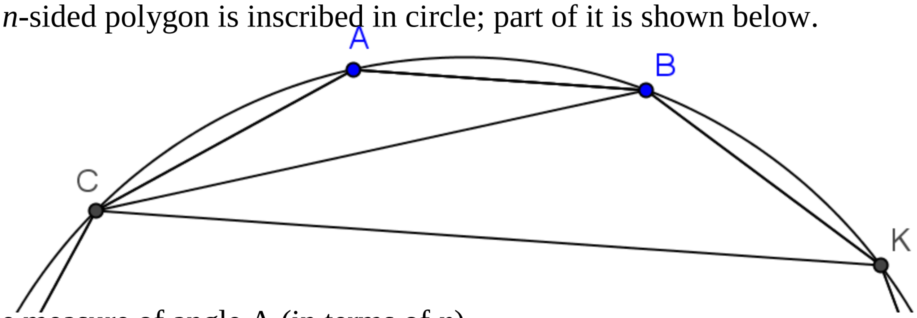
2. In the regular decagon below, show that angles 1, 2, and 3 are congruent. You may use the results the previous question.



3. A regular nine-sided polygon is inscribed in a circle, Why are angles 1-7 all congruent?



4. A regular *n*-sided polygon is inscribed in circle; part of it is shown below.



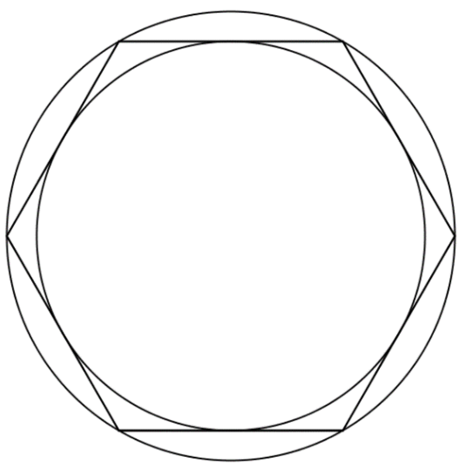
a. Find the measure of angle A (in terms of *n*).

b. Find the measures of angles ACB and BCK in terms of *n*.

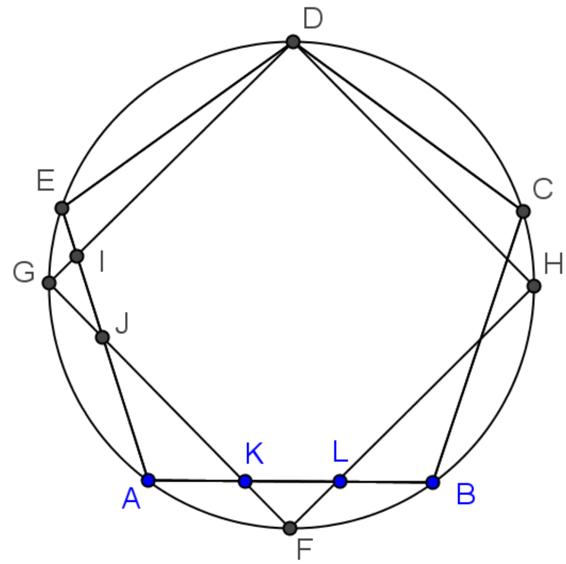
c. Must ABKC be an isosceles trapezoid? Explain.

d. Does this mean that any four consecutive vertices of a regular polygon with at least five sides forms as isosceles trapezoid?

5. A regular hexagon is inscribed in a circle of radius 6. A smaller circle is inscribed in the hexagon. Without using your calculator, find the perimeter of the hexagon and the radius of the smaller circle.



6. A regular pentagon and square are inscribed in a circle below. They share vertex D. Find the following:



a. The measure of minor arc CH.

b. The measure of minor arc AF.

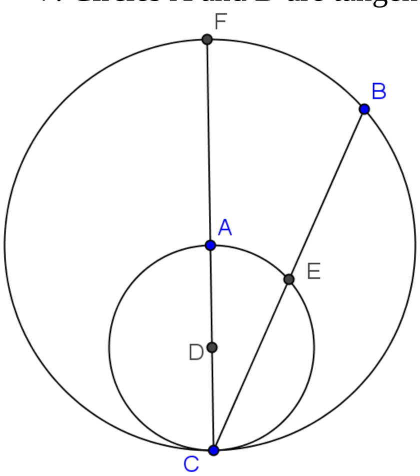
c. The measure of angle C.

d. The measure of angle EID.

e. The measure of angle IJK.

f. The measure of angle FKL.

7. Circles A and D are tangent to each other at point C. Explain why E is the midpoint of chord $\overline{CB}$.

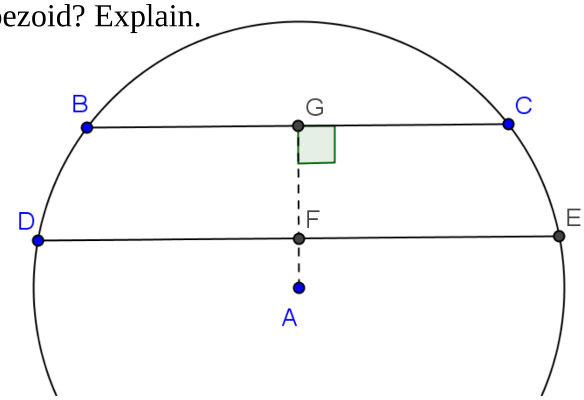


8. Determine whether each statement below is true or false; justify your answer.

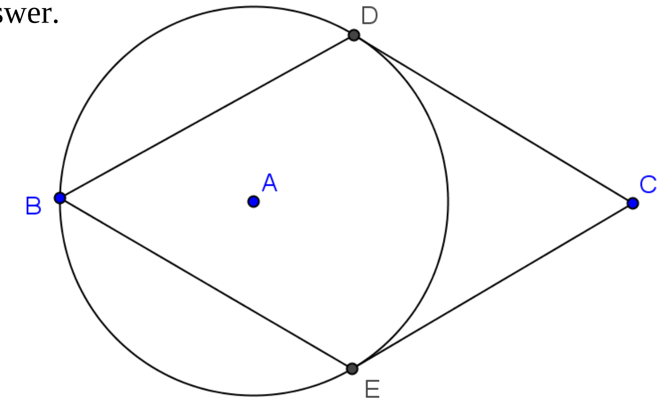
a. A kite inscribed in a circle must have two right angles.

b. A parallelogram inscribed in a circle must be a rectangle.

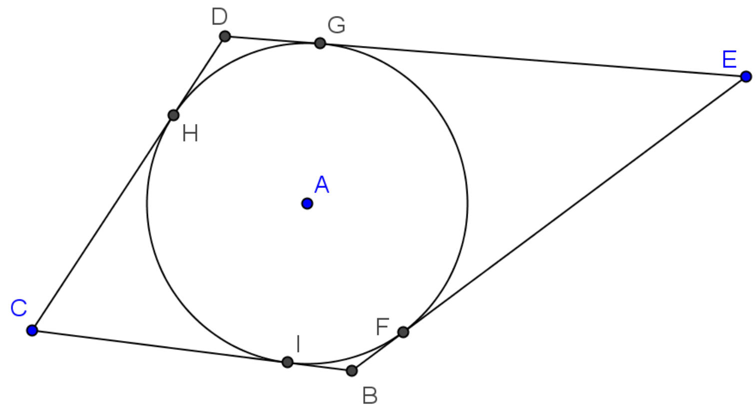
9. In circle A below, $\overline{BC} \parallel \overline{DE}$. Must BCED be an isosceles trapezoid? Explain.



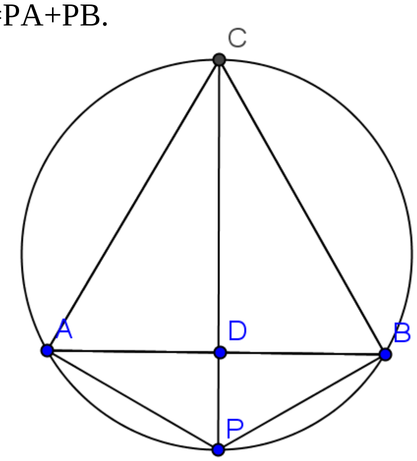
10. BECD is a rhombus, with B, E, and D on circle A. $\overline{CD}$ *and* $\overline{CE}$ are tangent to circle A. It looks like BDE is an equilateral triangle. Must it be? Justify your answer.



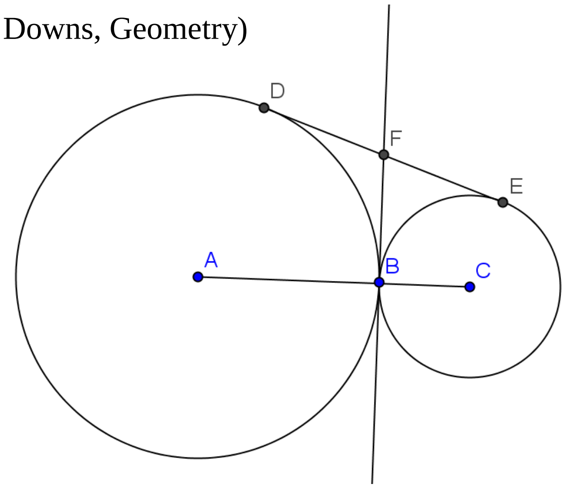
11. The sides of quadrilateral BCDE are tangent to circle A. Explain why BC+DE=CD+BE. (from Moise Downs, Geometry)



12. ABC is an equilateral triangle and ACBP is a kite. Explain why PC=PA+PB.



13. Circles A and C intersect at B, and line FB is tangent to both circles. Segment $\overline{DE}$ is also tangent to both circles. Explain why F is the midpoint of $\overline{DE}$ (from Moise Downs, Geometry)



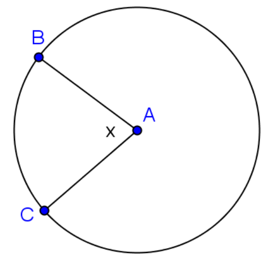
Answers

1. connect each vertex to the center of the polygon (I). Eight congruent triangles are created.
2. they intercept congruent arcs.. 3. They intercept congruent arcs
4a. 180-(360/n) b. ACB=180/n and BCK=180/n c. yes since angles ACB, ABC, and BCK are congruent (intercepting congruent arcs), we know AB//CK. And angles ACK and BKC are equal so it is an isosceles trapezoid. d. yes.. cool!
5. hexagon perim=36; radius of smaller circle is $3\sqrt{3}$
6a. 18° b. 36° c. 108° d. 63° e. 153° f. 45°
7. ΔFBC~ΔAEC because B and E are both right angles (intersect semi-circular arcs) and share angle C. Since AC= ½ CF (A is the center of a circle); CE must be ½ of CB, thus E is the midpoint.
8. both are true; any quadrilateral inscribed in a circle has opposite angles that are supplementary. That means the kite has two right angles. And since a parallelogram's opposite angles are congruent, if they are supplementary then each is 90° so it is a rectangle.
9. yes. Draw FC and FB. Triangles CFG an BFG are congruent by HL so angles GFC and GFB are equal. Since complements of congruent angles are congruent, angles CFE and BFD are congruent. Thus triangles CFE and BFD are congruent by SAS so legs CE and BD of trapezoid BCED are congruent.
Angles CFE and BFD are congruent since they are central angles intercepting arcs of equal measures.
10. Yes; draw AB and AE; angles AEB and ABE and ABD are equal and angles DBE and BEC sum to 180; also angle AEC is a right angle. So AEB=ABE=ABD=30 and angles B and C must measure 60°. Since DE bisects angle BDC and BEC, ΔBED is equiangular and thus equilateral.
11. two tangents to a circle from any point are the same length, so DG=DH, CH=CI, BI=BF, and EG=EF. Now just add up these four equations and substitute the long lengths for the 2 shorter ones four times…
12. PBC is a 30/60/90 triangle so CP=2*PB. And since it is a kite, PA=PB.
13. FE and FB are tangents to circle C, so congruent; DF and FB are tangents to A thus congruent. So y transitivity, DF=FE and thus F is the midpoint of DE.

Unit 7 Handout #6: Arc Length

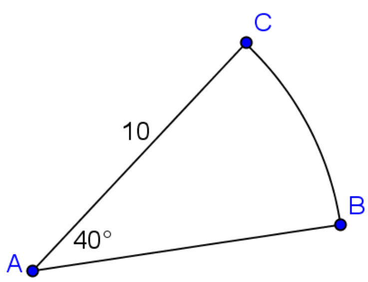
Earlier we measured arcs in degrees; now we will measure them in units of distance. We can use proportions based on central angles, as we did when measuring the areas of sectors.

An arc that consists of the entire circle measures the circumference and corresponds to a central angle of 360°. Arc length is proportional to the central angle. Thus the length of any arc can be computed as $\frac{arc\,length}{circumf} = \frac{central\ angle}{360}$ or $\frac{arc\,length}{2\pi r} = \frac{central\ angle}{360}$. Multiplying both sides of this proportion by the circumference, we get a length of $arc\,length = \left(\frac{x}{360}\right) \cdot 2\pi r$ for an arc intercepted by a central angle of x°.



Example #1: Find the length of arc BC in the diagram below.
The circumference of the circle is 20π, so set up a proportion:
$\frac{arc\,length}{2\pi r} = \frac{central\ angle}{360}$ or $\frac{x}{20\pi} = \frac{40}{360}$. Solve for x by either cross-multiplying or just multiplying both sides by 20π.

Thus $x = 20\pi \cdot \left(\frac{40}{360}\right) = \frac{20\pi}{9}$.



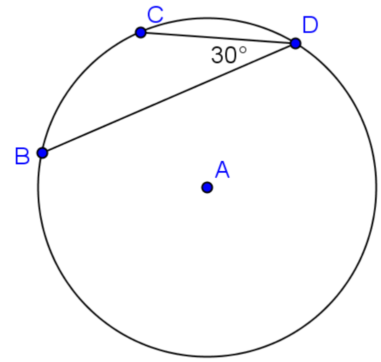
Example #2: Minor arc BC has length π. Find the area of the circle and the length of chord $\overline{BC}$.

We first need the radius of the circle. Either the central angle CAB or the degree measure of minor arc BC will enable us to find it. Since the inscribed angle measures one half of the intercepted arc, we know minor arc BC measures 60°, and so does angle CAB.

To find the radius we use the proportion $\frac{arc\,length}{2\pi r} = \frac{central\ angle}{360}$.

Entering in what we know, $\frac{\pi}{2\pi r} = \frac{60}{360}$. Solving this yields r=3.

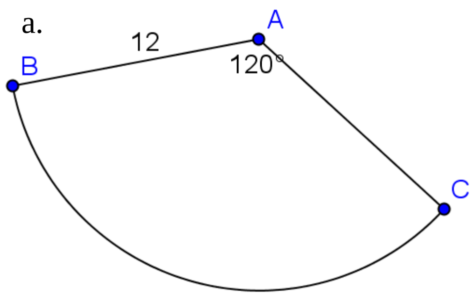
Thus the area of the circle is $\pi 3^2$ *or* 9π.



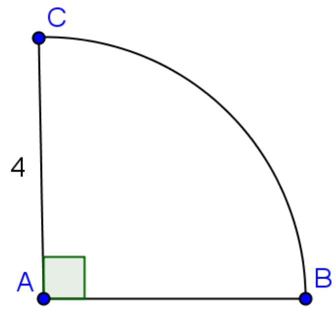
To find the length of chord $\overline{BC}$, we see that ΔABC is equilateral, since A measures 60° and angles B and C are congruent (because the sides opposite them are both radii). Therefore chord $\overline{BC}$ is congruent to the radius and measures 3.

1. Find the length of the arcs in the diagrams below. Point A is the center of all circles.

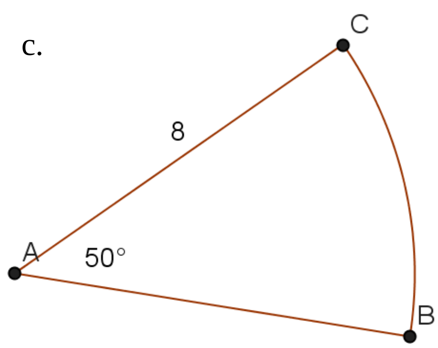
a.



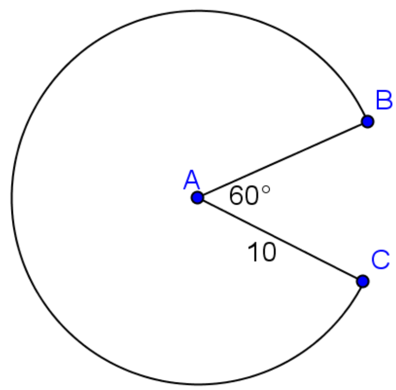
b.



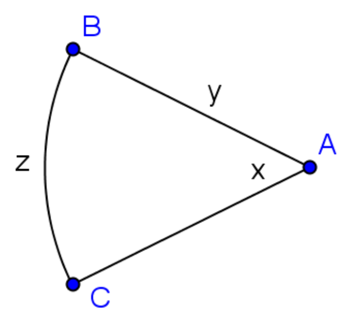
c.



d.



2. Given the diagram below, answer the following questions:



a. If the arc length (z) is 2 (not 2π) and the angle (x) is 72°; find the radius (y).

b. Instead, if y=10 and z=3π, then find x.

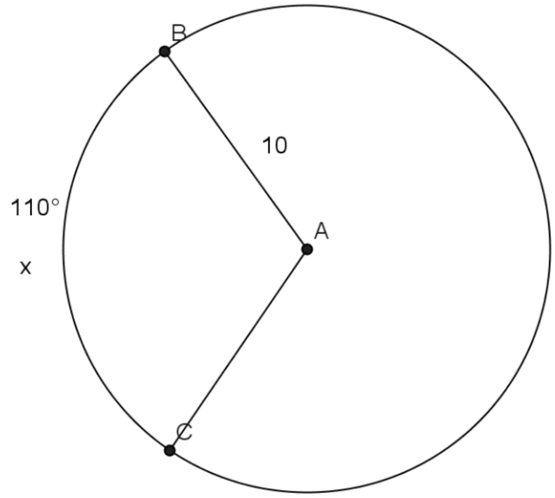
c. What is x if y=z (pick any number!)?

3. In a given circle, an arc of 100° degrees is 6 cm long. What is the radius of the circle?

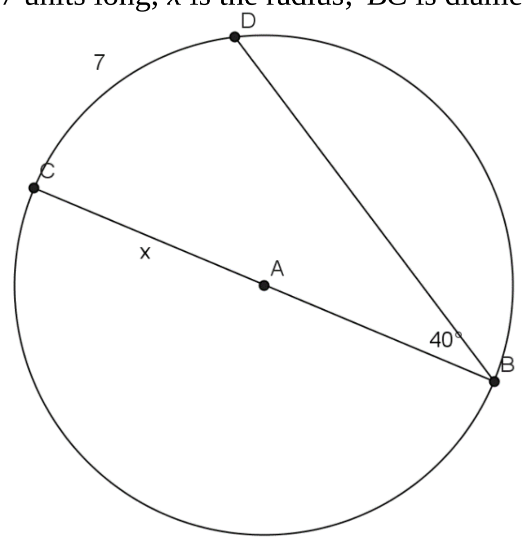
4. A circular pizza is cut into eight equal slices. One slice has a crust 2π inches long. What is the radius of the pizza?

5. Find the length *x* in each diagram. Leave answers in terms of π. Point A is the center of both circles.

a. *x* is minor arc BC

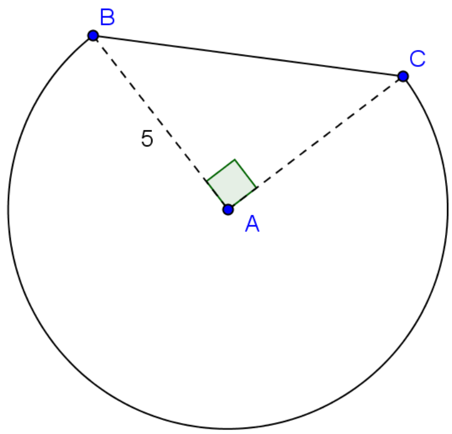


b. Arc CD is 7 units long; *x* is the radius; $\overline{BC}$ is diameter

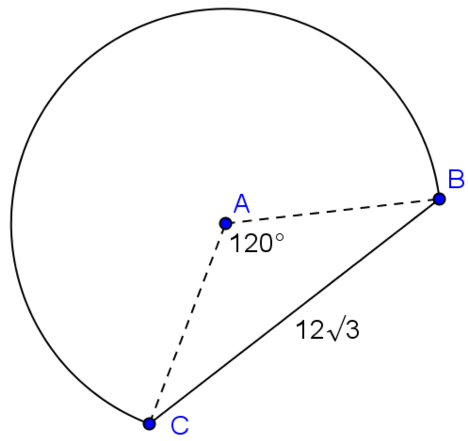


6. Find the perimeter of the objects below. Note: the dashed lines are not external so they should not be included in the perimeter.

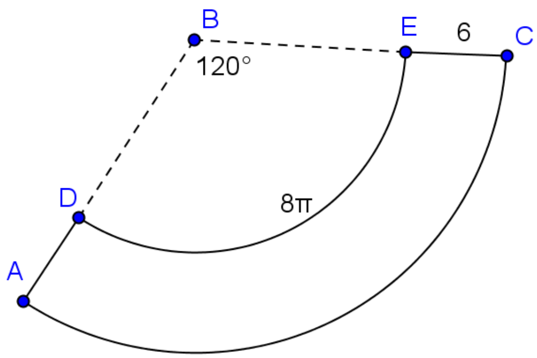
a.



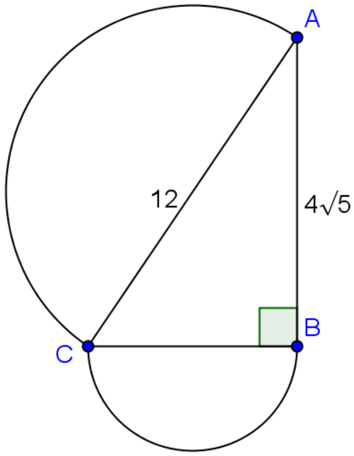
b.



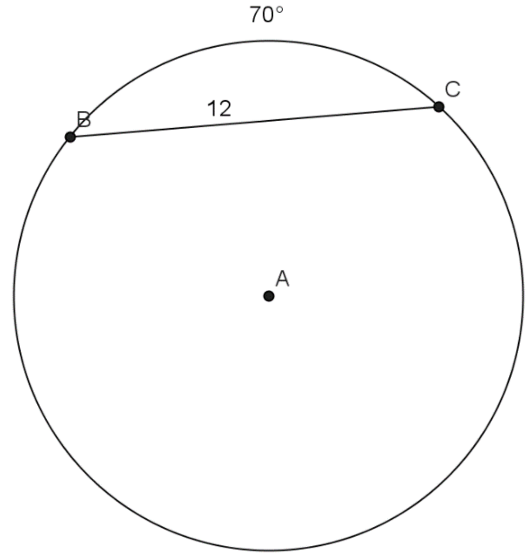
7. AC and DE are arcs of circles whose centers are B. Find the perimeter of ADEC.



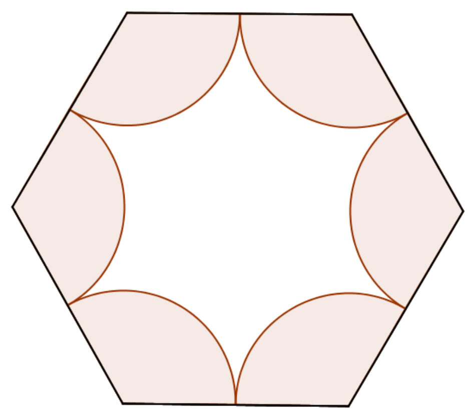
8. Find the perimeter of the object below; it consists of a right triangle and two semicircles.



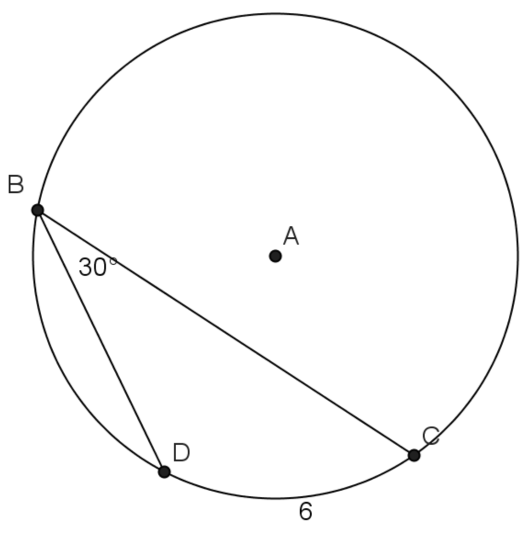
9. In the circle below, the chord measures 12 and the arc is 70°. Find the length of the arc. Hint: find the circle's radius first. Decimal approximation ok! 



10. Arcs are extended from each vertex of a regular hexagon with side length of 12. What is the perimeter of the six-pointed shape in the center?

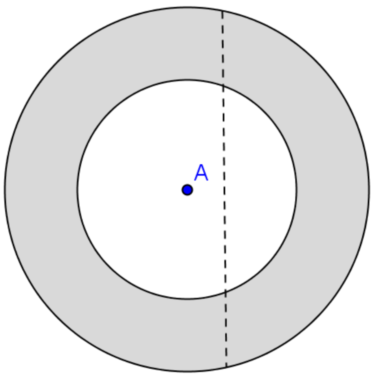


11. Find the radius of the circle.

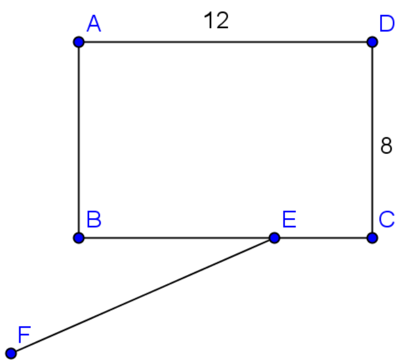


12. In a given circle, inscribed angle A is 40° and it intercepts an arc of length 6. Angle B is inscribed in the same circle and measures 80°. Must B intercept an arc that is 12 units long? Explain.

13. The shaded area below is between two circles centered at A; the larger one with a radius of five and the smaller one with a radius of three. The dashed line is one unit from A (measured along the perpendicular) and cuts the shaded area into two pieces. What is the perimeter of the smaller shaded piece? (decimal approximation) 

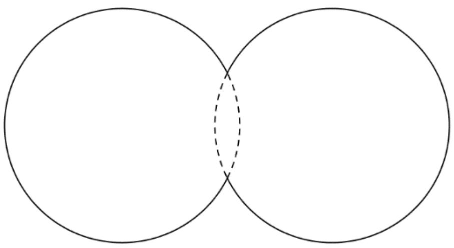


14. Milo the dog is attached to the side of a rectangular building with sides of 8 and 12 meters. He is attached to a long side, one-third of the way from a corner. His rope is 13 meters long. Find the perimeter of the area (outside the building) he can reach (include the sides of the building).



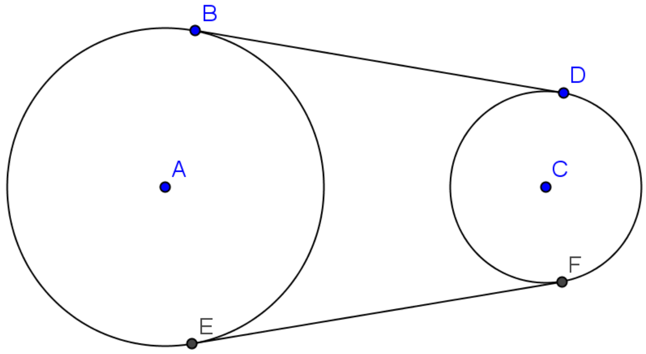
15. The circumference of the earth is about 25,000 miles. Somehow, a rope gets wrapped on the surface of the earth at the equator, going all the way around one time. The people who did this then decide that they want to wrap a second rope to be one foot above the earth's surface all the way around. How much longer is the second rope than the first rope?

16. Two circles with radii of 8 have their centers 14 units apart. What is the length of the solid curve?



17. Circles A and C have radii 5 and 3 respectively, and their centers are 12 units apart. $\overline{BD}$ *and* $\overline{EF}$ are both tangent to both circles. A rubber band is stretched around the circles so it contains major arc BE, minor arc DF, and $\overline{BD}$ *and* $\overline{EF}$. Your goal is to find the length of the rubber band.

a. Draw a segment through C parallel to $\overline{BD}$. Let it intersect radius $\overline{AB}$ at point G. Why is triangle AGC a right triangle?

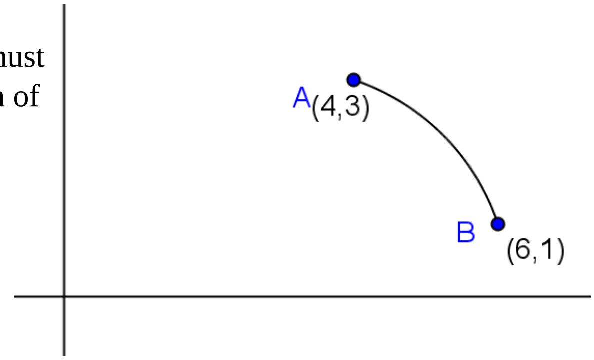


b. Use this triangle to find the length BD.

c. Now find the length of the rubber band. Give a decimal approximation.

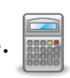
18. Arc AB is part of a circle whose center, C, is somewhere on the *x*-axis. Your goal is to find its length, following these steps:

a. Given the coordinates of A and B, the center of the circle must be on some line (besides for the *x*-axis!). What is the equation of this line? Hint: the definition of a circle is….



b. Now that you know two lines that C must be on, find C's coordinates.

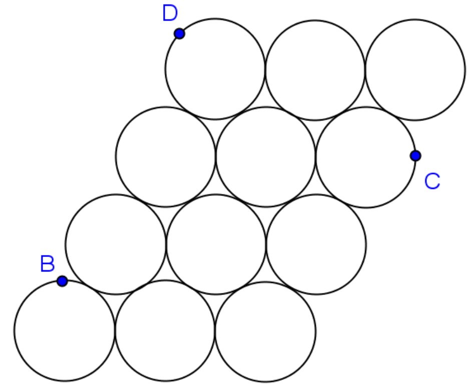
c. What is the radius of the circle?

d. Draw some right triangles to find the central angle for arc AB. 

e. Now find the length of the arc. Write it as a decimal. Check that it is slightly longer than segment $\overline{AB}$.

19. Twelve circles of radius one are stacked so they are mutually tangent to adjacent circles. B is at the top of its circle; C is at the right of its circle; and D is midway between the top and left of its circle.

a. What is the shortest distance from B to D staying on the edges of the circles?



b. What is the shortest distance from B to C staying on the edges of the circles?

c. Steve and Amy start at B and race to C. Steve must stay on the edges of the circles and goes 3 units per minute. Amy can go 2.4 units per minute but can cut through the interiors. Who wins the race, and by how much Hint: place the origin at the center of B's circle and find the coordinates of B and C.

Answers

1a. 8π b. 2π c. 20π/9 d. 50π/3 2a. 5/π b. 54° c. 180/π degrees

3. 10.8/π 4. 8 inches 5a. 55π/9 b. 63/(4π) 6a. $7.5\pi + 5\sqrt{2}$ b. $16\pi + 12\sqrt{3}$

7. 20π+12 8. $4\sqrt{5} + 10\pi$ 9. r= 10.46 so length= 12.78 10. 24π

11. 18/π 12. yes; the arc has twice the degree measure so twice the length

13. the dashed parts are each $\sqrt{24} - \sqrt{8}$ *or* $2\sqrt{6} - 2\sqrt{2}$. The central angle for the small circle is 141° and for the larger circle is 157°. So approximately 25.22

14. 26 + 20.5π meters 15. 2π feet—hard to believe but true! 16. 84.36

17a. since ABD is a right angle (tangent to circle) b. AG=2 (since it is 5-3) so $2\sqrt{35}$ c. 49.47

18a. the perp bisector of segment AB, which is y=x-3 b. (3,0) c. $\sqrt{10}$

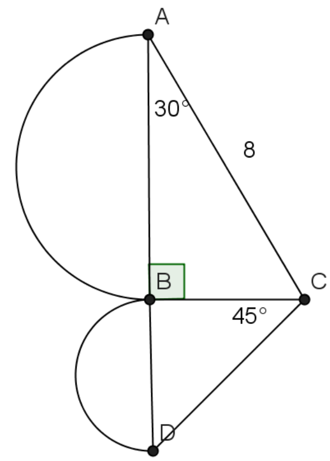
d. $90 - \tan^{-1}(1/3) - \tan^{-1}(1/3) \approx 53^{\circ}$ e. 2.93; segment AB is $2\sqrt{2}$, which is 2.82, so reasonable

19a. 11π/4 b. 17π/6 c. Steve takes 17π/18 minutes (2.967)

If center of B's circle is (0,0) then center of C's circle is $(6, 2\sqrt{3})$ → 4 right and then 4 diagonally at an angle of 60° → so B is (0,1) and C is $(7, 2\sqrt{3})$ and the distance between them is 7.421. This takes Amy 3.092 minutes. So Steve wins by 0.125 minutes.. close!

Unit 7 Handout #7: Review Problems

1. What is the perimeter of the object below? It is composed of two semi-circles and two right triangles.

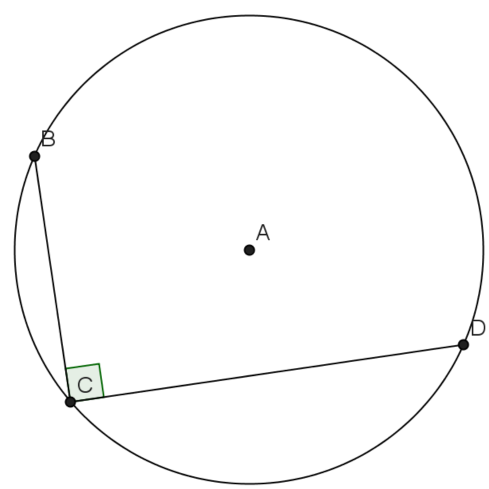


2. Chords $\overline{BC}$ and $\overline{CD}$ meet at a right angle on circle A, whose radius is $\sqrt{13}$. If chord $\overline{CD}$ is two units longer than chord $\overline{BC}$, then find the following:

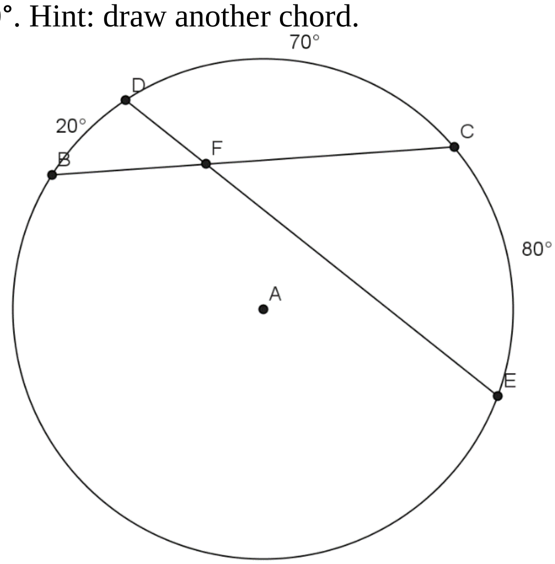
a. The length of chord $\overline{BC}$.

b. The measure of minor arc BC in degrees. 

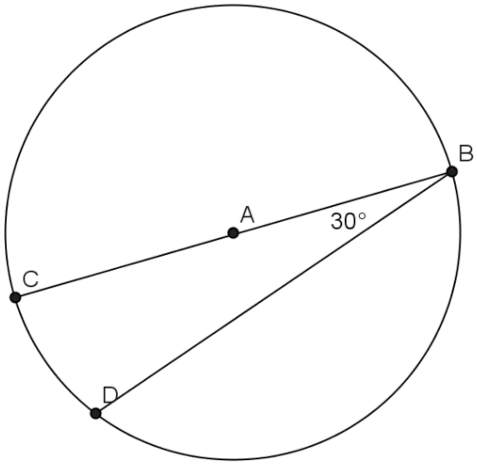
c. The length of minor arc BC. 



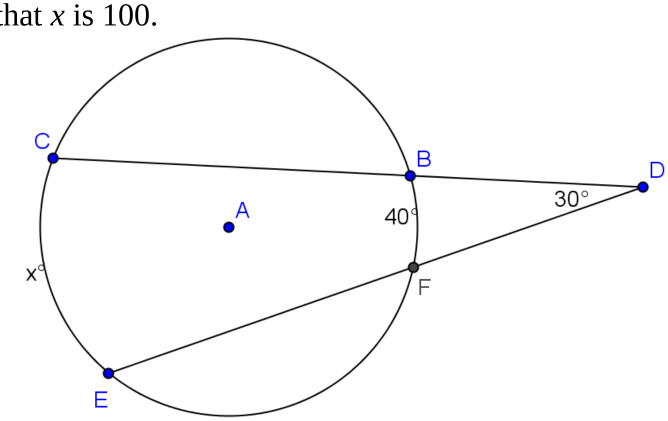
3. In the diagram below, show that the measure of angle CFE is 50°. Hint: draw another chord.



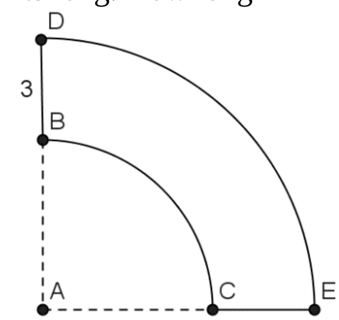
4. Why is an inscribed angle half of the intercepted arc? In circle A below, show that the measure of minor arc CD is 60°. You know that that the central angle is equal to the arc measure (in degrees),…Note: you can't use the fact that the inscribed angle is half of the intercepted arc; that's what you are trying to show! Hint: draw $\overline{AD}$.



5. Draw one line segment in the diagram below to show that *x* is 100.



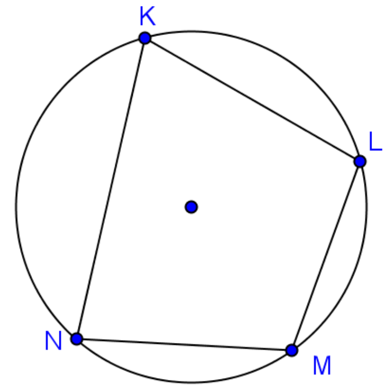
6. DE and BC are quarter-circles centered at A. Arc DE is 12 units long and DB is 3 units long. How long is arc BC? Hint: find AD first.



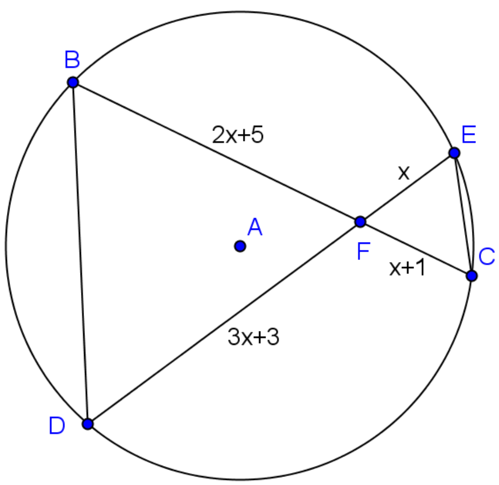
7. In the diagram below, do the following:

a. Explain why angles K and M must be supplementary.

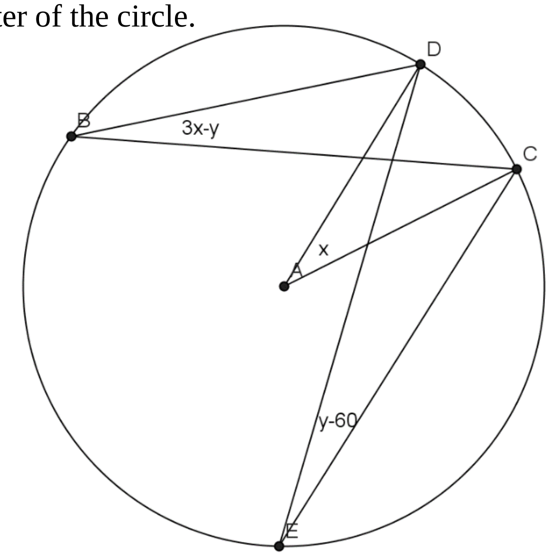
b. Given that K measures 65° and L measures 105° and arc KN measures 130°, find the measure of minor arc LM.



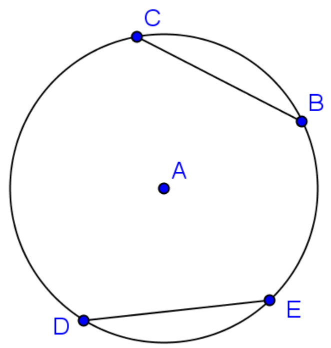
8. Identify a pair of similar triangles in the diagram below (justifying your answer). Then use algebra to find the value of *x*.



9. Find the values of *x* and *y* in the diagram below —A is the center of the circle.



10. In circle A below. $\overline{BC} \cong \overline{DE}$. Explain why minor arcs BC and DE must be the same length.



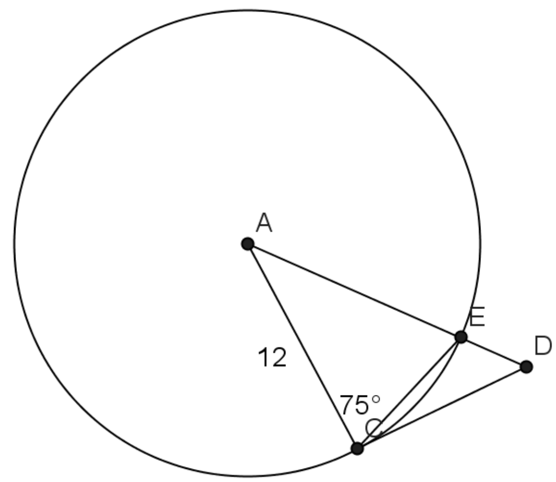
11. Given tangent segment CD in circle A where ACE measures 75°. Find the following:

a. The measure of angle A

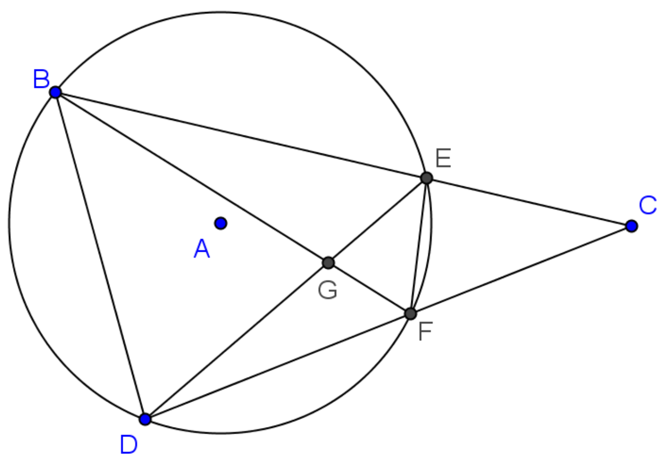
b. The length of $\overline{CD}$.

c. The length of $\overline{ED}$.

d. The length of minor arc CE



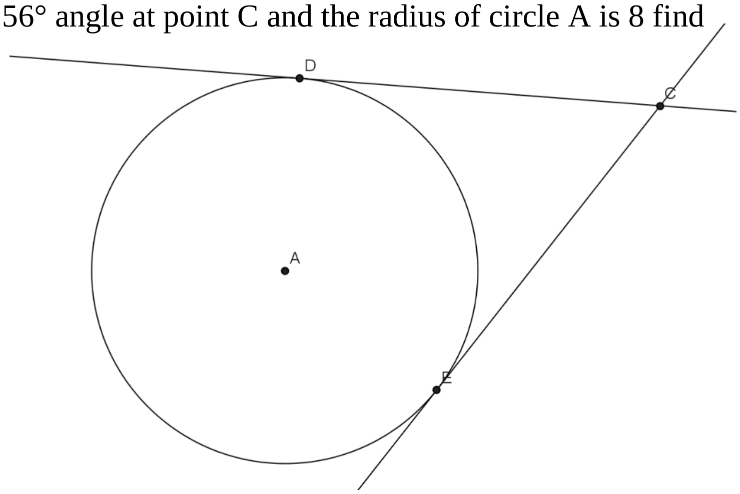
12. Given circle A and line segments $\overline{BC}$ *and* $\overline{CD}$. Identify four pairs of similar triangles and explain why they must be similar. Do not draw any new line segments.



13. Given that tangent lines to circle A meet at a 56° angle at point C and the radius of circle A is 8 find the following.

a. The length of $\overline{CD}$.

b. The length of major arc DE



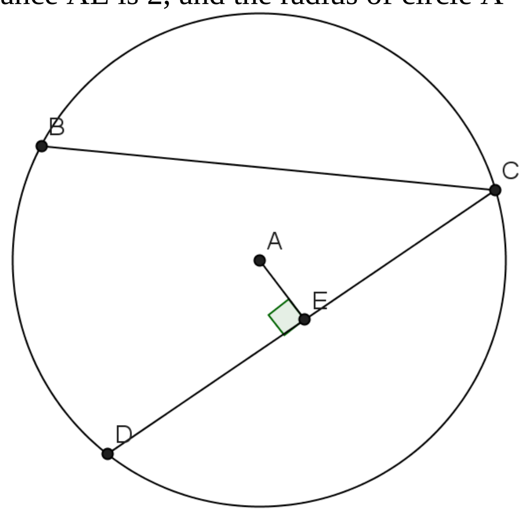
14. In circle A below, minor arc BC measures 140 degrees, the distance AE is 2, and the radius of circle A is 6. Answer the following:

a. How long is chord $\overline{CD}$?

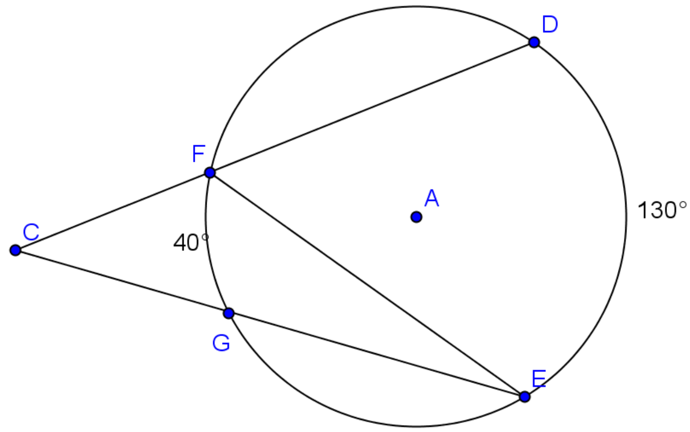
b. How long is chord $\overline{BC}$?

c. What is the length of minor arc CD?

d. What is the measure of angle C?



15. Given the measures of arc DE and FG below, find the measure of angle C. Hint: one way is to look at triangle CEF.



16. Given circle A with tangents $\overline{CD}$ *and* $\overline{EF}$. Answer the following questions:

a. Why must $\overline{EF} \cong \overline{CF}$?

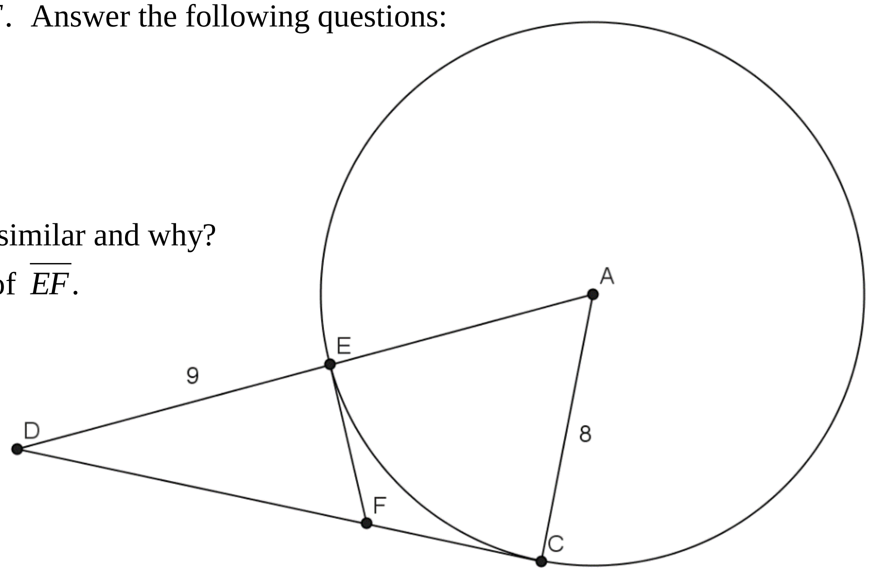
b. Find the length of $\overline{CD}$.

c. Find the measure of angle A.

d. Find the length of minor arc EC.

e. Which two non-congruent triangles are similar and why?

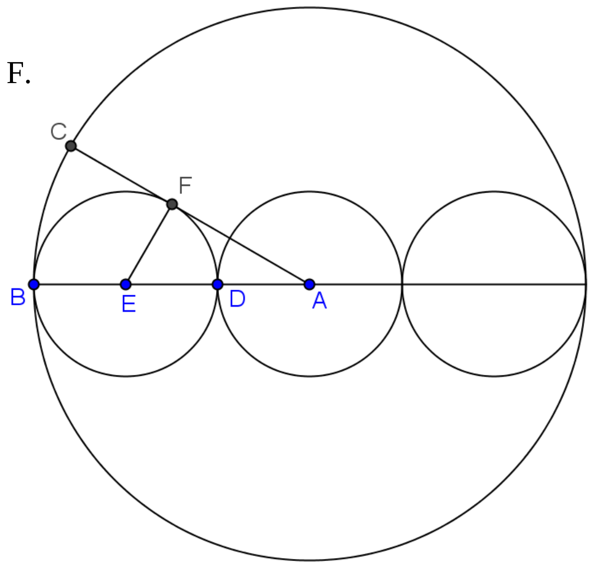
f. Use your answer to *e* to find the length of $\overline{EF}$.



17. The large circle below centered at A has a radius of 3 units. Three congruent smaller circles are placed so that they are internally tangent to the large circle and externally tangent to each other. Radius $\overline{AC}$ of the large circle is tangent to circle E at point F. Without your calculator, do the following:

a. Explain why minor arc BC must measure 30°.

b. Find the perimeter of the region bound by points B, C and F. (decimal approximation)



Answers

1. $8+4\sqrt{2}+2\pi+2\pi\sqrt{3}$ 2a. 4 b. 67° c. 4.24 3. Draw BE: $\angle BCE = 95^\circ$ & $\angle CEF = 35^\circ$ so 180-95-35

4. $\angle ADB = \angle ABD = 30$ *so* $\angle DAB = 120$ *and* $CAD = 60 = arc\ CD$

5. Draw BE: angle BEF is 20° so angle EBD is (180-20-30)=130° so angle EBC=50° so x=100 (can draw CF instead)

6. 12-1.5π 7a. between them, they intercept the whole circle b. 50°

8. ΔBFD~ΔEFC since angles B & E intercept same arc and there are vertical angles; x=5

9. x=30; y=75 10. Triangles ABC and ADE are congruent by SSS. Therefore the central angles are congruent. And since the arc length is a function of central angle, the arcs have the same length.

11a. 30° b. $12/\sqrt{3} = 4\sqrt{3}$ c. $8\sqrt{3}-12$ (or 24/sqrt(3) – 12) d. 2π

12. ΔGDF~ΔGBE since angles D and B intercept the same arc and angle G's are vertical
ΔGEF~ΔGBD since angles E and B intercept the same arc and angle G's are vertical
ΔCED~ΔCFB since angles D and B intercept the same arc and angle C's are same
ΔCBD~ΔCFE angle C's are same and angles CBD and CFE are both supplements of angle DFE

13a. 15.05 b. 32.95 or 472π/45 14a. $8\sqrt{2}$ b. 11.3 c. 14.8 d. 39.5° 15. 45°

16. CPCTC and HL b. 15 c. 62° d. 8.7 e. DCA & DEF f. 4.8 13.5.

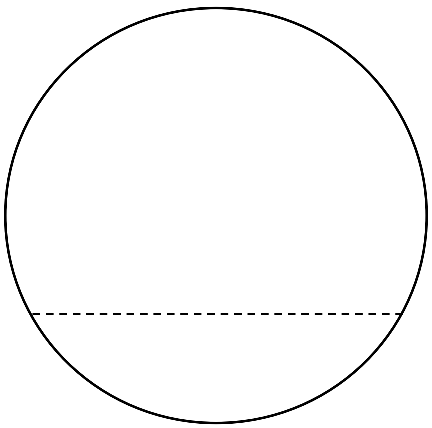
17a.ΔAEF is 30/60/90 because EF=1 and EA=2 so angle FAE is 30° and the measure of an arc is equal to the central angle that intercepts it, so minor arc CB is also 30°

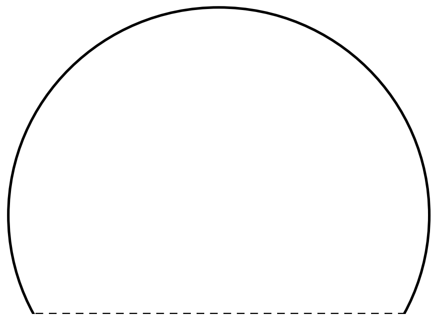
b. minor arc BC is 0.5π; minor arc BF is 2π/3 ; CF=3-$\sqrt{3}$ so perimeter is about 4.93

Area Introduction: Paper-Folding Activity

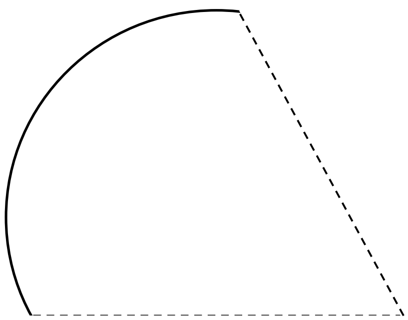
Start with a large circle cut from an 8.5-by-11 inch piece of paper. Have the center marked.

1. Fold it so a point on the edge of the circle hits the center.

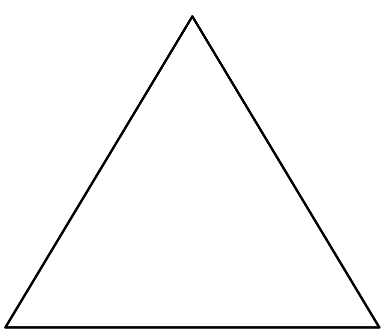




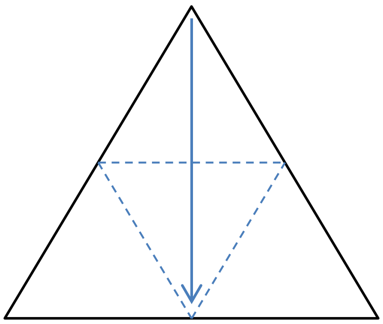
2. Now fold another point on the circle to the center such that the two chords have a common end point. This gives us a sno-cone!



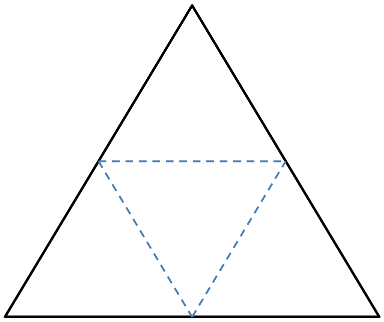
3. Now fold a third point on the circle to the center, yielding a triangle. It's okay if it does not go exactly to the center.



4. Fold one vertex across to midpoint of opposite side and crease. Unfold.

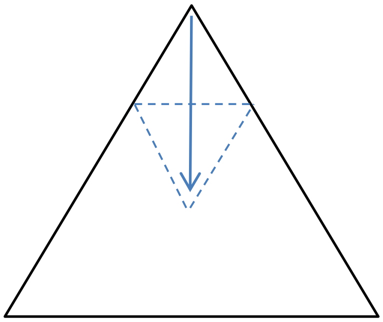


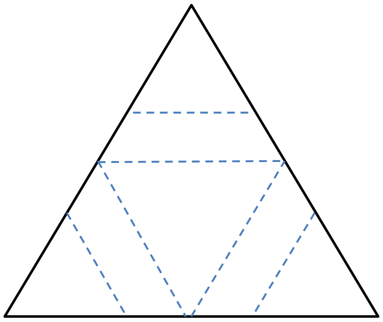
5. Do the same with the other two vertices. If you fold them all over, you get a smaller equilateral triangle. Unfold to have original equilateral triangle.



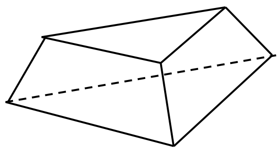
6. Fold all three vertices up until they meet at a point above the center of the middle triangle to create a pyramid (technically called a tetrahedron). Now unfold to have the large equilateral triangle.

7. Now, while unfolded, fold one vertex to the center of the original circle and crease. Unfold. Do the same with the other two vertices. Unfold each time. Your triangle should now look like the one on the right below, where the dashed lines should correspond to crease marks.



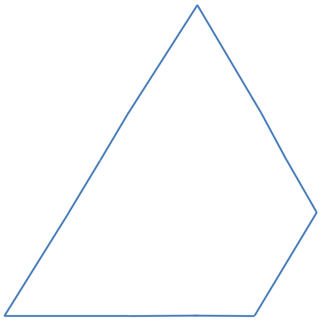


8. Now create a ***frustum*** by recreating the tetrahedron from step #6 and then folding the three vertices down along the smaller crease lines.

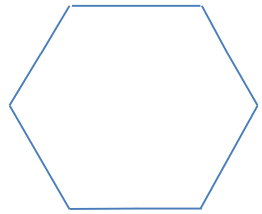


9. Unfold to get the original (large) equilateral triangle. *Assume its area is equal to one*. Now fold along the crease lines to make each of the shapes below and compute the area of each shape.

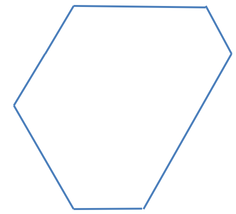
a.



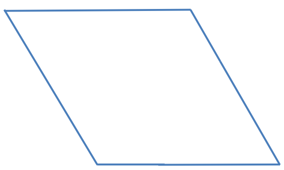
b.



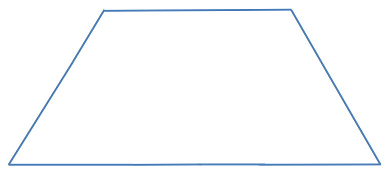
c.



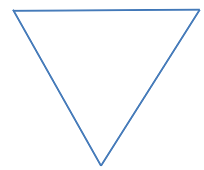
d.



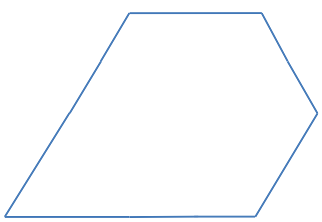
e.



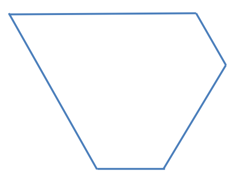
f.



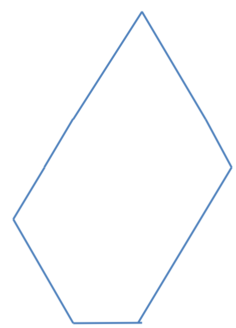
g.



h.



i.



10. Answer the following:

a. What is the area of the original circle?

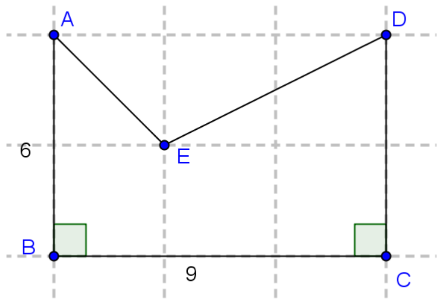
b. The volume of a tetrahedron is equal to one third of the product of the height and the area of the base. What is the volume of the tetrahedron from step 6?

c. What is the volume and surface area of the frustum you created in step 8?

Unit 8 Handout #2: Area of Rectangles and Triangles

The area of a rectangle is the product of its base and its height. The area of a triangle is equal to one half of the product of its base and its height, where its height is measured along a perpendicular (an altitude).

Example #1: Find the area of polygon ABCDE on the left below.



We can subtract the area of triangle ADE from the area of rectangle ABCD. The area of the rectangle is (6)(9)=54.

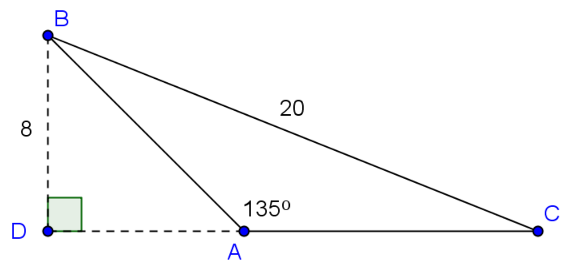
Using the grid lines, we can assume that the altitude from E to side AD has length of three, which is half the height of the rectangle. Thus the triangle's area is (½)(3)(9)=13.5

So the area of ABCDE is 54-13.5=40.5.

Example #2: Find the area of triangle ABC below.

The height is 8; we need the length of base $\overline{AC}$. We can use the Pythagorean Theorem in ΔBCD to find the length of $\overline{CD}$ and then subtract the length of $\overline{AD}$.

The length of $\overline{CD}$ is $\sqrt{20^2 - 8^2} = \sqrt{336} = 4\sqrt{21}$.
Since ΔADB is an isosceles right triangle, AD=8.

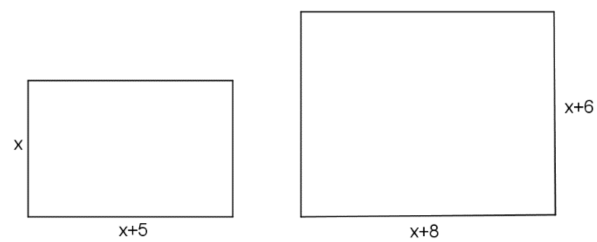


Therefore, $AC = 4\sqrt{21} - 8$ and the area of ΔABC is $\frac{1}{2}(8)(4\sqrt{21} - 8) = 16\sqrt{21} - 32$

Example #3: A rectangle's length exceeds its width by five units. If its length were increased by three units and its width increased by six units, then its area would increase by 80. Find its length.

If x is its original width, then x+5 is its original length, and its original area is $(x)(x+5)$.

Its new length is three greater than its original length, so it is x+8. Its new width is x+6. Thus its new area is $(x+6)(x+8)$.



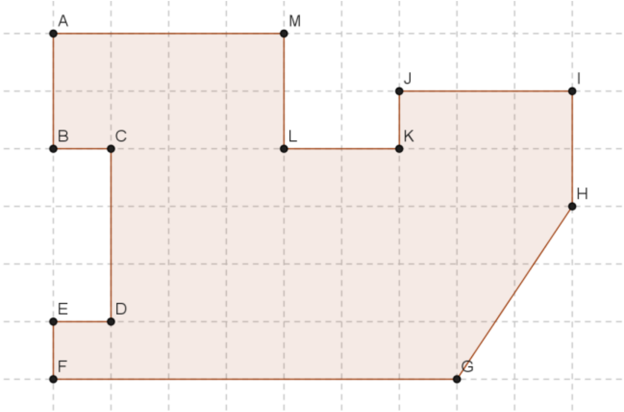
Since its new area is 80 square units more than its original area, $(x+6)(x+8) = 80 + (x)(x+5)$

Thus $x^2 + 14x + 48 = x^2 + 5x + 80 \quad and \quad 9x = 32 \quad and \quad x = 32/9$

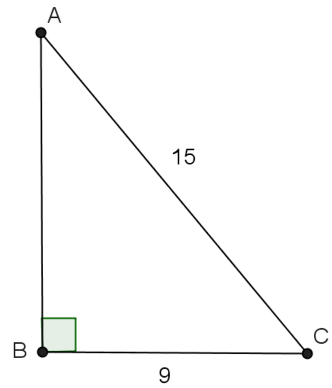
Therefore its original length is $x + 5 \text{ or } \frac{32}{9} + \frac{45}{9} = \frac{77}{9}$

1. Find the area of the following objects. Some may require applications of special triangles. If you are not sure what to do with a shape, try dividing it into pieces that are triangles and rectangles.

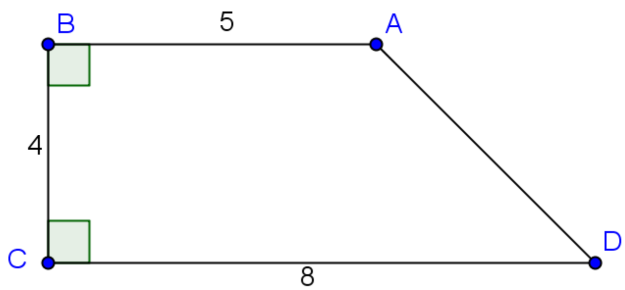
a. assume each square is 2 units by 2 units.



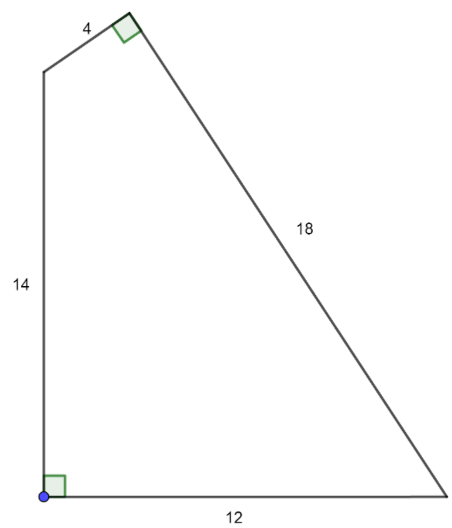
b.



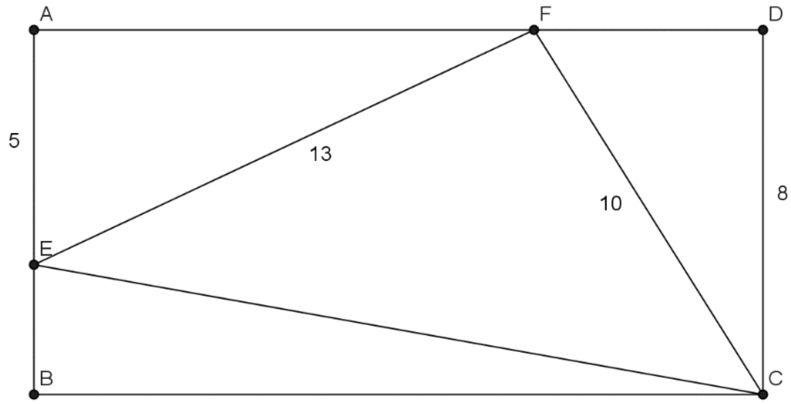
c.



d.



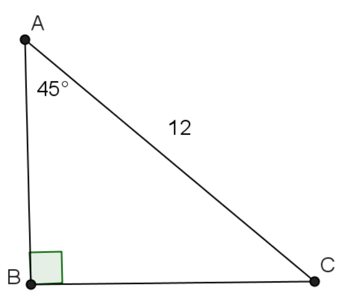
2. Find the area of triangle CEF given that ABCD is a rectangle. Note: do not assume that any angles in the triangle are right angles!



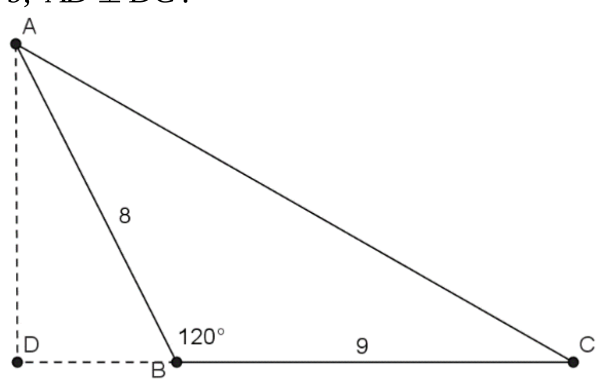
3. In question #2 above, is angle EFC a right angle? Justify your answer.

4. Find the area of ΔABC in each diagram below. In part *b*, $\overline{AD} \perp \overline{DC}$.

a.

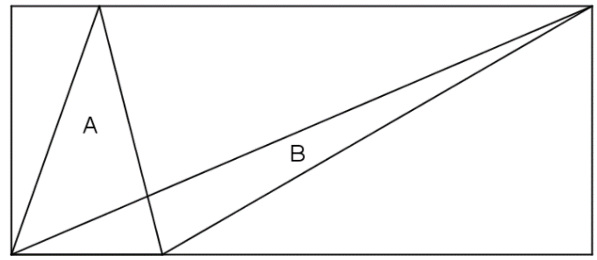


b.



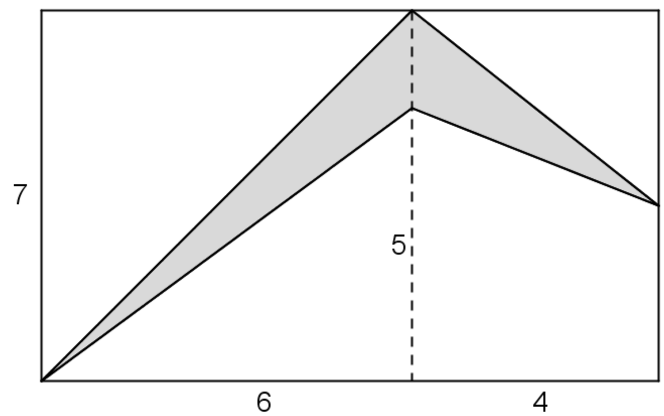
5. Triangles A and B are both inscribed inside the rectangle below.

a. Which has a larger perimeter?



b. Which has a larger area?

6. Find the area of the shaded region; the larger shape is a rectangle.



7. Logic statements:

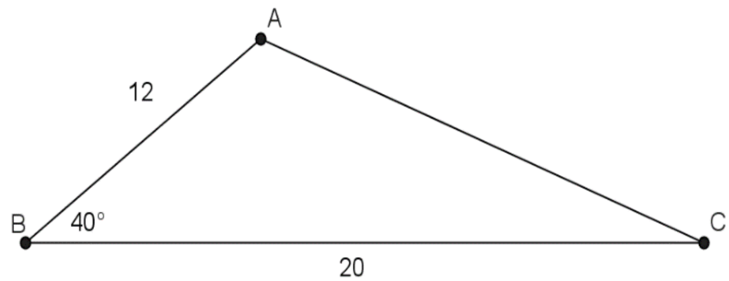
a. Is the statement, “If two triangles are congruent then they have the same area” true?

b. What is the converse of the statement? Is it true?

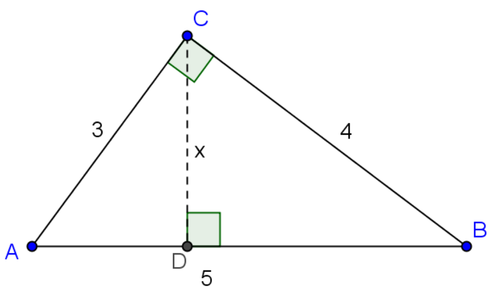
c. What is the inverse of the statement? Is it true?

d. What is the contrapositive of the original statement, and is it true?

8. Find the area of the triangle below. 



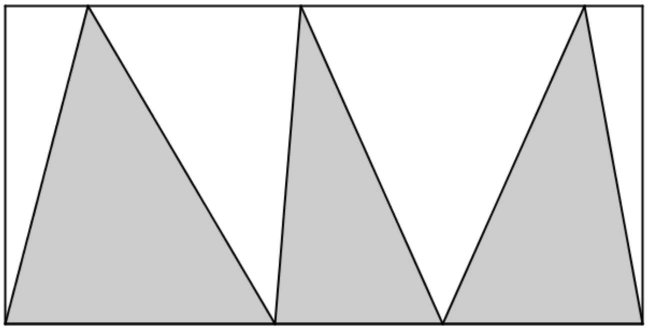
9a. Use similar triangles (or perhaps trig—with no calculator) to find *x* in the diagram below:



b. Instead, use area to find *x*. Note that the area of a triangle is the same whichever side you consider the base.

c. Why is the length of the altitide to the hypotenuse of a right triangle always equal to (*ab/c*), where *c* is the length of the hypotenuse and *a* and *b* are the lengths of the legs?

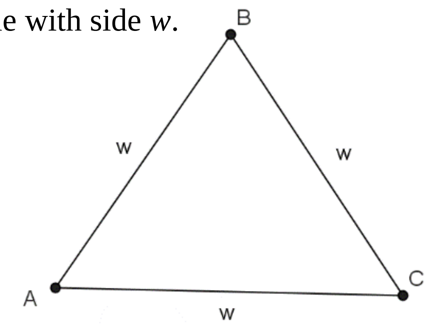
10. Given that rectangle below has dimensions of 20-by-12, find the area of the shaded region.



11. Find the area of:

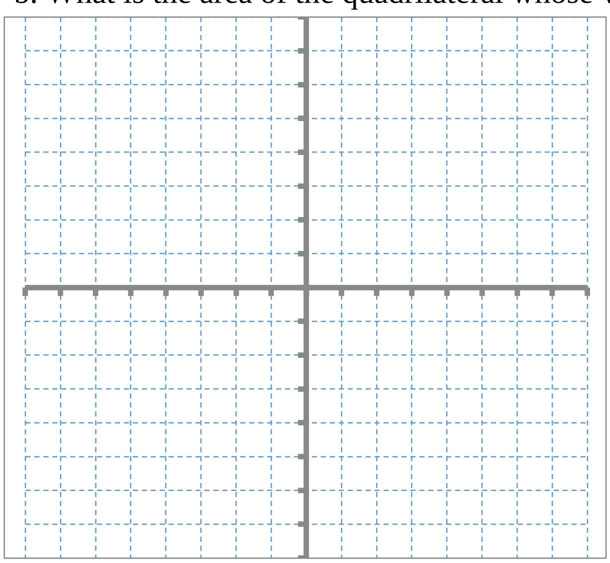
a. An equilateral triangle with side 6.

b. An equilateral triangle with side *w*.

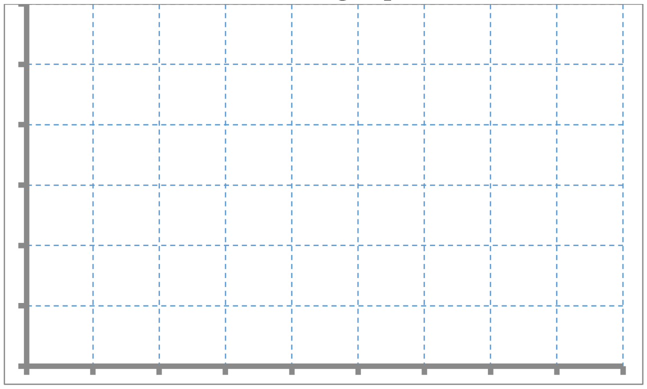


12a. A triangle's vertices are (-7,5), (-5,-2), and (-3,1). Find its area. Hint: look at #2 above!

b. What is the area of the quadrilateral whose vertices are (-1,6), (7,4), (6,-7), and (-2,-2)?



13. A rectangle's vertices are (1,0), (7,0), (1,4), and (7.4). The line $y = 0.4x + 2$ breaks it into two pieces. What is the area of the larger piece?

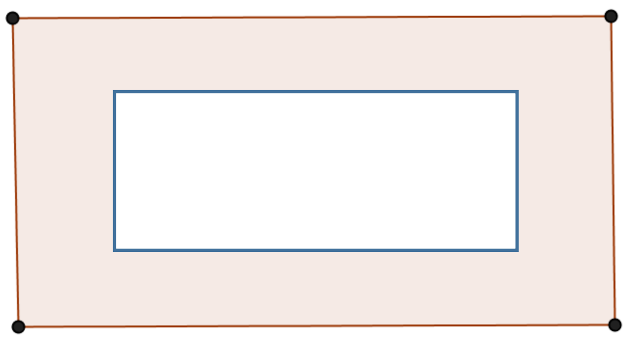


14. A rectangle has a perimeter of 40 and an area of 96. What are its dimensions? Hint: write a system of two equations and two unknowns and solve by substitution.

15. The larger rectangle below is 20-by-12. The smaller rectangle's dimensions are 14-by-7.

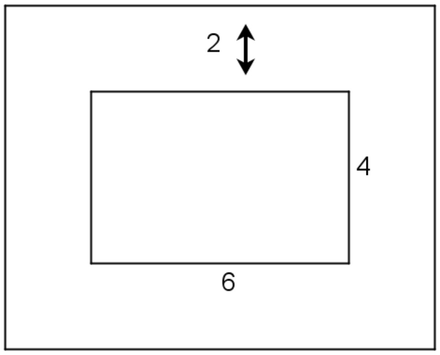
a. Assume the lower base of the small rectangle is parallel to the lower base of the larger rectangle. What is the area of the region inside the larger rectangle and outside of the smaller rectangle?

b. Would your answer to part *a* be different if the lower bases of the two rectangles were not parallel? Explain.



16. Given a rectangular photograph that is 4-by-6, Sandy puts a frame around it that is 2 units wide.

a. What is the area of the frame itself (not including the area of the photograph)?

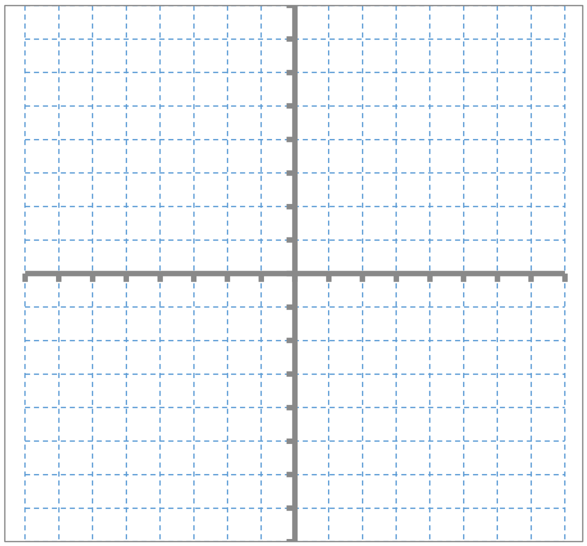


b. If instead of the frame being one unit wide, it was x units wide, then what would the area of the frame be (in terms of x)?

c. How wide must the frame be for the area of the frame to be equal to the area of the photograph?

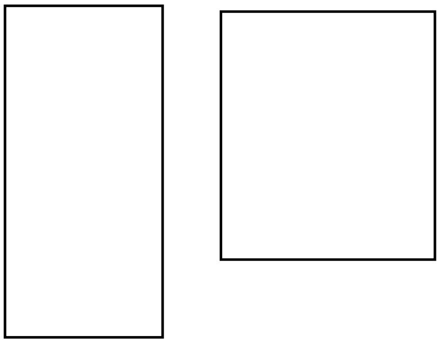
17. Two vertices of a tringle are (1,3) and (5,3). The triangle's area is 12.

a. If the x-coordinate of the third vertex is -2, then what could the y-coordinate be?



b. Instead, the third vertex is on the line $y = 2x + 4$. Now find all possible coordinates of the third vertex.

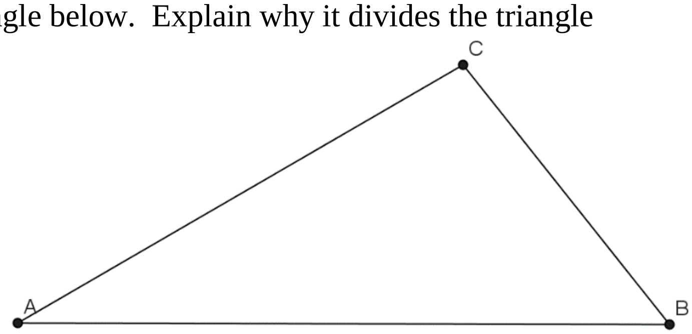
18. A rectangle's width is twice its length. If its width was decreased by five units and its length increased by 3 units, then its area would not change. What is its area? Hint: define x as its length.



19. A rectangle's length exceeds its width by three units. It its width were doubled and its length reduced by two units, then its area would grow by 12 units. What were its original dimensions?

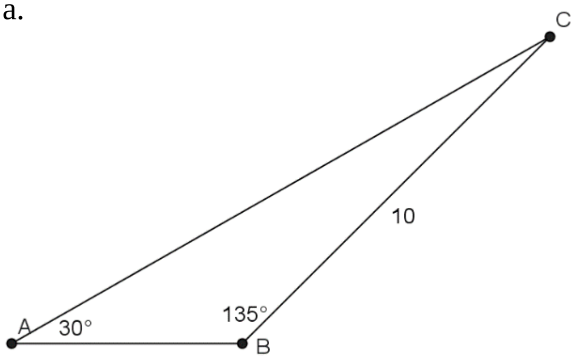
20. The ratio of a rectangle's length to its width is 3:2. Its area is 120. What is its perimeter?

21. Draw a median from vertex C to side $\overline{AB}$ in the triangle below. Explain why it divides the triangle into two triangles that have equal area.

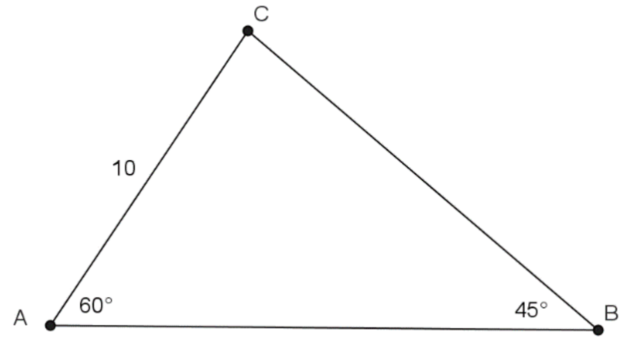


22. Find the area of the triangles below. Draw altitudes.

a.

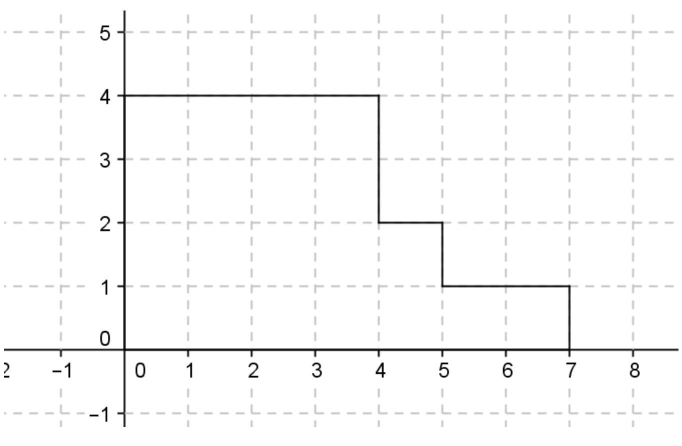


b.



23. Given the shape below:

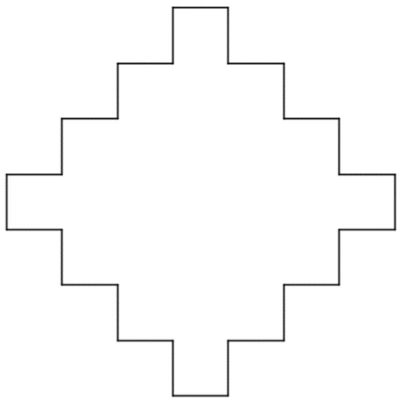
a. A line through the origin breaks this into two pieces of equal area. What is its slope?



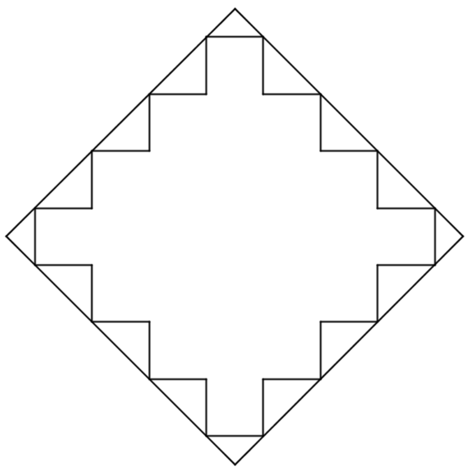
b. A vertical line divides this shape in a way that three-fourths of the area is to the left of the line and one-fourth is to the right. What is the line's equation?

24. The perimeter of the figure below is 56 and all sides are congruent and meet at right angles.

a. What is its area?



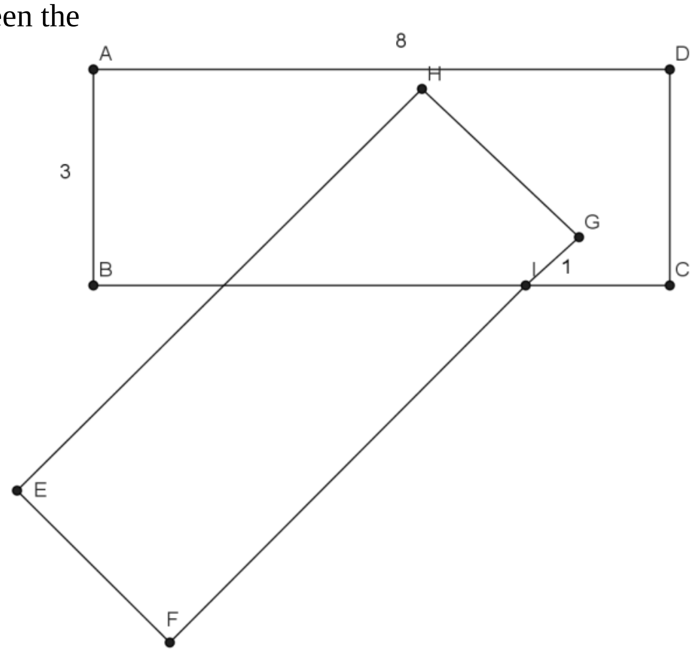
b. Below, the figure is placed into a square box. What is the area outside the figure but inside the box?



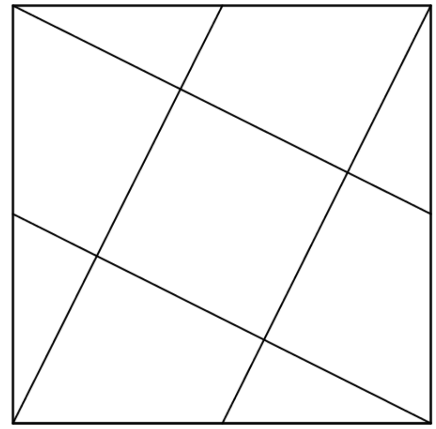
25. Rectangles ABCD and EFGH are congruent, but rectangle EFGH has been rotated 45° counter-clockwise and moved. Segment $\overline{GI}$ is one unit long. Answer the following (no calculator):

a. What is the area of the region of overlap between the rectangles?

b. How far is point H below side $\overline{AD}$ (measured vertically)?



26. The midpoints of the sides of square with area 36 are connected to an opposite vertex. What is the area of the square in the center?



Answers

1a. 164 b. 54 c. 26 d. 120 2. 63

3. No- use the Pythagorean Theorem. Or, if it were a right angle, area of Δ would be (1/2)(10)(13)=65

4a. 36 b. $18\sqrt{3}$ 5a. triangle B b. same area since same base and same height! 6. 10

7a. yes b. If two triangles have the same area then they are congruent. Not always

c. If two triangles are not congruent then they do not have the same area. Not always.

d. If two triangles do not have the same area then they are not congruent. Always true.

8. 77.1 9a. sinB=(3/5)=(x/4) so x=2.4

9b. Area is (½)(3)(4)=6 so alt to hypot: (½)(5)(x)=6 and x=2.4 c. Area=(1/2)(ab)=(1/2)(x)(c)

10. 120; it must be half of the rectangle's area since the base is the same and so is the height

11a. $9\sqrt{3}$ b. $w^2 \cdot \sqrt{3}/4$ 12a. 10 b. 79.5 13. 20.8

14. $2x+2y=40$ *and* $xy=96$ *so* $x+y=20$ *and* $y=20-x$ *so* $x(20-x)=96$ which yields x=12 or x=8 .. if you plug them in to find y, you see that it must be an 8x12 rectangle.

15a. 142 b. no.. just subtract smaller area from larger one

16a. 56 b. $(6+2x)(4+2x)-24=4x^2+20x$ c. $4x^2+20x=24$ *so* $x^2+5x-6=0$ *and* $x=1$

17a. 9 or -3 b. base= 4 so hgt=6 → y=9 when x=2.5 and y=-3 when x=-3.5

18. 450 19. Length=7; width=4 20. 20*sqrt(5)

21. median bisects the base, so both triangles have the same height and equal bases

22a. $\frac{1}{2}\left(5\sqrt{6}-5\sqrt{2}\right)\left(5\sqrt{2}\right)=25\sqrt{3}-25$ b. $\frac{1}{2}\left(5+5\sqrt{3}\right)\left(5\sqrt{3}\right)=12.5\sqrt{3}+37.5$

23a. ¾ b. x=3.75 24a. 100 b. 28 25a. 7.5 b. $3-2\sqrt{2}$

26. 7.2; one approach is to use coordinate geometry and put the shape on the coordinate plane. You can also show that each small triangle plus small trapezoid is equal to the square in the center. So the area of the central square in then one-fifth of the area of the large square.

Unit 8 Handout #3: Scale and Area

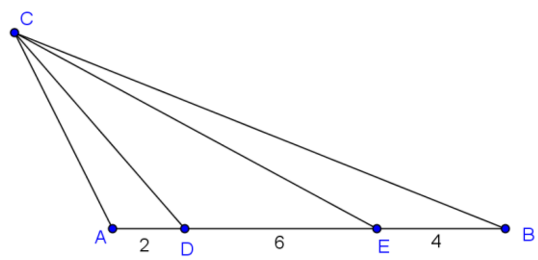
Given that the area of a rectangle is the product of the base and the height, if we multiply either by a constant, the area also gets multiplied by that constant. [Note: this works for triangles as well.]

Base	Height	Area	notes
10	6	60	original
20	6	120	double base → double area
10	3	30	halve height → halve area
x	y	xy	
$3x$	y	$3xy$	triple base → triple area
x	by	bxy	multiply height by b → multiply area by b

If we multiply ***both*** the base and height by the same constant, the area gets multiplied by that constant ***twice***. This means it gets multiplied by that constant squared.

Base	Height	Area	notes
10	6	60	original
20	12	240	double both → quadruple area
5	3	15	halve both → multiply area by $(1/2)^2 = 1/4$
x	y	xy	
$3x$	$3y$	$9xy$	triple both → multiply area by 9
bx	by	$(bx)(by) = b^2xy$	multiply both by b → multiply area by b^2

Example #1: The area of ΔABC is 48. Find the area of triangles ACD, CDE, and DCB.



Since ΔACD and ΔABC have the same height and the base of ΔADC is one-sixth (2/12) of the base of ΔABC, its area must also by one-sixth as large. So its area is (1/6)(48)=8.

ΔCDE has the same height as ΔABC and half the base, so it has half the area, or (1/2)(48)=24.

ΔDCB has the same height as ΔABC and 10/12$^{\text{th}}$ the base, so it has 10/12$^{\text{th}}$ (or 5/6$^{\text{th}}$) of the area. So its area is (5/6)(48)=40.

Example #2: ΔABC~ΔDEF. The perimeters of ΔABC and ΔDEF are 10 and 30, respectively. If the area of ΔABC is 4, then what is the area of ΔDEF?

Since the triangles are similar and the ratio of the perimeters is three, each side of ΔDEF must by three times as large as the corresponding side of ΔABC. Therefore the base and height of ΔDEF must be each three times as large as the base and height of ΔABC. Thus the area is three-squared or nine times as large, so the area of ΔDEF is 36.

1. Triangle A has an area of 20.

a. Triangle A and triangle B have the same base. If B's altitude is three times A's altitude, then what is the area of triangle B?

b. Triangle A and triangle C have the same altitude. If C's base is 1.5 times A's base, then what is the area of triangle C?

c. Triangle D's base is twice triangle A's base. Triangle D's altitude is three times triangle A's altitude. What is the area of triangle D?

d. Triangle E's base is *n* times triangle A's base. Triangle E's altitude is *m* times triangle A's altitude. What is the area of triangle E (in terms of *m* and *n*)?

e. Are any of triangles B, C, and D similar to triangle A? Explain.

f. Triangle F is similar to triangle A. Triangle F's base is 7 times triangle A's base.

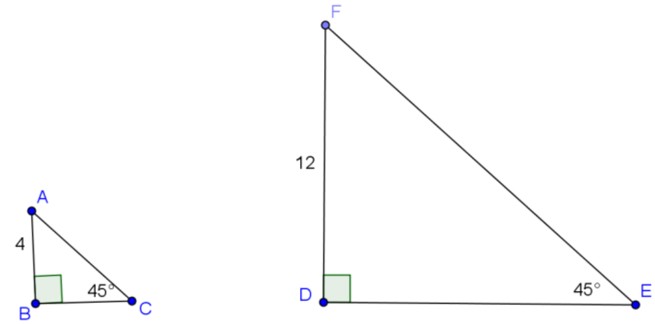
i. Triangle F's altitude is ____ times triangle A's altitude.

ii. What is triangle F's area?

2. The two triangles below are similar; one's sides are three times as large as the other's sides.

a. Find their perimeters and then calculate the ratio of the perimeter of the larger one to that of smaller one.

b. How could you have known this ratio without finding their perimeters?

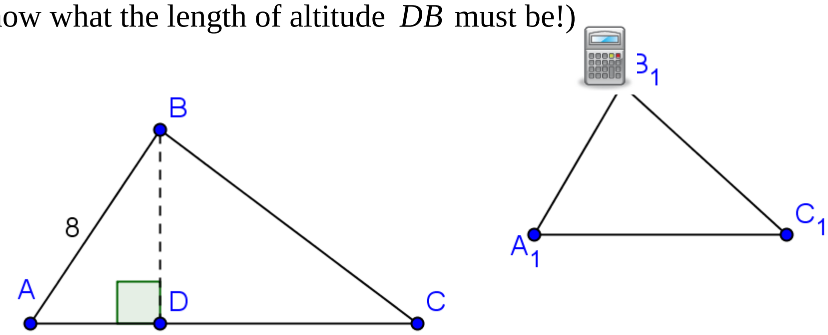


c. Find their areas and then calculate the ratio of the area of the larger one to that of the smaller one.

d. How could you have known this ratio without finding their areas?

3. The two triangles below (not to scale) are similar. The length of $\overline{AC}$ is 12 and the area of ABC is 40.

a. Find the measure of angle A (hint: you know what the length of altitude $\overline{DB}$ must be!)

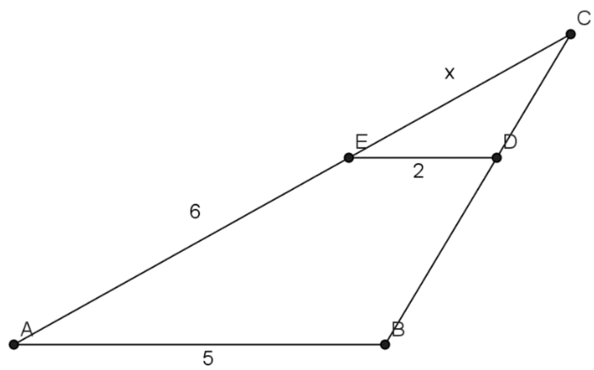


b. If the area of triangle $A_1B_1C_1$ is 10, then find the length of side $\overline{A_1B_1}$.

c. Instead, if the area of $A_1B_1C_1$ is 20, then find the length of side $\overline{A_1B_1}$. Don't be afraid of radicals.

4. Triangles ABC and DEF are similar and the perimeter of ABC is twice the perimeter of DEF. If the area of ΔABC is 20, then what is the area of ΔDEF? Explain.

5. In the diagram below, $\overline{DE} \parallel \overline{AB}$.

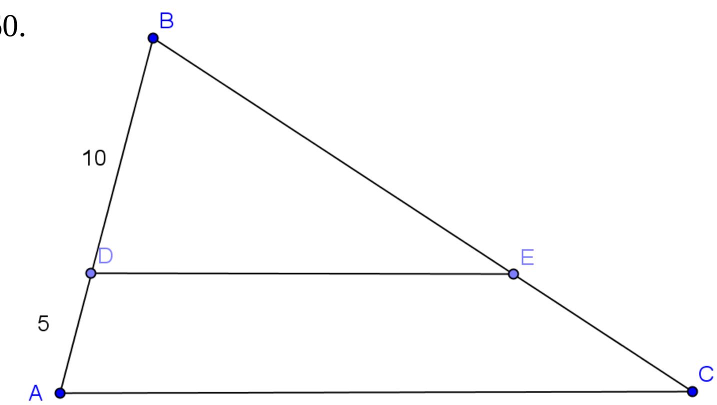


a. Jay says x is 2.4. How did he get his answer and why is it wrong? What is x?

b. If the area of ΔABC is 15, then find the area of ΔCDE and trapezoid ABDE.

c. Instead, if the area of trapezoid ABDE is y, then find the area of triangle ABC in terms of y.

6. In the diagram below, $\overline{AC} \parallel \overline{DE}$. The area of ΔBDE is 60.



a. What is the area of ΔBAC?

b. What is the area of trapezoid ACED?

c. What is the ratio of side lengths $\overline{AC}$ *to* $\overline{DE}$?

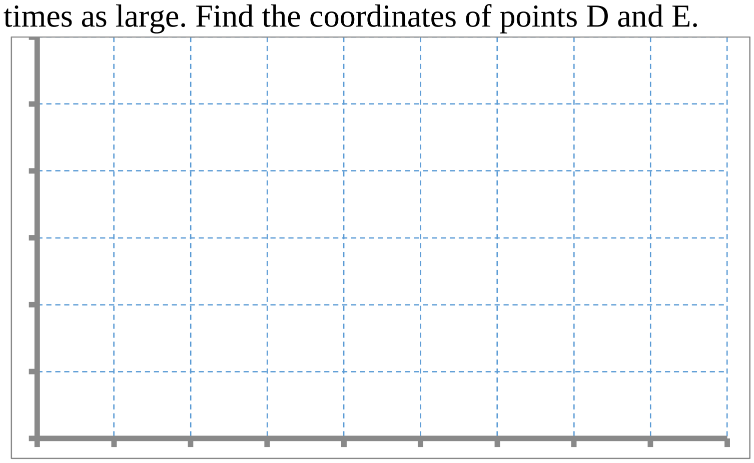
7. Given ΔABC~ΔDEF and the areas of the two triangles are 100 and 60 respectively. If AC=15, then what is DF?

8. A square has an area of 20.

a. How long is its diagonal?

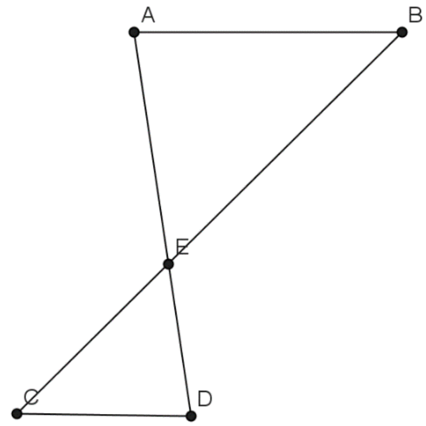
b. If you doubled its diagonal, what would happen to the area?

9. Given points A(0,5), B(3,4), and C(4,5). Triangle ADE is similar to triangle ABC but its area is four times as large. Find the coordinates of points D and E.



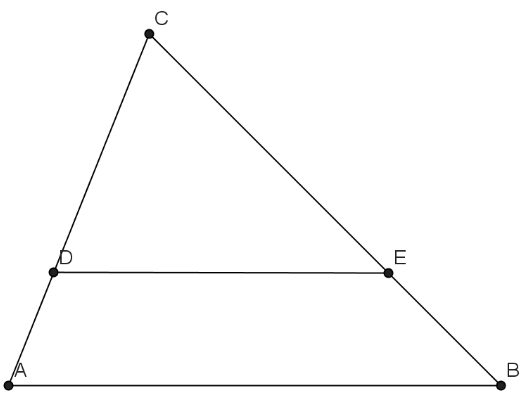
10. In the diagram below, $\overline{AB} \parallel \overline{CD}$. The area of ΔABE is three times the area of ΔCDE.

a. If $\overline{CE}$ measures 5 units, then find the length of $\overline{BE}$.

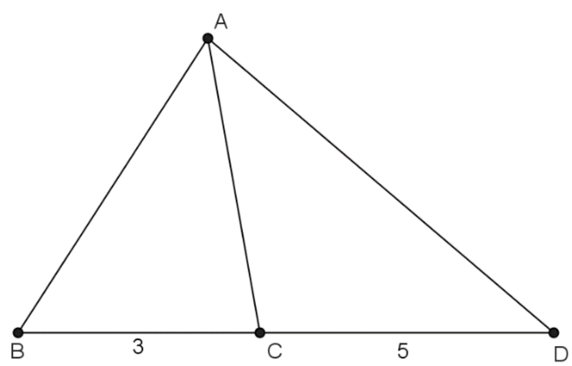


b. If $\overline{DA}$ measures 12 units, then find the length of $\overline{DE}$.

11. In triangle ABC, $\overline{AB} \parallel \overline{DE}$. If $\overline{AC}$ measures 12 units and the area of ΔCDE is equal to the area of trapezoid ADEB, then what it the length of $\overline{CD}$? Don't be afraid of radicals!

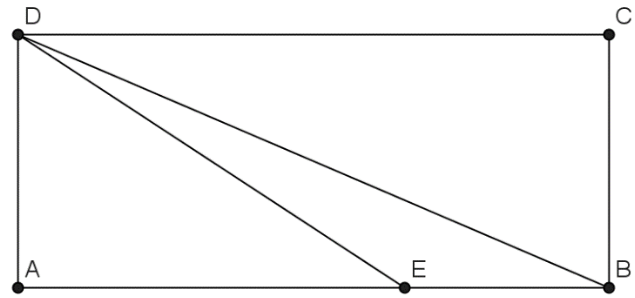


12. In the diagram below, the area of ΔABC is 6. What is the area of ΔACD? Why?



13. The area of rectangle ABCD below is 60, and the length of $\overline{AE}$ is twice the length of $\overline{BE}$.

a. Find the area of ΔDCB.



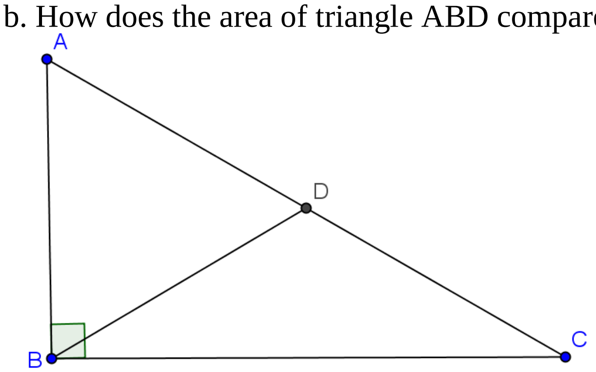
b. Find the area of ΔADE.

c. Find the area of ΔEDB.

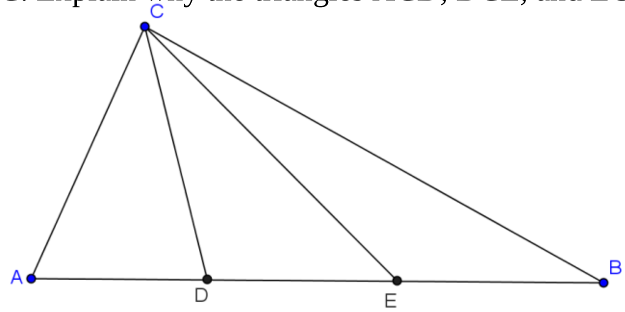
14. In right triangle ABC, the median is draw from B to side $\overline{AC}$.

a. What does this mean? (ie, what is the median of a triangle?)

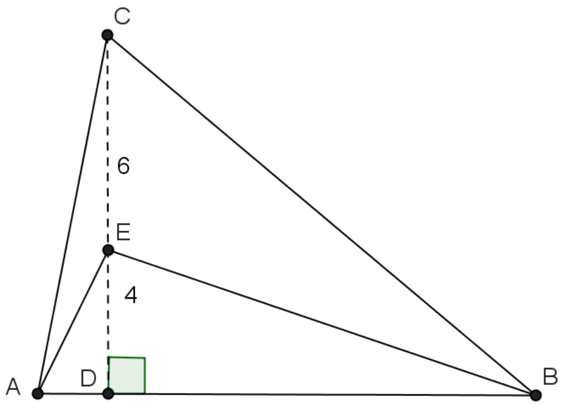
b. How does the area of triangle ABD compare to the area of triangle BCD? Explain why.



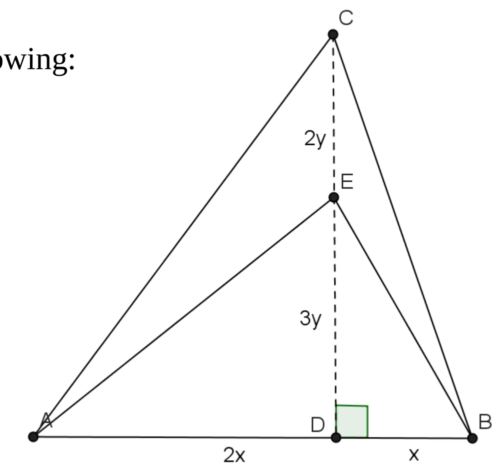
15. Points D and E trisect side $\overline{AB}$ of triangle ABC. Explain why the triangles ACD, DCE, and ECB all have the same area.



16. Assume the area of ΔABC is 50. Find the area of triangle ABE and quadrilateral AEBC.



17. Given that area of triangle ABC is 120, find the area of the following:



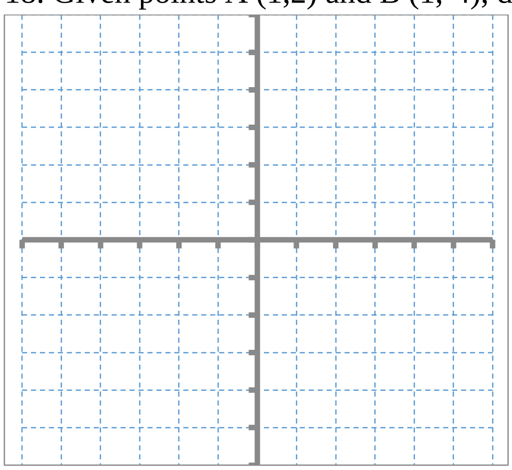
a. ΔADE

b. ΔAEC

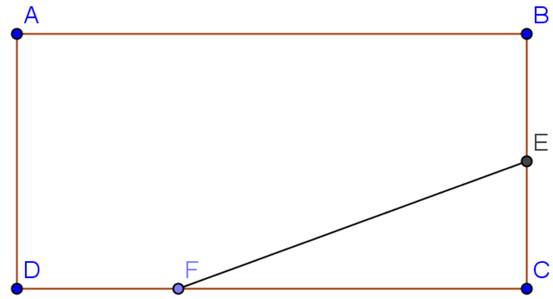
c. ΔBDE

d. ΔBEC

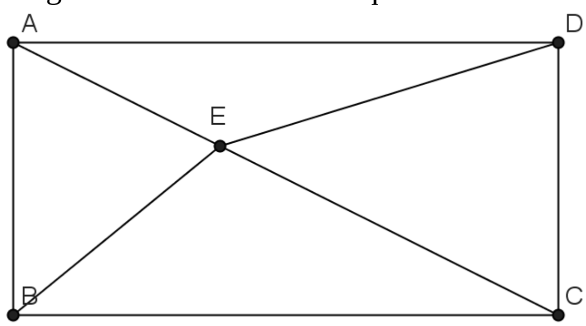
18. Given points A (1,2) and B (1,-4), describe the set of points C that makes ΔABC have an area of 12.



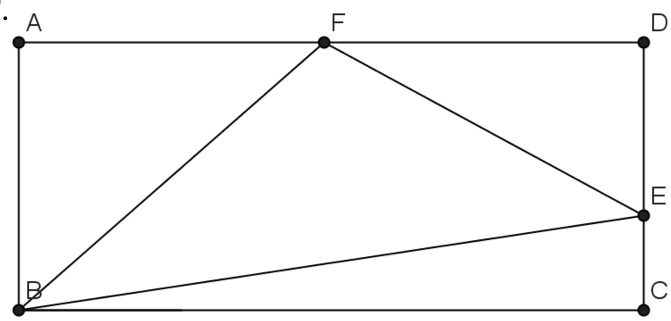
19. In rectangle ABCD below, E is midpoint of BC; if the area of triangle EFC is 1/6 the area of the rectangle, then what is the value of the ratio DF/DC?



20. Diagonal $\overline{AC}$ is drawn in rectangle ABCD. Let E be any point on the diagonal. Explain why the areas of triangles DEC and BEC are equal. Hint: think about altitudes.



21. The area of rectangle ABCD below is 100. The areas of triangles ABF and BCE are 25 and 20, respectively. Your goal is to find the area of triangle BEF.



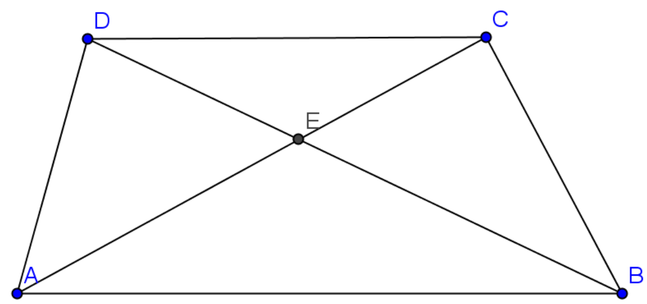
a. Why must F be the midpoint of $\overline{AD}$?

b. Point E is what percentage of the way from C to D? In other words, what is the ratio of the length of $\overline{CE}$ to the length of $\overline{CD}$?

c. Now find the area of triangles DEF and BEF.

22. In the trapezoid below show the following;

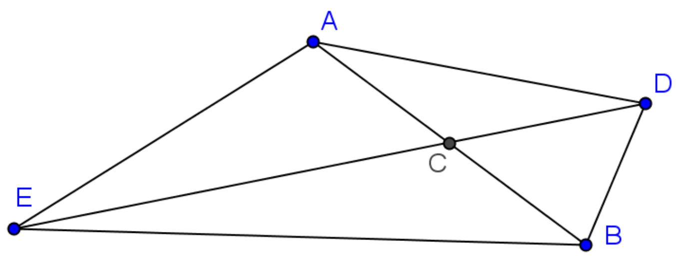
a. ΔCDA and ΔCDB have same area



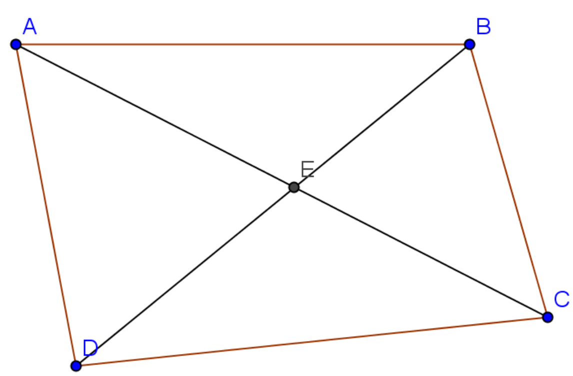
b. ΔBCE and ΔADE have same area (you will probably use part *a* above for this)

c. ΔABE and ΔCDE are similar

23. In quadrilateral ADBE, $\overline{DE}$ bisects $\overline{AB}$. Prove that $\overline{DE}$ splits ADBE into two regions of equal area. (from Moise, Downs *Geometry*)

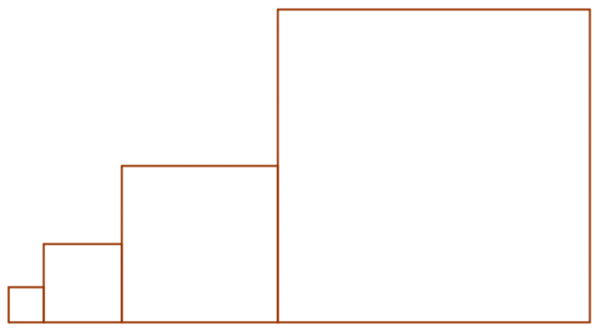


24. Quadrilateral ABCD below has no two parallel sides. The areas of ΔABE, ΔBEC, and ΔDEC are 20, 15, and 22 respectively. Find area of ΔAED. Think proportions!

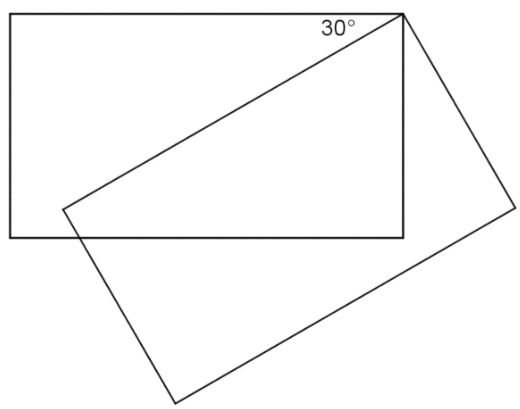


25. You have a square birthday cake (well, it's actually three-dimensional so it can't be a square—but it is a right rectangular prism with a square base!). It has frosting on its top and sides. You want to divide it into eleven pieces of equal area. And you want each piece to have the same amount of frosting. How can you do this?

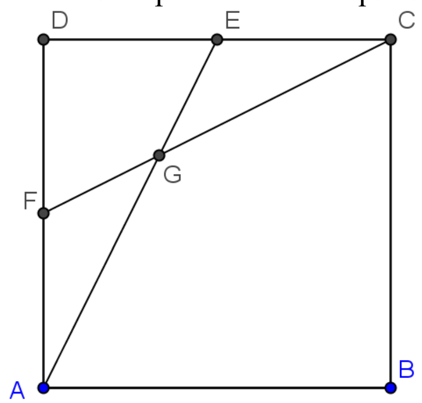
26. Each square on the in the diagram below has sides twice long as the square on its left. If the outer perimeter of the object is 115, then find its area. (from the Art of Problem Solving)

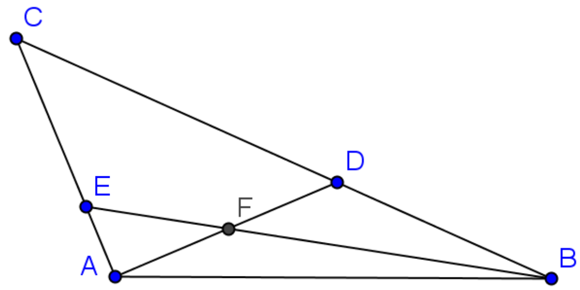


27. A 4-by-7 rectangle is rotated 30° counter-clockwise around one corner. What is the area of the region of overlap between the original and rotated rectangles? Express your answer with radicals; no calculator!



28. In square ABCD (on the left below), E and F are midpoints of two sides. Quadrilateral DEFG represents what portion of the square's area





29. In the diagram on the right above, the area of triangles AFB and DFB are each seven, and the area of ΔAEF is 3. Find the area of ΔABC. This one is tough!

Answers

1 a. 60 b. 30 c. 120 d. 20*mn* e. no; the base and altitude would have to scale by the same amount for them to be similar and preserve their shape. f. i.7 ii. 980

2a. $8+4\sqrt{2}$ and $24+12\sqrt{2}$ so ratio=3 b. each side is three times as large so perimeter is 3x
c. 8 and 72, so ratio is 9 d. each leg is three times as large so product of the legs is 9 times as large

3a. altitude is 20/3 so angle A measures about 56° b. 4 c. $8/\sqrt{2}$ *or* $4\sqrt{2}$

4. 5; each side is half as large so area is one fourth as large 5a. 4 b. 12/5 = 2.4 c. 25y/21

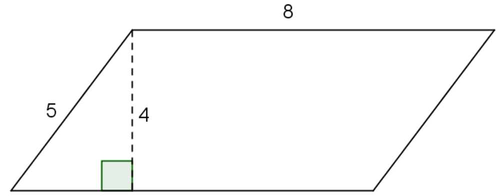
6a. 135 b. 75 c. 3:2 7. $3\sqrt{15}$ 8a. $2\sqrt{10}$ b. it would quadruple

9. D is (6,3) and E is (8,5) 10a. $5\sqrt{3}$ b. $x+x\sqrt{3}=12$ *so* $x=\dfrac{12}{1+\sqrt{3}}$ 11. $\dfrac{12}{\sqrt{2}}=6\sqrt{2}$

12. 10; same height and base is 5/3 as large 13a. 30 b. 20 c. 10 14a. Bisects AC b. same
15. equal base and same altitude… 16. ABE has area 20 so AEBC has area 50-20=30
17a. 48 b. 32 c. 24 d. 16 18. Anywhere on vertical lines x=5 or x=-3 19. 1/3
20. triangles ABC and CDA are congruent so have congruent altitudes; thus DEC and BEC have the same base and equal altitudes
21a. area of ABF is ¼ of whole rectangle; LW=100 and ABF=(1/2)(W)(xL) and thus x=1/2
b. 0.40 c. 15 and 40 22a. same base/same altitude b. subtract DEC from both triangles in part *a*…
c. vertical angles and alternate interior angles..
23. draw perps from A to ED and from B to ED. The 2 right triangles created have AC and BC as hypotenuses are congruent by AAS… thus the lengths of the perp segments are the same; so EBD and EDA have the same base (DE) and height and thus the same area…
24. CE must be ¾ of AE since ABE and BEC have same altitude; so AED must be 4/3 of 22 or 88/3
25. Divide the perimeter by 11 and make slices from each 11^{th} of the perimeter to the center
26. 531.25 27. $47\sqrt{3}/6$ 28. 1/6 since 4x+0.5A – x = A (A=square area and x=quad area)
29. draw FC AF=DF b/c same area and same alt from B; FB=7/3 of EF b/c same altitude from A
Area of CFB = 7/3 area of CFE and area of CFD = area of CFA
Call area of CFE x…. then CFB = 7x/3 and CFD = 7x/3 – 7 and CFA = x+3
So 7x/3 – 7 = x+3 and x=7.5 so CFD = 10.5 and total area = 35

Unit 8 Handout #4: Areas of Parallelograms and Trapezoids

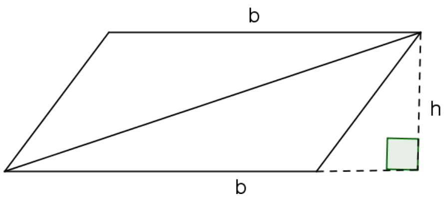
This section covers the areas of two kinds of quadrilaterals: parallelograms and trapezoids.

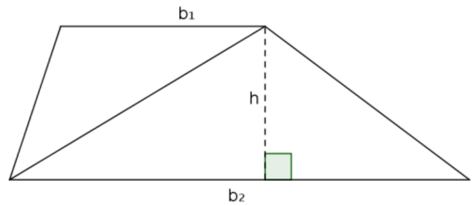


The area of a parallelogram is defined as the product of the base and the height, where the height is perpendicular to the base.

Thus the area of the parallelogram on the right is 32, the product of eight and four.

One way to derive this area formula is to divide a parallelogram into two congruent triangles with a diagonal, as shown below. (The triangles are congruent by SSS). The area of each triangle is one half of the product of the base and the height, so the area of the parallelogram is $\frac{1}{2}bh + \frac{1}{2}bh = bh$.

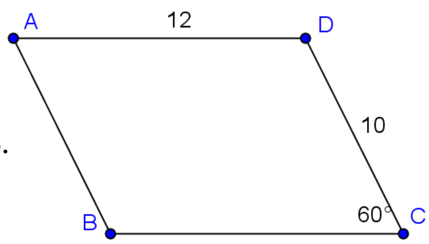




The area of a trapezoid is half the product of the sum of the bases and the height, or $A = \frac{1}{2}(b_1 + b_2) \cdot h$.

One way to derive this formula is to divide the trapezoid into two triangles with a diagonal, as on the right above. The area the triangles are $\frac{1}{2}b_1 \cdot h$ and $\frac{1}{2}b_2 \cdot h$; adding them and factoring gives $A = \frac{1}{2}(b_1 + b_2) \cdot h$.

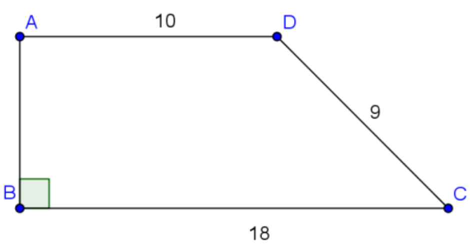
Example #1: Find the area of the parallelogram below.



While it is tempting to multiply the 12 and the 10 to get an area of 120, it is incorrect because the height must be measured perpendicular to the base.

Drop an altitude from D perpendicular to $\overline{BC}$. It creates a 30/60/90 triangle. The side opposite the 30° angle is one-half the hypotenuse, or 5. And the side opposite the 60° angle is $\sqrt{3}$ times this large. So the height is $5\sqrt{3}$ and the area is $A = b \cdot h = 12 \cdot 5\sqrt{3} = 60\sqrt{3}$.

Example #2: Find the area of the trapezoid below.

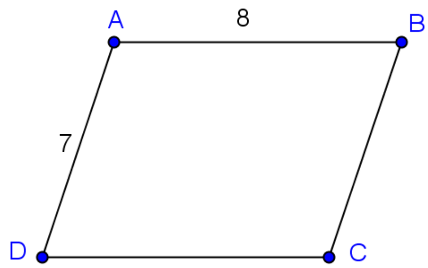


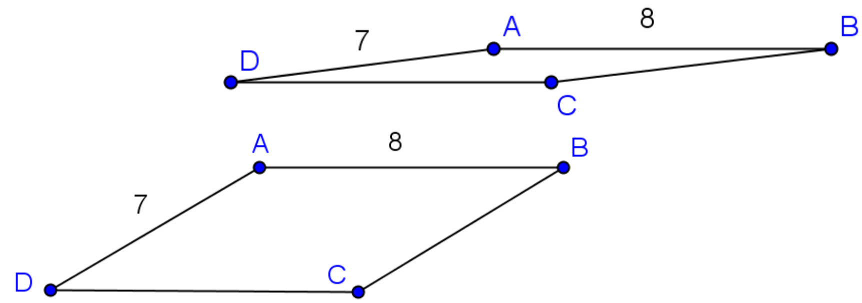
We have the bases and need the height. We can find it by dropping an altitude from D to $\overline{BC}$. The portion of the lower base to the right of this altitude has length 8, which is 18-10. Now use the Pythagorean Theorem to find the altitude a: $a^2 + 8^2 = 9^2$

Thus $a = \sqrt{81 - 64} = \sqrt{17}$.

So the area of the trapezoid is $A = \frac{1}{2}(b_1 + b_2) \cdot h = \frac{1}{2}(18 + 10) \cdot \sqrt{17} = 14\sqrt{17}$

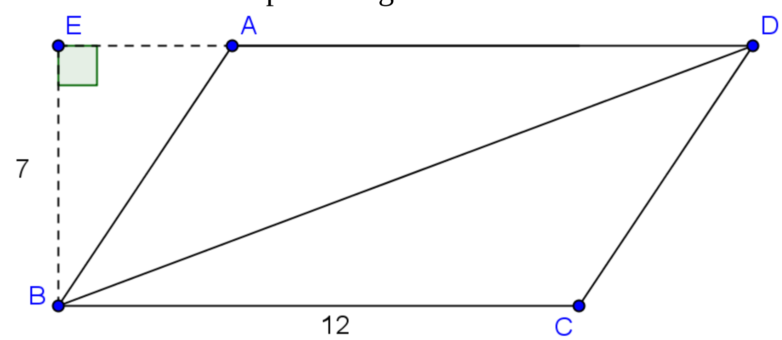
1. ***Area of a parallelogram, Part I***: Erica thinks that the three parallelograms below all have areas of 56. Is she right? Explain.





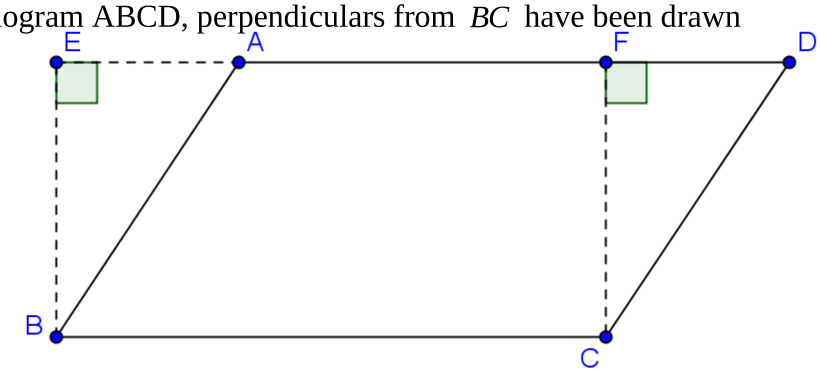
2. ***Area of a parallelogram, Part II***: Sarah wants to find the area of parallelogram ABCD so she divides in into triangles ABD and BCD.

a. Why are the areas of the two triangles the same? Will this be the case for all parallelograms?



b. Explain why the formula $A = b \cdot h$ works for a parallelogram, where *b* is a base and *h* is the height to that base, measured along a perpendicular.

3. ***Area of a parallelogram, Part III***: In parallelogram ABCD, perpendiculars from $\overline{BC}$ have been drawn to the line containing base $\overline{AD}$.



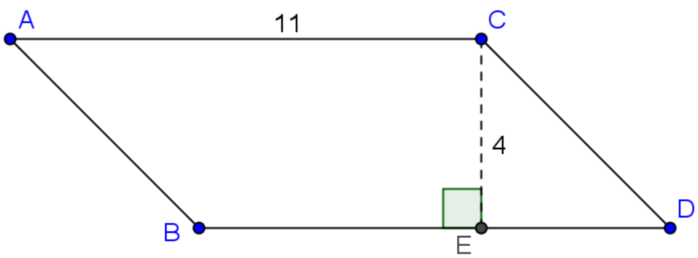
a. Explain why $\Delta AEB \cong \Delta DFC$.

b. Explain why the area of ABCD is the same as the area of rectangle FEBC.

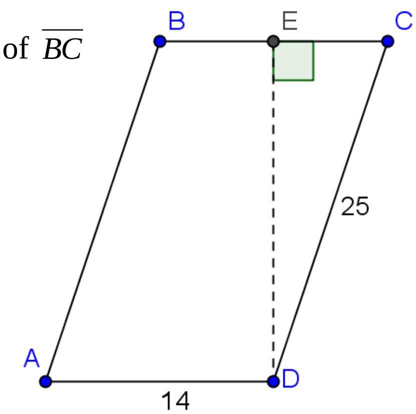
c. Explain why the formula $A = b \cdot h$ works for a parallelogram, where *b* is a base and *h* is the height to that base, measured along a perpendicular.

4. Find the area of each parallelogram below.

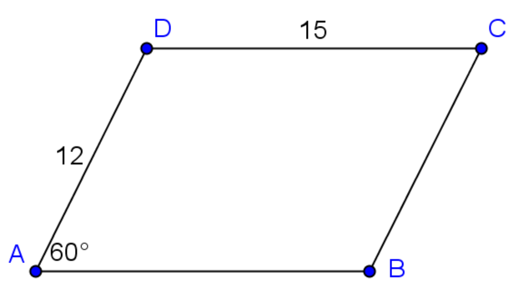
a.

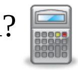


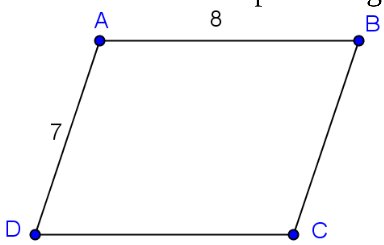
b. E is the midpoint of $\overline{BC}$



c.

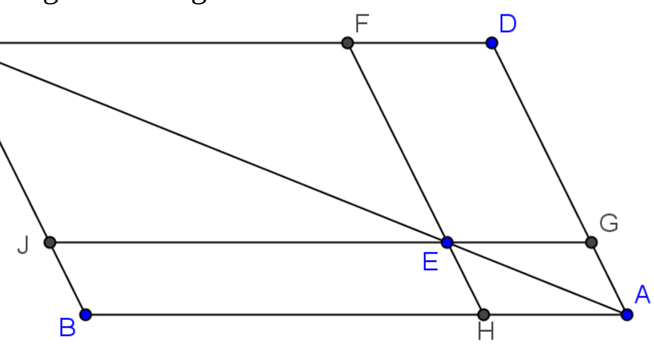


5. If the area of parallelogram ABCD is 50, then what is the measure of angle A? 



6. Diagonal $\overline{AC}$ is drawn in parallelogram ABCD. An arbitrary point E on the diagonal is selected and line segments $\overline{HF}$ *and* $\overline{JG}$ are drawn through E parallel to the sides of the parallelogram.

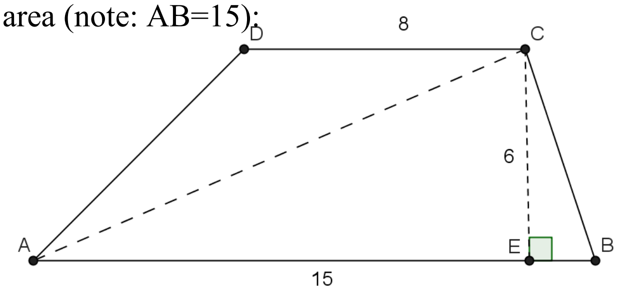
It turns out that parallelograms FEGD and JEHB, though not congruent, must have the same area. Explain why. Hint: the diagonal of a parallelogram splits it up into two congruent triangles.



7. ***Area of a trapezoid, part I:*** One way to think of a trapezoid's area (note: AB=15):

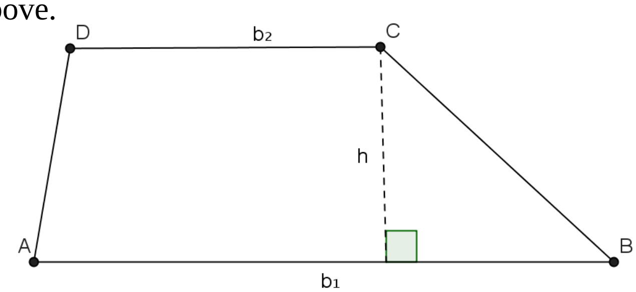
a. Divide trapezoid ABCD into triangles ABC and ACD.

b. If $\overline{AB}$ *and* $\overline{CD}$ are the bases of these two triangles, why must they have the same height?

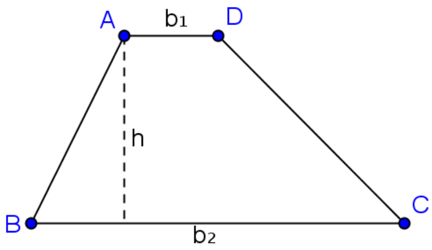


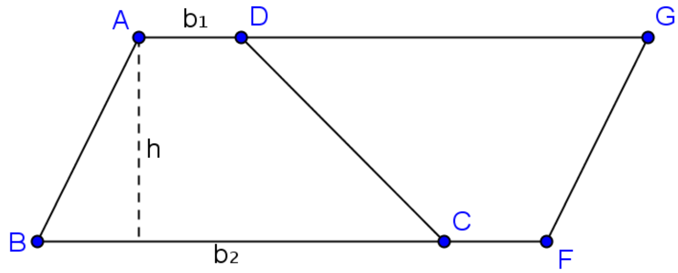
c. What is the area of trapezoid ABCD?

8. ***Area of a trapezoid, part II:*** Find the area of trapezoid ABCD in terms of its two bases and its height. Divide it by a diagonal into two triangles as in the question above.

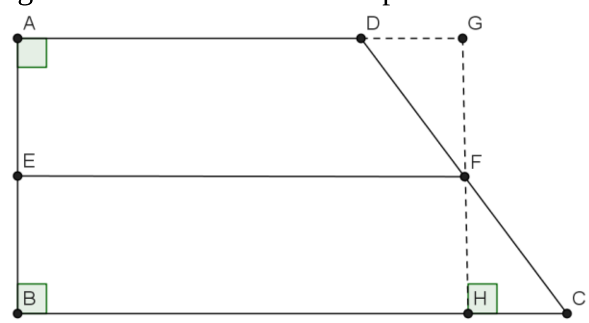


9. ***Area of a trapezoid, part III***. Kerry knows how to find the area of a parallelogram but not how to find the area of a trapezoid. To find the area of trapezoid ABCD on the left below, she makes a congruent one (FGDC) and flips it as in the diagram on the right so that A, D, and G are collinear and so are B, C, and F. How can she use her knowledge of parallelogram areas to find the area of trapezoid ABCD?





10. ***Area of a trapezoid, part IV***. In right trapezoid ABCD below, segment $\overline{EF}$ connects the midpoints of sides $\overline{AB}$ *and* $\overline{CD}$.



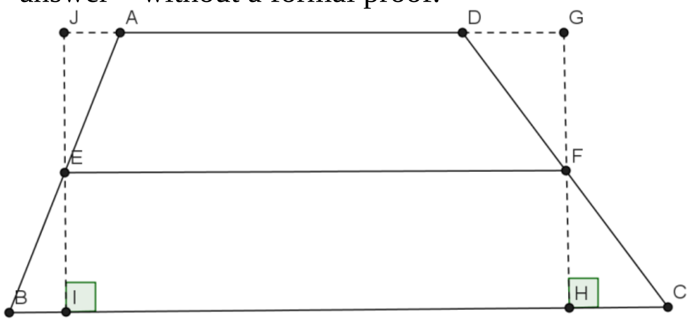
a. Explain why $\Delta FHC \cong \Delta FGD$.

b. Use this result to show that the area of ABCD must be equal to the area of rectangle ABHG.

c. Explain why $\overline{EF}$ must be the average of the lengths of $\overline{AD}$ *and* $\overline{BC}$.

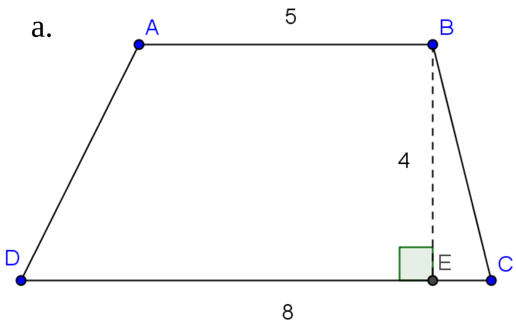
d. Explain why the formula $A = \frac{(b_1 + b_2)}{2} \cdot h$ represents the area of a trapezoid with bases b_1 and b_2 and height h.

11. ***Area of a trapezoid, part V***. In trapezoid ABCD below, $\overline{EF}$ connects the midpoints of the opposite sides. If the area of rectangle GHIJ is 100, then determine the area of trapezoid ABCD. Justify your answer—without a formal proof.

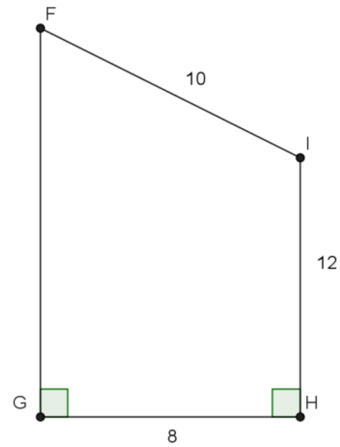


12. Find the area of the following:

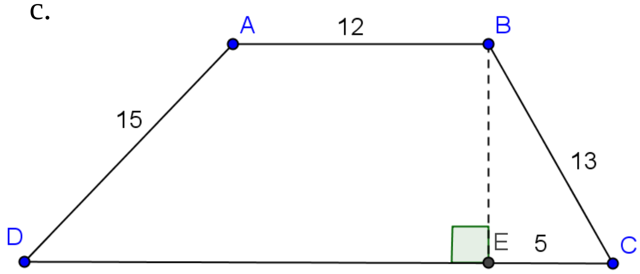
a.



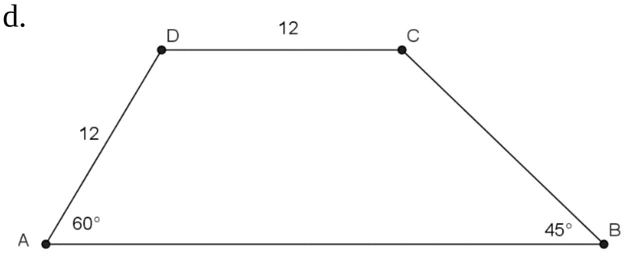
b.



c.

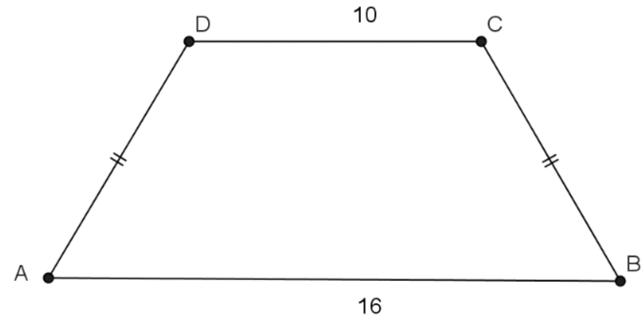


d.



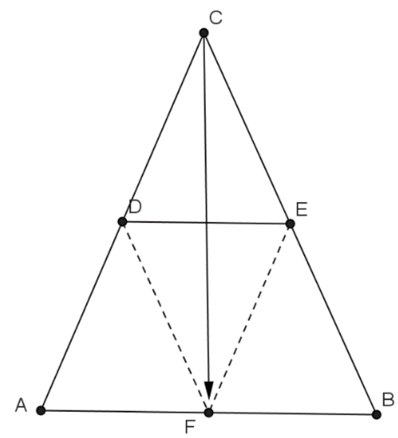
13. Given that the area of the isosceles trapezoid below is 91, do the following:

a. Find its perimeter.



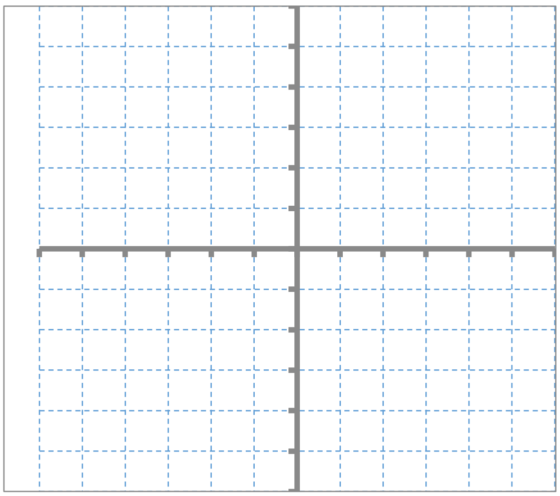
b. Find the measure of angle A. 

14. The vertex (C) of an isosceles triangle ABC is folded down to touch the midpoint of the opposite side (F)—as shown in the diagram below. If the area of the trapezoid ABED is 30, then was the area of the original triangle ABC?

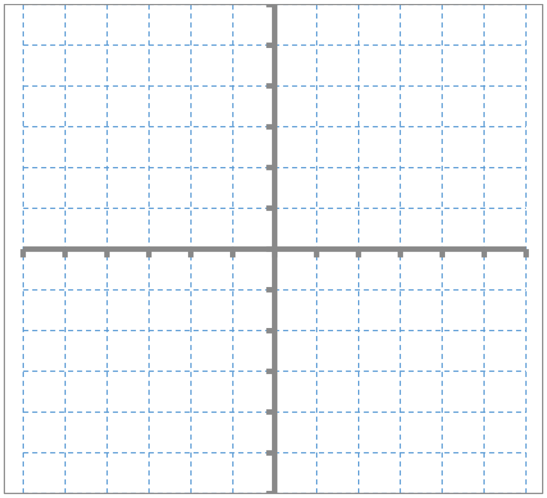


15. In each part below, find the area of the quadrilateral bound by the four given lines.

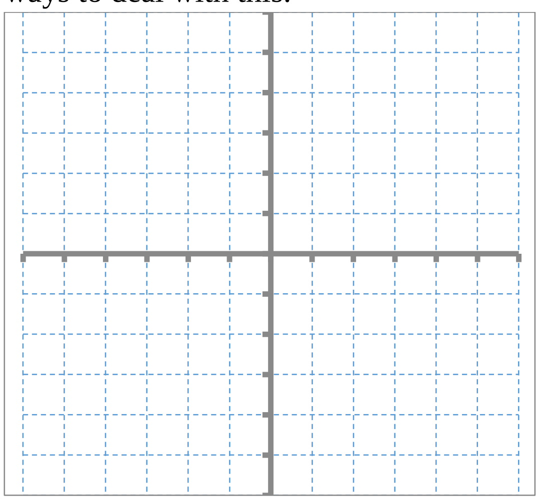
a. $y = 0,\ x = -2,\ x = 4,\ y = -0.5x + 4$



b. $y=-2,\ y=3,\ y=2x+6,\ y=-x+4$



16. What is the area of the trapezoid whose vertices are (-3,-3), (1,5), (4,5), and (-1,-5)? There are many ways to deal with this.



17. Given trapezoid ABCD with midsegment $\overline{EF}$ (meaning E and F are midpoints of the sides).

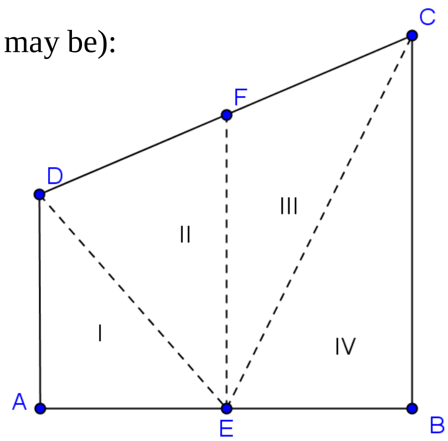
Determine whether each statement below must be true (more than one may be):

a. Area of II = Area of III

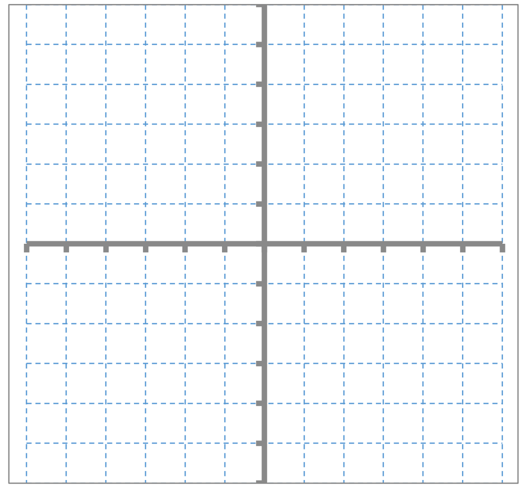
b. Area of I = Area of II

c. Area of I + Area of II = area of III + area of IV

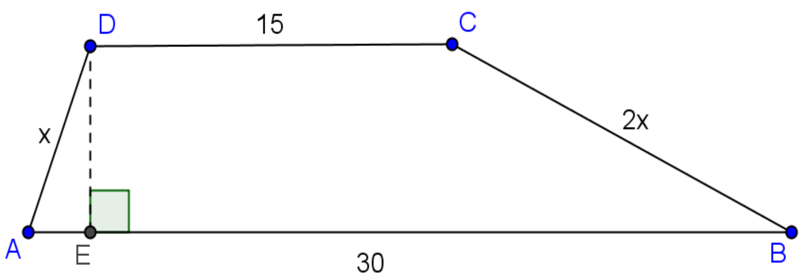
d. Area of I + Area of IV = area of II + area of III



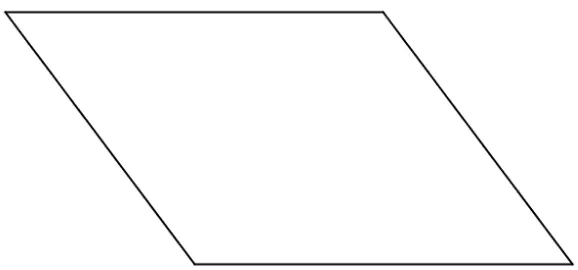
18. Three vertices of a parallelogram are (-2,-5), (3,-2), and (0,2). Find its area. There are many ways to approach this!



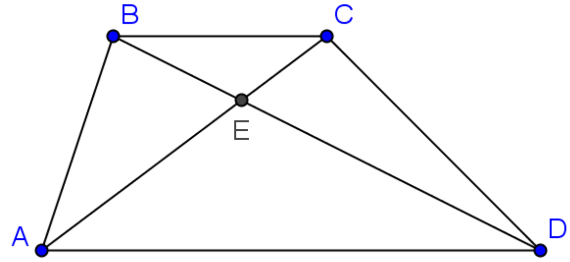
19. In trapezoid ABCD, one base is twice the other and one leg is twice the other. If the area is 135, find the value of *x*. Hint: define *y* as the length of segment $\overline{AE}$.



20. Given a rhombus with area of 80 and perimeter of 40. Find the lengths of its diagonals.



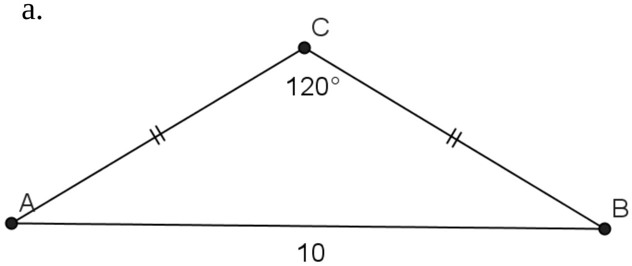
21. Show that, in the trapezoid below, the area of triangles ABE and CED must be equal. Hint: start with the areas of triangles ABC and BCD.



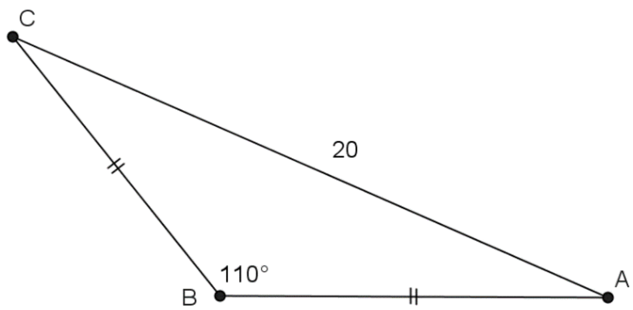
More general area questions:

22. Find the area of the two triangles below.

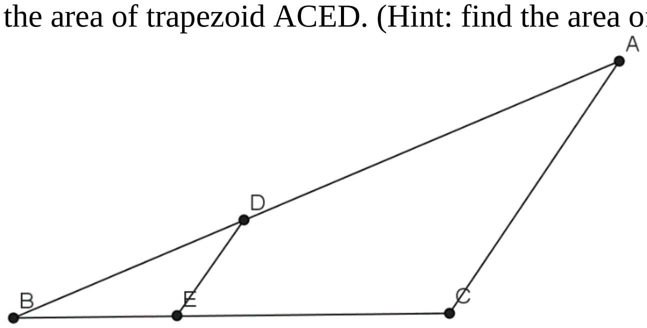
a.



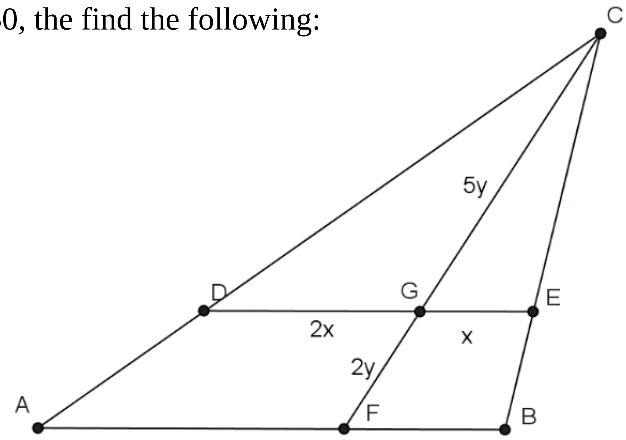
b.



23. In the diagram below, $\overline{DE} // \overline{AC}$. If $\overline{EC}$ is twice the distance $\overline{BE}$ and the area ΔBED is 20, then find the area of trapezoid ACED. (Hint: find the area of ΔABC first!)



24. In the diagram below, $\overline{DE} \parallel \overline{AB}$. If the area of ΔCEG is 50, the find the following:



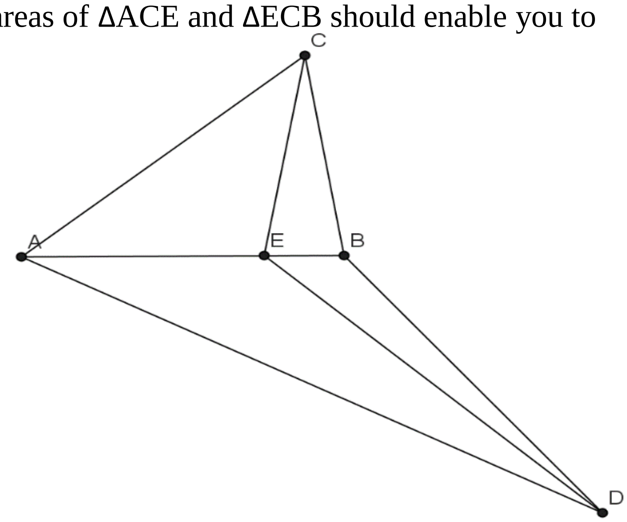
a. The area of ΔCGD.

b. The area of ΔCBF.

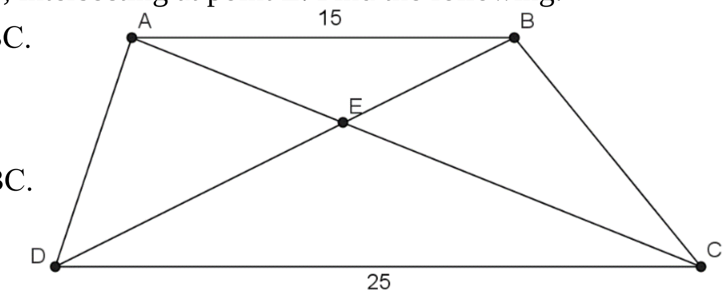
c. The area of quadrilateral BFGE.

d. The area of quadrilateral ADGF.

25. In the diagram on the left below, the areas of triangles ACE, ECB, and EBD are 70, 20, and 30 respectively. What is the area of triangle AED? Hint: the areas of ΔACE and ΔECB should enable you to determine the ratio of segment $\overline{AE}$ to $\overline{EB}$.



26. The diagonals of trapezoid ABCD are drawn below, intersecting at point E. Find the following:



a. The ratio of the area of ΔADB to the area of ΔABC.

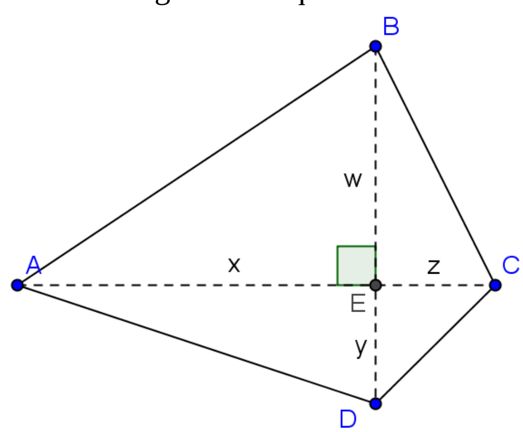
b. The ratio of the area of ΔABD to the area of ΔDBC.

c. The ratio of the area of ΔDEC to the area of ΔABE.

d. The ratio of the area of ΔBEC to the area of ΔABE.

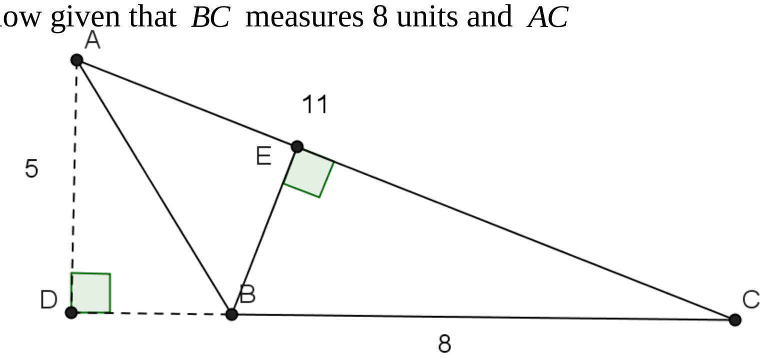
e. The ratio of the area of ΔBEC to the area of ΔADE

27a. Show that the area of any convex quadrilateral with perpendicular diagonals is equal to one half of the product of the diagonals. Use the diagram below.

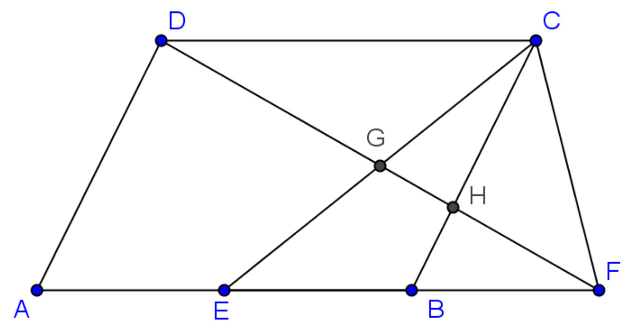


b. Which special quadrilaterals can you use this to find the area of?

28. Find the length of segment $\overline{BE}$ in the diagram below given that $\overline{BC}$ measures 8 units and $\overline{AC}$ measures 11. Hint: think area!



29. In the figure below, ABCD is a parallelogram and E and B trisect $\overline{AF}$. The area of ABCD is 36. Find the area of the following:



a. ΔBFC

b. Trapezoid AFCD

c. Trapezoid AECD

d. ΔCDF

e. ΔAFD

f. ΔBHF

g. ΔCHF

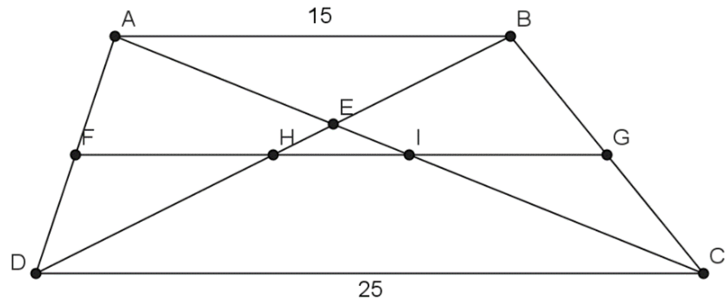
h. Explain why G must be the midpoint of $\overline{DF}$.

i. ΔCGF

j. Quadrilateral EBHG

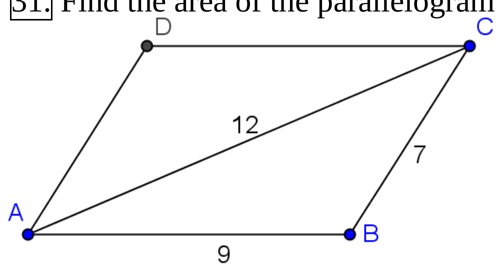
30. The diagonals of trapezoid ABCD are drawn below, as is the mid-line (the line joining the midpoint of the legs—which is parallel to the bases).

a. What is the ratio of the area of ΔEHI to the area of ΔABE?

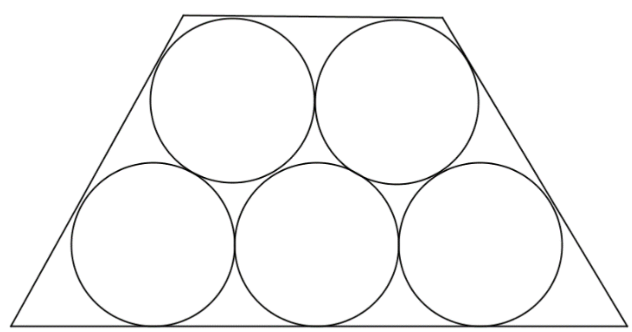


b. What is the ratio of the area of ΔEHI to the area of quadrilateral GIEB?

31. Find the area of the parallelogram below. Hint: add some line segments.



32. A trapezoid is drawn around 5 congruent and mutually-tangent circles. The top and sides of the trapezoid are each tangent to two circles, and the bottom is tangent to three. If the radius of each circle is one, the find the area of the trapezoid. Leave your answer as a simplified radical expression.



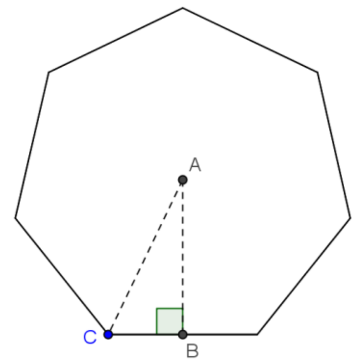
Answers

1. no; you can't just multiply adjacent sides – the flat one's area is much smaller than the squarer one
2a. opposite sides of a parallelogram are congruent b. ½ b*h + ½ b*h = b*h
3a. AAS b. triangles AEB and DFC have = area…just moving it over 4a. 44 b. 336 c. $90\sqrt{3}$ 5. 117°
6. the areas of triangles ABC and ADC are equal since the diagonal of a parallelogram divides it into 2 congruent triangles (by SSS). For the same reason, area ΔCJE=area ΔCFE and area ΔAHE=area ΔAGE. Using the subtraction property twice means that parallelograms FEGD and JEHB have equal areas.
7b. they are parallel so the altitude to one is the altitude to the other c. 69 8. ½ b_1*h + ½ b_2 *h
9. ABFG is a parallelogram whose bases BF and AG are (b_1+b_2). So its area is h(b_1+b_2). Since the two trapezoids are congruent, the area of each one is half this, so $A = \frac{(b_1 + b_2)}{2} \cdot h$
10a. AAS b. just moving part away c. adding DG to one and subtract = amount from other d. area is the average base times the height
11. 100 since it is average base times height—same as rectangle; just moving two triangles around
12a. 26 b. 120 c. 228 d. $54 + 90\sqrt{3}$ 13a. $26 + 2\sqrt{58}$ b. 67° 14. 40
15a. 21 b. 31.25 16. 27 (can draw rectangle around it and subtract 3 right triangles; can also draw segment from (-3,-3) to (0,-3) and add area of parallelogram and triangle; can also use algebra to find the equation of an altitude and its length…) 17a. yes b. no c. no d. yes
18. 29; find the area of the Δ with those 3 vertices and double it…. 19. $3\sqrt{5}$ 20. $4\sqrt{5}$ *and* $8\sqrt{5}$
21. The areas of ABC and BCD are equal (same base, same height). Subtract the area of BCE from both and thus the areas of ABE and CED are also equal. Cool!
22a. $25/\sqrt{3}$ b. about 70 23. 160 24a. 100 b. 98 c. 48 d. 96 25. 105
26. 1:1 (same base; same height) b. 3:5 (same height) c. 25:9 (similar with sides in 5:3 ratio) d. 5:3 since base EC is 5/3 base AE e. 1:1 (alt from D to AC is 5/3 alt from B to AC; AE is 3/5 EC)
27a. divide it into right Δ's: $A = 0.5(wx) + 0.5(wz) + 0.5(xy) + 0.5(xz) = 0.5(x+z)(y+w)$ b. rhombi
28. 40/11 29a. 9 b. 45 c. 27 d. 18 e. 27 f. 3 (similar to ADF in 1:3 ratio) g. 6 h. draw DE… FEDC is a parallelogram and diagonals bisect each other i. 9 j. 6
30a. 1/9: FG=20 and FH and IG are each 7.5 (midlines of Δ's with base 15) so HI=5; so ~ Δ's b. also 1/9
31. $28\sqrt{5}$: drop a perpendicular from C and write two equations with two unknowns.
32. Base angles must be 60° because sides are parallel to equilateral triangles formed by connecting circle centers. Bottom is $4 + 2\sqrt{3}$; top is $2 + \frac{2}{\sqrt{3}}$ *or* $2 + \frac{2}{3}\sqrt{3}$; height is 2+altitude of an equilateral triangle with side of 2 so $2 + \sqrt{3}$ so area = $(2 + \sqrt{3})\left(3 + \frac{4}{3}\sqrt{3}\right) = 10 + \frac{20}{3}\sqrt{3}$

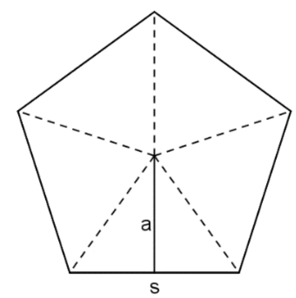
Unit 8 Handout #5: Area of Regular Polygons

The ***apothem*** of a regular polygon is the length of a segment connected its center to the midpoint of any side, such as $\overline{AB}$ in the diagram to the right. In a regular polygon, each vertex is equidistant from the center, so using SSS, it can be shown that the apothem is perpendicular to the side.

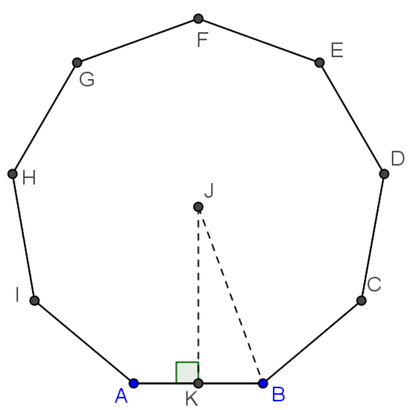
The ***radius*** of a regular polygon is the distance from the center to any vertex, such as $\overline{AB}$ in the diagram to the right. This is called the radius because it is the radius of the circle drawn through all vertices.



One way to find the area of a regular polygon is to divide it into congruent triangles, where each triangle's vertices are the center of the polygon and two consecutive vertices. For an *n*-sided polygon, there will thus be *n* triangles. Each one's area is $\frac{1}{2}a \cdot s$, where *a* is the apothem and *s* is the side length. The total area of the *n*-sided polygon will thus be $n\left(\frac{1}{2}a \cdot s\right)$. Since $n \cdot s$ is the perimeter, we can use the formula that the area of a regular polygon is $\frac{1}{2}ap$, where *a* is the apothem and *p* is the perimeter.



Example #1: Find the area of a regular nonagon (nine-sided polygon) with a perimeter of 180.



First we need to find the length of the apothem $\overline{JK}$.

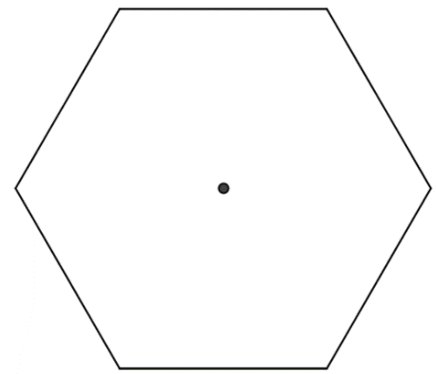
ΔAJB is one of nine congruent triangles making up the nonagon, each consisting of point J and two consecutive vertices. Thus $\angle AJB$ is one ninth of 360°, or 40°. And since ΔAJB is isosceles, the altitude $\overline{JK}$ bisects $\angle AJB$ so $\angle KJB$ measures 20°.

In ΔAJB, KB is one-half of side $\overline{AB}$, which is 180/9 or 20.

Therefore KB=10. Using trigonometry, $\tan(20^\circ) = 10/JK$. This can be solved with a calculator, yielding JK≈27.47.

So the area of ΔAJB is $(0.5)(20)(27.47) = 274.7$. And the area of the entire nonagon is nine times this large, so $274.7 \cdot 9 = 2472.3$.

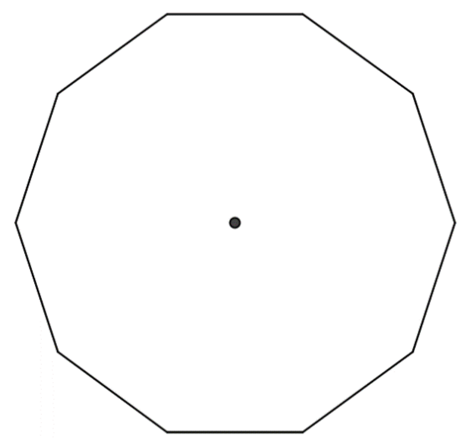
1. A regular hexagon has a radius of 10. *Remember, the radius of a regular polygon is the distance from the center to any vertex.*



a. What is its apothem?

b. What is its area?

2. A regular decagon has 10 sides, each of length 12.



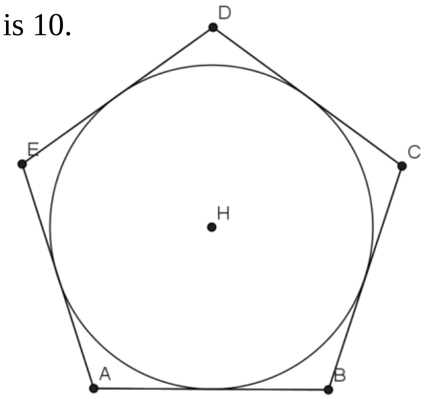
a. What is its radius? 

b. What is its apothem? 

c. What is its area?

d. If a different regular decagon had a side length of 18, what would its area be? (use “scale” rather than re-computing its apothem!)

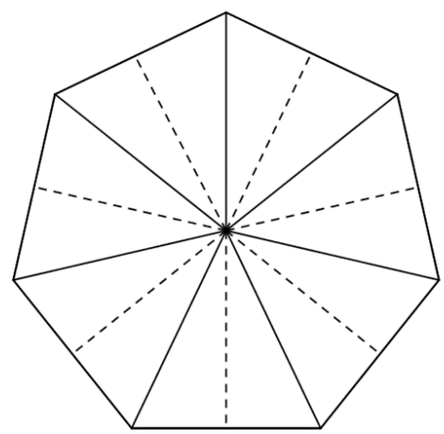
3. A circle is inscribed in a regular pentagon below. The side of the pentagon is 10.



a. What is the radius of the circle? 

b. What is the area of the pentagon?

4. Explain why the area of a regular polygon can be calculated by the formula $Area = \frac{1}{2}ap$, where a is the apothem and p is the perimeter. The diagram below may help!

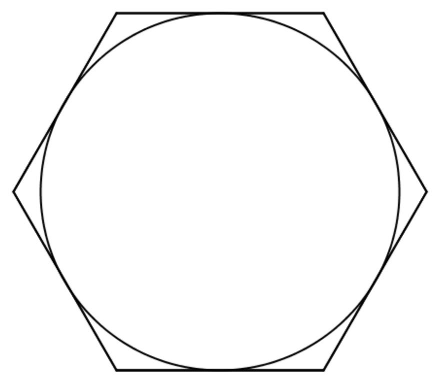


5. A regular twelve-sided polygon has an apothem of 10.

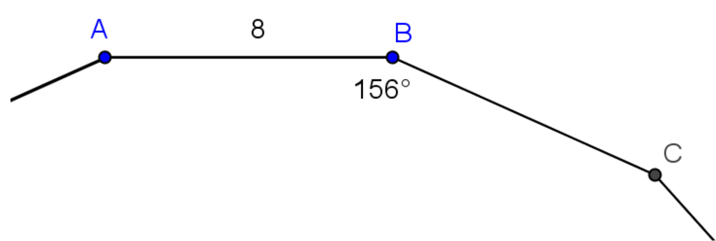
a. What is its side length? 

b. What is its area?

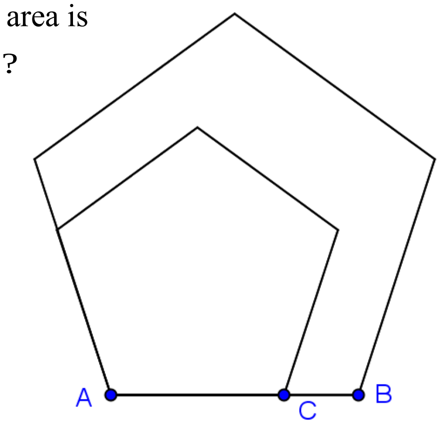
6. A regular hexagon has a circle with radius 8 inscribed in it. What is the hexagon's area?



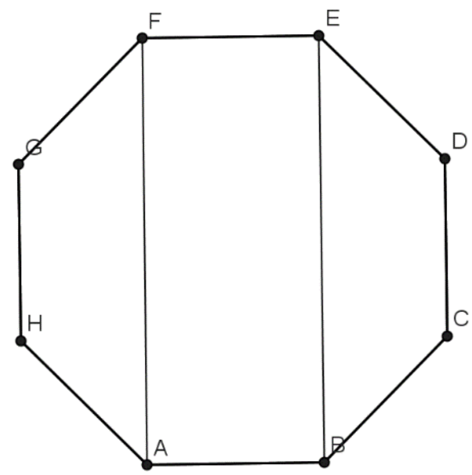
7. A part of a regular polygon is shown below. What is the polygon's area? 



8. The two pentagons in the diagram are both regular. The larger pentagon's area is twice the smaller pentagon's area. If AB=10, then what is the length of $\overline{BC}$? Leave your answer as a radical.



9. A regular octagon of side length 6 is drawn on the right.



a. Can you find its apothem easily without a calculator?

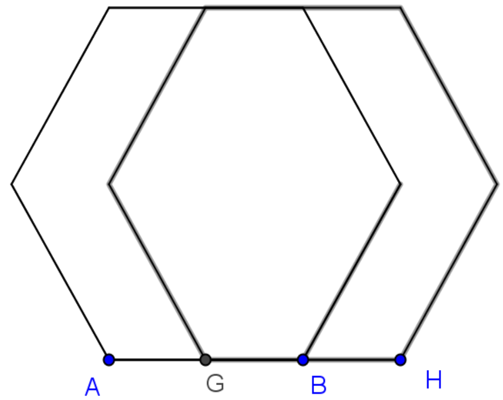
b. What is the length of segment $\overline{AF}$ (hint: divide it into 3 pieces and use special triangles)?

c. What is the area of trapezoid AFGH? Simplify the radical.

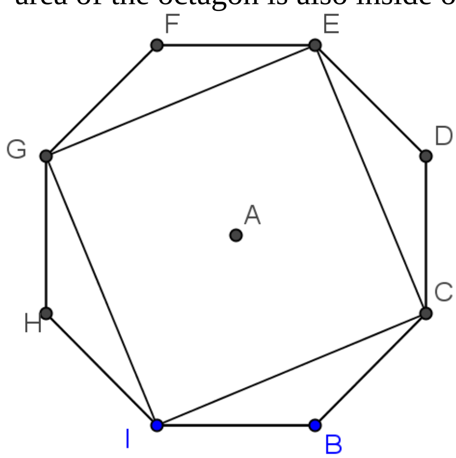
d. What is the area of the octagon? Add up the parts!

e. What is the length of the apothem of the octagon? Use the area formula "backwards"!

10. In the diagram below, $\overline{AB}$ *and* $\overline{GH}$ are sides of congruent regular hexagons and have length 12. G is the midpoint of $\overline{AB}$. What is the area of the region common to both hexagons (it is an equiangular, but not equilateral, hexagon).



11. A regular octagon has square inside; sharing every second vertex of the octagon. What percent of the area of the octagon is also inside of the square? Hint: draw $\overline{AF}$ and work in ΔAEF.

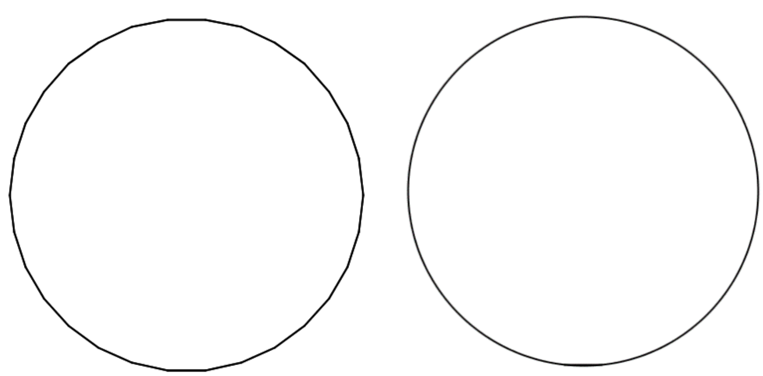


12. One of the shapes below is a regular 30-sided polygon and the other is a circle (whose radius is equal to the apothem of the polygon).

a. Is it obvious which is which?

b. How would you compare the areas of the two?

c. Based on this, do you think the $Area = \frac{1}{2}ap$ formula works for a circle? If so, what is the area of a circle?



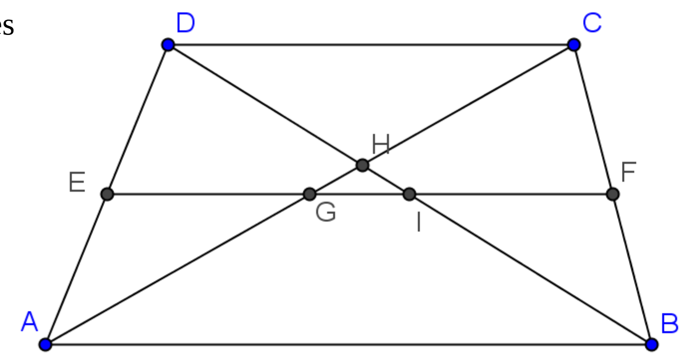
13. Two regular polygons are similar; the larger one's sides are 1.5 times the smaller one's sides. If the sum of the areas is 360, then what is the area of the smaller one?

14. The sum of the area of two squares is 170 and the sum of the perimeters is 72. What are their side lengths? Hint: write a system of two equations with two unknowns and then solve by substitution.

15. $\overline{EF}$ is the midline of trapezoid ABCD below. $\overline{CD}$ is 12 units long, $\overline{AB}$ is 18, and $\overline{AC}$ is 20.

a. Find the ratios of the areas of the following triangles

i. $\frac{\Delta ADC}{\Delta ABC}$ ii. $\frac{\Delta CGF}{\Delta DEI}$ iii. $\frac{\Delta CDH}{\Delta AHB}$



iv. $\frac{\Delta GHI}{\Delta AHB}$ v. $\frac{\Delta CDH}{\Delta GHI}$ vi. $\frac{\Delta FBI}{\Delta BDC}$

b. What is the ratio of the area of ΔCGF to the area of trapezoid AGFB? To ABCD?

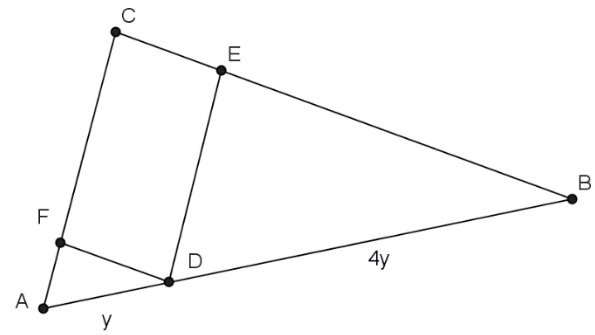
c. What is the ratio of the area of ΔDBC to the area of ΔBGD?

d. What is the ratio of the area of ΔBDG to the area of trapezoid ABCD?

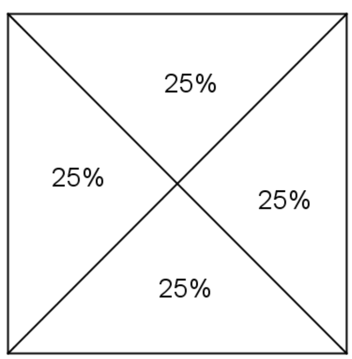
e. Triangle ABG makes up what percentage of the area of trapezoid ABCD?

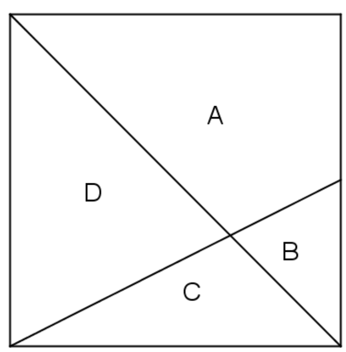
16. A length of rope is cut in half. One half is folded to make a square. The other is folded to make a regular hexagon. The sum of the areas of the two shapes is 300 square inches. How long is the rope? A decimal approximation is fine.

17. Scalene triangle ABC, point D is on AB such that AD:DB = 1:4. DE is parallel to AC and DF is parallel to BC. What fraction of the area of triangle ABC does parallelogram CEDF represent?



18. In the diagram on the left below, the diagonals of a square break it into four regions, each consisting of 25% of the square's area. In the diagram on the right, one diagonal is drawn, as well as a segment from a corner to the midpoint of an opposite side. What percent of the square's area does region A represent? Hint: call region A's area *x*, solve for the others in terms of *x*, and use similar triangles to write an equation that can be solved for *x*.





answers

1a. $5\sqrt{3}$ b. $150\sqrt{3}$

2a. 19.4 b. 18.5 c. 12*18.5/2*10 or about 1110 d. 1.5^2 as large so about 2498

3a. 6.88 b. (1/2)*(10)*(6.88)*5 or about 172

4. divide it into triangles where each side is the base of a triangle. Each triangle has area ½ * b * apothem. Add up all of the bases and you get the perimeter, so area = ½ *perim * apothem

5a. 5.36 b. 321.5 6. Apothem =8 perim= $96/\sqrt{3}$ so area = $384/\sqrt{3}=128\sqrt{3}$

7. 15-sided polygon; apothem is 18.82 so area is about 1129 8. $10-\frac{10}{\sqrt{2}}=10-5\sqrt{2}$

9a. no- you need trig functions of 22.5° b. $6+6\sqrt{2}$ c. $18+18\sqrt{2}$

d. $2\left(18+18\sqrt{2}\right)+6\left(6+6\sqrt{2}\right)=$ $72+72\sqrt{2}$ … interesting that the rectangle is exactly ½ the area!

e. $Area=\frac{1}{2}ap$ so $72+72\sqrt{2}=\frac{1}{2}a(48)$ and $a=3+3\sqrt{2}$

10. Regular hexagon with side 12 minus rectangle with base 6 and height $12\sqrt{3}$ so $144\sqrt{3}$

11. Divide the shape into 4 kites. The % of area the square takes is the ratio of the apothem of the square to the radius of octagon, which is $1/\sqrt{2}$

12a. somewhat b. very close c. why not! You get $\frac{1}{2}r(2\pi r)=\pi r^2$ .. neat! 13. 1440/13

14. $x^2+y^2=170$ and $4x+4y=72$ or $x+y=18$ so $y=18-x$ and $x^2+(18-x)^2=170$; this simplifies to $2x^2-36x-154=0$ or $x^2-18x-77=0$ so $(x-7)(x-11)=0$ and sides are 7 and 11.

15a. i. 2:3 ii. 1:1 iii. 4:9 iv. 1:36 v. 16:1 vi. 1:4 b. to AGFB 1:3; to ABCD it is 1/4th of 3/5th so 3:20

15c. AC is 20 so CH=8 and AH=12 so HG=2; so 4:1 d. 1:10 e. 30%

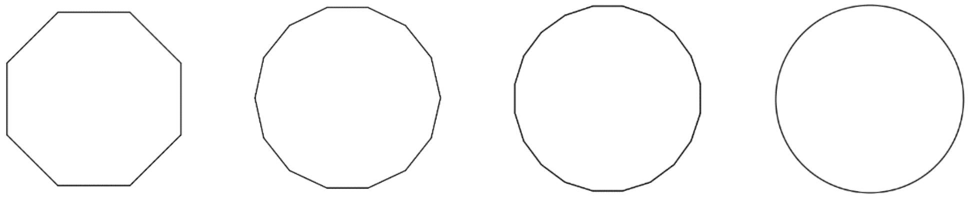
16. call length 24x: $9x^2+6x^2\sqrt{3}=300$ so $x=\sqrt{\frac{100}{3+2\sqrt{3}}}\approx 3.93$ → so rope is 94.4 inches 17. 8/25

18. A=x so B=50-x and D=75-x B~D with sides 1:2 so area is 1:4 so $\frac{50-x}{75-x}=\frac{1}{4}$ and x=125/3

Or draw horiz line thru center of the square: big part of A is (3/4)(1/2) of square, so 3/8. Smaller part of A is 1/3 of 1/8 or 1/24 so … 3/8 + 1/24 = 10/24 = 5/12

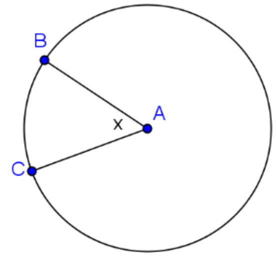
Unit 8 Handout #6: Area of Circles, Sectors, and Segments

A circle can be thought of as a regular polygon with an infinite number of sides. The diagram below shows regular polygons with 8, 14, and 20 sides and then a circle.

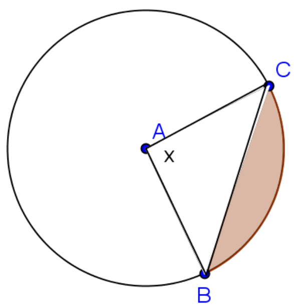


If we conceive of a circle this way, then it makes sense to use the area formula for a regular polygon, $A = (1/2)ap$, where *a* is the apothem and *p* is the perimeter. The apothem of a circle is its radius, and the perimeter of a circle is its circumference, or 2πr. Thus the area is $A = (1/2)ap = (1/2)r \cdot 2\pi r = \pi r^2$.

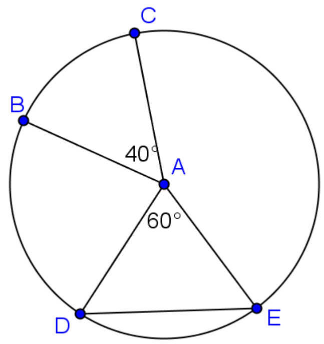
A ***sector*** of a circle is part of a circle enclosed by two radii and an arc. The area of a sector is proportional to its central angle. The entire circle's area is πr^2, so $\frac{sec\_area}{\pi r^2} = \frac{x}{360^\circ}$.



A ***segment*** of a circle is part of a circle on one side of a chord. So the shaded area of the circle on the right is a segment. When computing the area of the segment, it is useful to think of it as a sector minus a triangle. Trigonometry is often required to compute the triangle's area.



Example #1: In circle A below with radius 12, find the area of sector ABC and of the small segment created by chord $\overline{DE}$.



First sector ABC: The area of the entire circle is $\pi r^2 = 144\pi$.

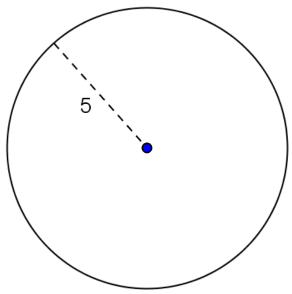
Use propotions: $\frac{sec\_area}{circle\_area} = \frac{40^\circ}{360^\circ}$ we get the sector area is 16π.

Now segment created by $\overline{DE}$. The area of a segment is the area of a sector minus the area of the triangle. The area of the sector is $\frac{sec\_area}{144\pi} = \frac{60^\circ}{360^\circ}$, so the sector area is 24π.

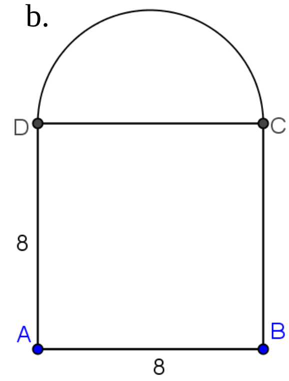
To find the area of ΔADE, we can drop an altitude, creating a 30/60/90 triangle. The length of the altitude is $6\sqrt{3}$, so the area of ΔADE is $0.5 \cdot 12 \cdot 6\sqrt{3} = 36\sqrt{3}$. Thus the segment's area is $24\pi - 36\sqrt{3}$. This answer is exact, but a calculator can give you a decimal approximation of 13.04.

1. Find the area of the following objects. Parts *b* and *d* include semi-circles.

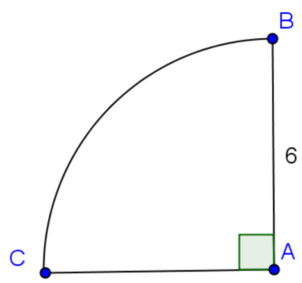
a.



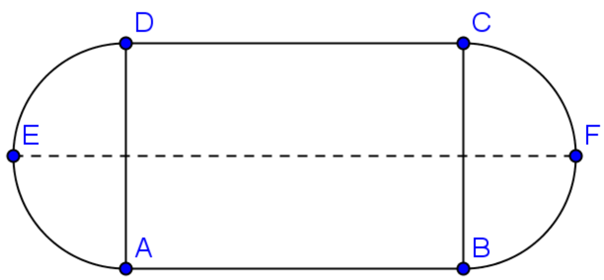
b.



c. A is the center of the circle



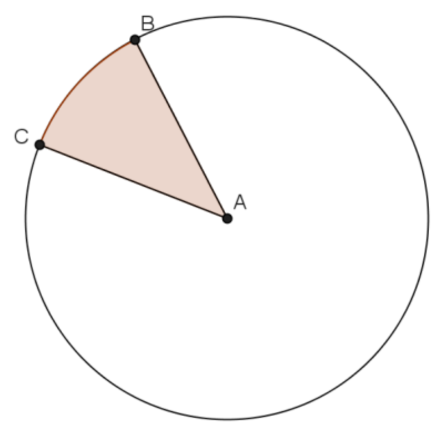
d. ABCD is a rectangle; CD=12 and EF=20.



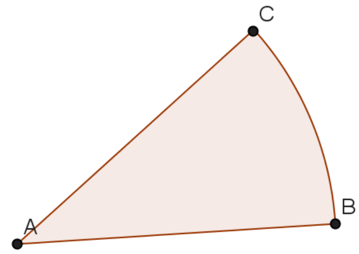
2. A sector of a circle with radius 6 has a central angle of 40°.

a. What is its area?

b. What is its perimeter?

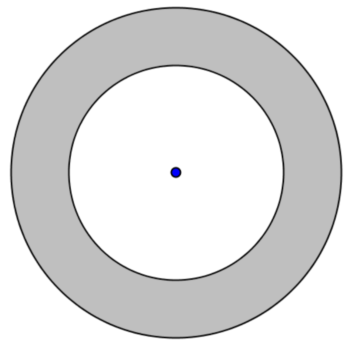


3. A circle has a radius of 10 and a sector has an arc length of 6.
   a. Find the central angle.

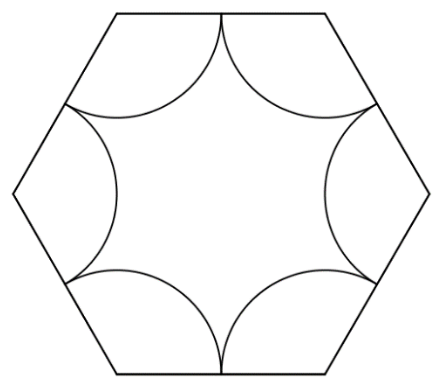


   b. Find the area of the sector.

4. The area of the shaded region below is 18π and the larger of the two concentric circles has a radius of 6. What is the radius of the smaller circle?

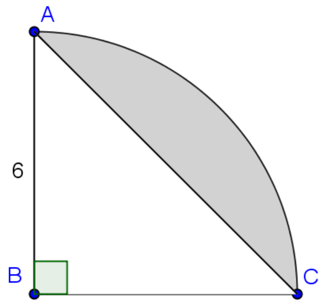


5. The regular hexagon below has a side length of 8. Circular arcs are drawn from each corner. What is the perimeter and area of the star-like shape in the center? No calculator.

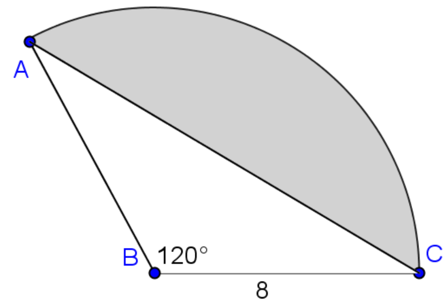


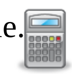
6. Find the area of the following segments. Hint: segment = sector – triangle!

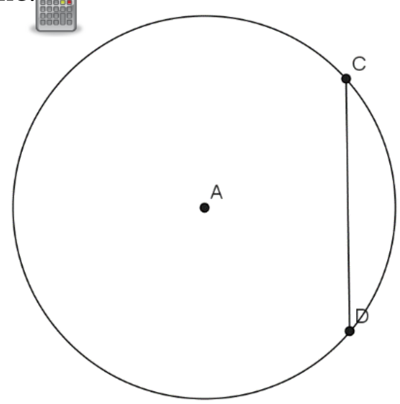
a.

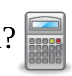


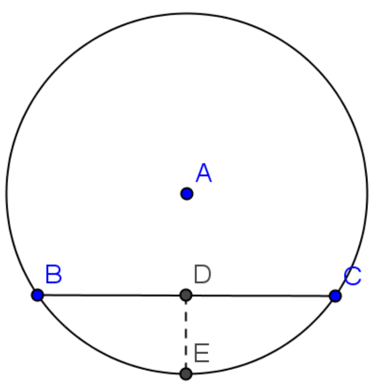
b.



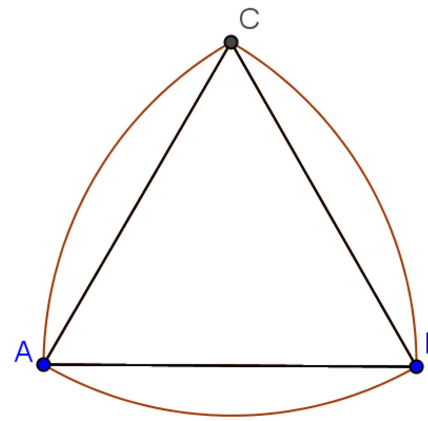
7. The circle below has a radius of 10. A chord of length 12 is drawn. Find the area and perimeter of the circular segment to the right of the chord. Decimal approximations are fine. 



8. The cross-section of a barrel (lying on its side) is a circle, shown below. Its radius is 18 inches. The level of oil in the barrel ($\overline{DE}$) is 7 inches. What percent full is the barrel? 

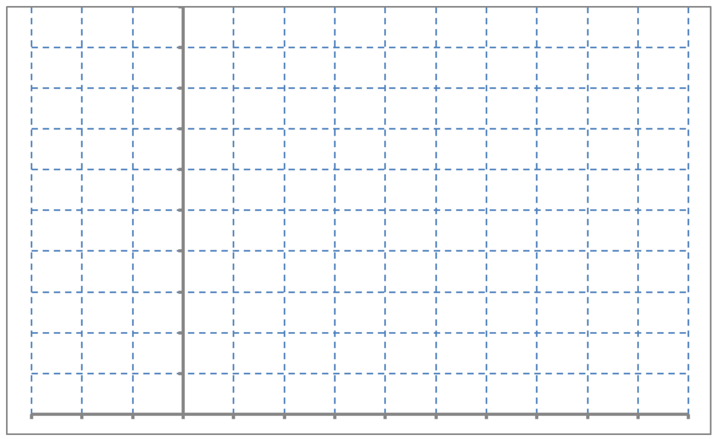


9. The shape below is an equilateral triangle of side 12 with circular arcs from each vertex through the other two vertices. Find its area and perimeter.



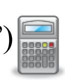
10. The endpoints of the diameter of a circle are (-1,3) and (7,9).

a. What are the coordinates of the circle's center?

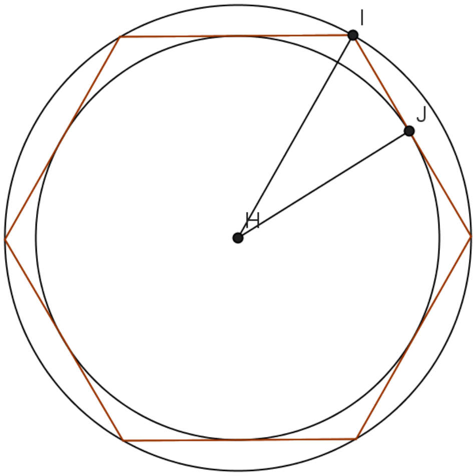


b. What is the area of the circle?

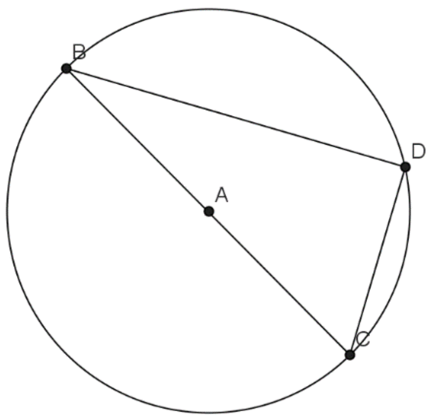
c. Find the coordinates of the points where the circle intersects the *y*-axis.

d. What is the area of the part of the circle to the left of the *y*-axis? ("in the second quadrant") 

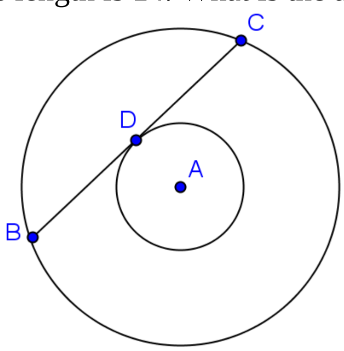
11. A regular hexagon of side 2 is drawn below, along with inscribed and circumscribed circles. What is the area of the region between the two circles? Hint: focus on ΔHIJ.



12. In the diagram below, ΔBCD is inscribed in a circle with a radius of 8. Minor arc BD is twice the measure of minor arc CD. Find the area of the region inside the circle but outside the triangle.

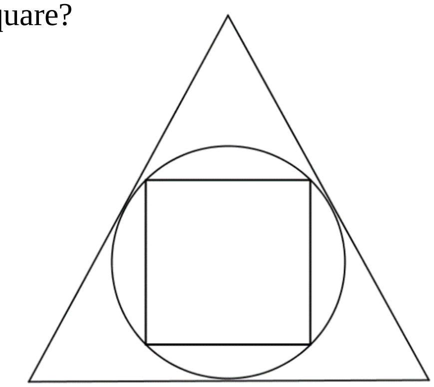


13. The radius of the smaller circle is 3. Chord $\overline{CB}$ of the larger circle is tangent to the smaller circle and its length is 14. What is the area of the region inside the larger circle but outside the smaller circle?



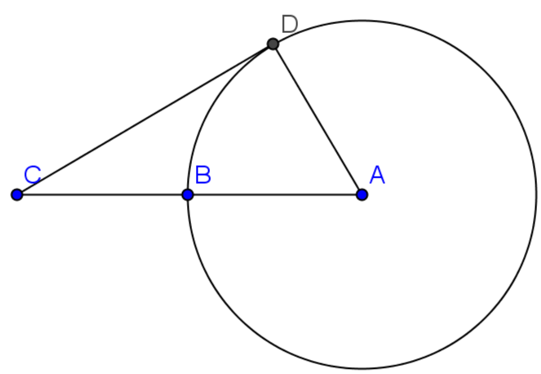
14. In a circle of radius 12, a chord of length 12 divides the circle into two regions. What is the area of the larger one?

15. In the diagram below, a circle is inscribed in an equilateral triangle, and a square is inscribed in the circle. The sides of the triangle are 2 units long. What it the area of the square?



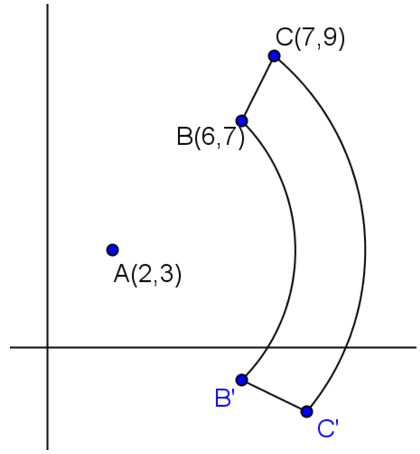
16. If the radius of a circle was two units larger, then its area would increase by 42π. Find its radius.

17. In the diagram below, circle A has radius of 10. Segment $\overline{CD}$ is tangent to circle A, and B is the midpoint of $\overline{AC}$. Find the area and perimeter of the region inside triangle ACD but outside the circle.



18. Segment $\overline{BC}$ is rotated 90° clockwise around point A.

a. What are the endpoints of the resulting segment, $\overline{B'C'}$?



b. What is the perimeter of the region swept by $\overline{BC}$ as it is rotated?

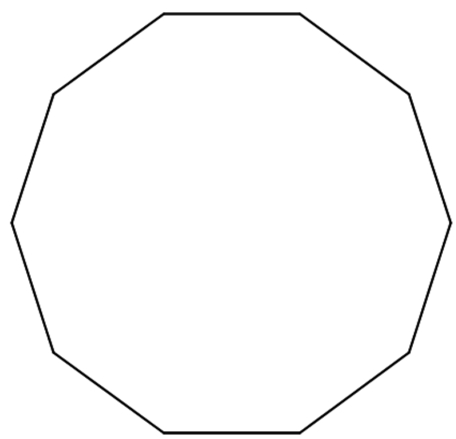
c. What is the area of the region swept by $\overline{BC}$ as it is rotated? Hint: what would happen if you did it three more times?

19. A regular decagon has a perimeter of 60.

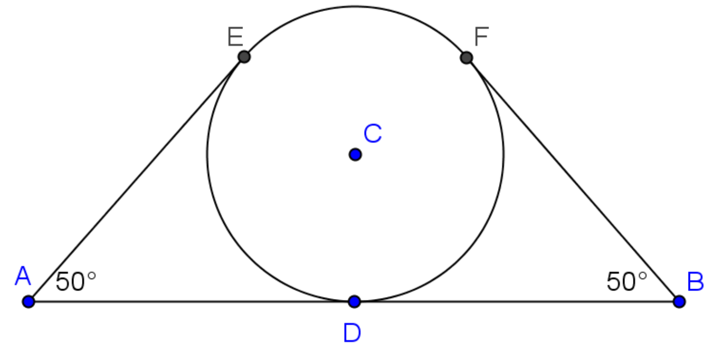
a. Find the area of the inscribed circle. Decimal OK.

b. Find the area of the circumscribed circle.

c. Find the area of the decagon; if it is not between your answers to *a* and *b*, then double-check your work!



20. In the diagram below, $\overline{AB}$, $\overline{AE}$, *and* $\overline{BF}$ are tangent to circle A. $\overline{AB}$ measures 24 units, with D being the midpoint. 



a. Find the radius of the circle.

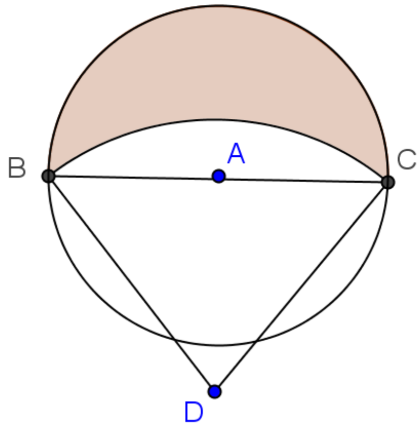
b. Find the measure of angle DCE.

c. Find the perimeter of the shape (like two ice-cream cones sharing one scoop of ice cream!) A decimal approximation is fine.

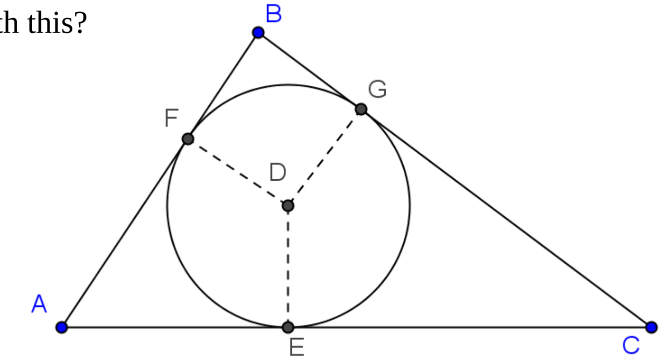
d. Find the area of the entire shape.

21. One circle's radius is twice as long as another circle's radius. The area of the larger circle is 200 square cm more than the area of the small circle. What is the radius of the small circle?

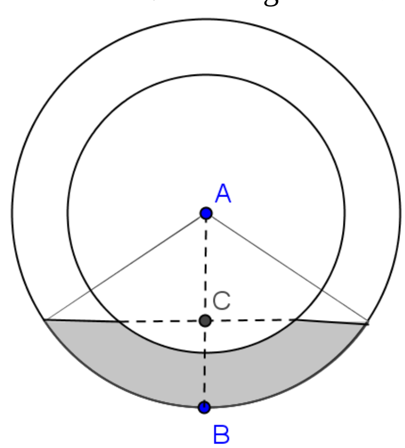
22. $\overline{BC}$, a diameter of a circle A, is 10 units long. D is the center of a circle with radius 8 on which points B and C lie. Minor arc BC of that circle is drawn. What is the area of the shaded region?



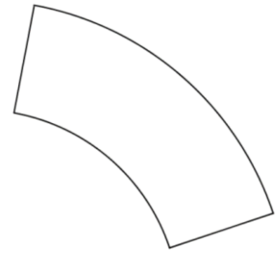
23. Angela says that the radius of the circle inscribed in any trangle is just twice the ratio of the triangle's area to its perimeter. Is she right? How did she come up with this?



24. In the diagram below, concentric circles with radii 5 and 7 have centers A. The length of $\overline{BC}$ is 4 units. Find the area of the shaded region.

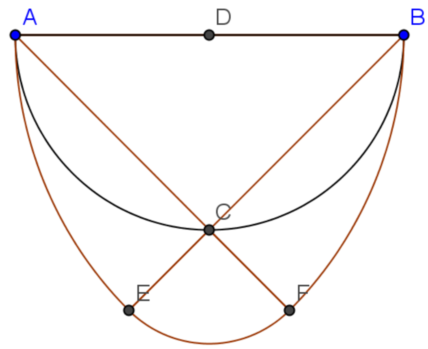


25. The shape below is formed by two circular arcs with the same center and 2 line segments; the arcs are 12 and 8 units long and the segments are 3 units long. What is the area of the shape?



26. Circles with radii of 10 and 8 have centers 15 units apart. What is the area of the region common to both circles? 

27. $\overline{AB}$ is a diameter of circle D, with radius one. ACB is an isosceles triangle with point C on circle D. ABE is a sector of a circle with center B; FAB is a sector of a circle with center A, and ECF is a sector of a circle with center C. What is the area of the "smile": the part of the figure outside the semicircle centered at D? (from AHMSE exam)



Answers

1a. 25π b. $64+8\pi$ c. 9π d. $96+16\pi$ 2a. 4π b. $12+4\pi/3$ 3a. 34.4° b. 30 4. $3\sqrt{2}$

5. perimeter is 16π; area is $96\sqrt{3}-32\pi$ 6a. $9\pi-18$ b. $\frac{64\pi}{3}-16\sqrt{3}$

7. area is approx. 16.35; perim is about 24.9 8. Segment/circle = 139.2 /(324π) or about 13.7%

9. perim = 12π and area $=72\pi-72\sqrt{3}$ 10a. (3,6) b. 25π c. (0,2) and (0,10) d. about 11.18

11. π.. cool! What's really cool for ***any*** regular polygon with side 2, the area between the inscribed and circumscribed circles is always π → as the polygon gets more sides the shape gets larger but thinner….

12. $64\pi-32\sqrt{3}$ 13. 49π (use Pythag to get radius of big guy) 14. $120\pi+36\sqrt{3}$

15. 2/3: The circle's radius is $1/\sqrt{3}$ (connect the circle's center to a vertex and the midpoint of a side to get a 30/60/90 triangle). Connect the circle's center to a 2 adjacent vertices of the square to get a 45/45/90 triangle—the squares side is thus $\sqrt{2}/\sqrt{3}$ *or* $2/\sqrt{6}$.

16. $\pi(x+2)^2-\pi x^2=42\pi$ so x=9.5 17. area is $50\sqrt{3}-\frac{100\pi}{6}$; perimeter is $10+10\sqrt{3}+10\pi/3$

18a. B'(6,-1) and C'(8,-2) b. $2\sqrt{5}+2\pi\sqrt{2}+\frac{\pi\sqrt{61}}{2}$ c. ¼ of a "rainbow" so $29\pi/4$

19a. 267.8 b. 296.1 c. 277.0 20a. 5.6 b. 130° c. 57.8 d. 161.8

21. Area small = 200/3 so radius≈4.61 22. Semicircle – sector + trianlge = 12.5π – 43.21 + $5\sqrt{39}\approx 27.29$

23. She is! Draw segments AD, CD, and BD. The area of the triangle is the sum of the areas of triangles ACD, BCD, and BAD. The altitudes of all of these is the radius of the inscribed circle.. so ½ (P)(r)=A And r=2A/P 24. 25.1 which is (55.24-18.97) – (23.17-12) 25. 30

26. 20.14; segments are 8.72 + 11.42; common chord is 8.7 units from one center and 6.3 from the other

27. $\left(2-\sqrt{2}\right)\pi-1$

Unit 8 Handout #7: Area Practice Problems

Key unit ideas:

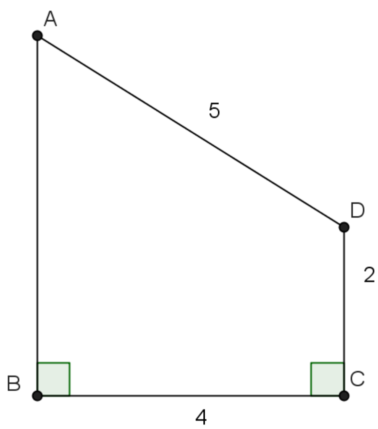
1. Rectangles and triangles
2. Parallelograms
3. Trapezoids
4. Regular polygons: apothems
5. Scale: similar and "stretched" – ie, same height and different bases
6. Circles, sectors, and segments

Trig table or calculator required for problems with  .

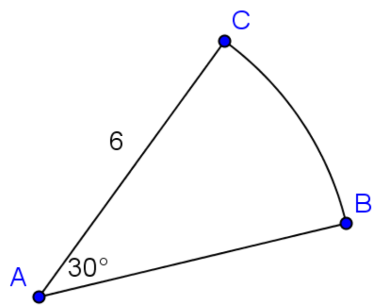
There are too many problems to do all of them; pick a sampling of them!

1. Find the area of each shape below:

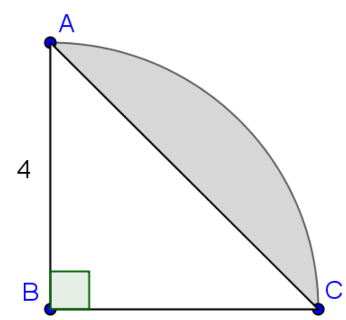
a.



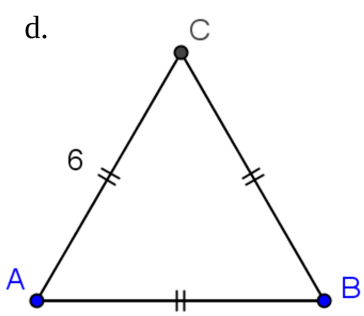
b.



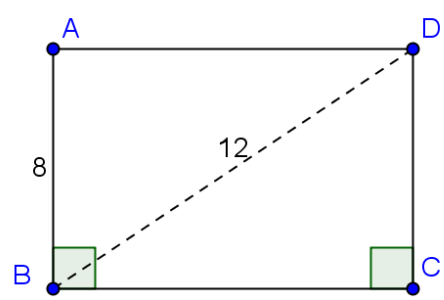
c. find the shaded area



d.

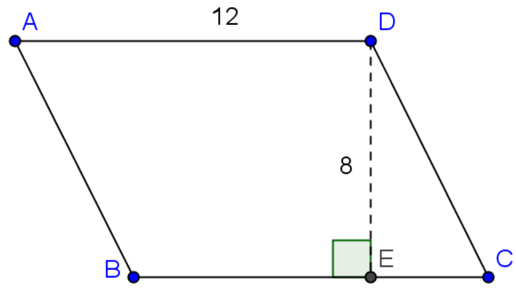


e.

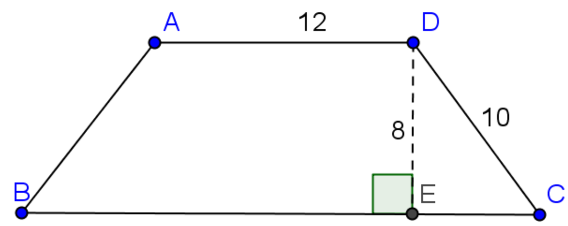


2. Find the area of each shape below; part *a* is a parallelogram and part *b* is an isosceles trapezoid.

a.



b.

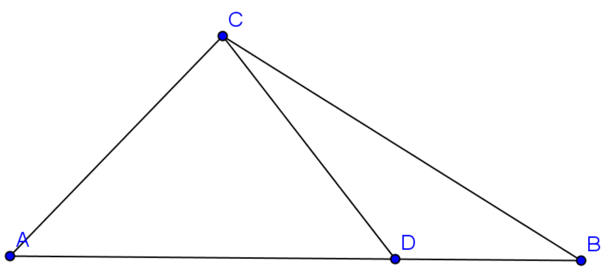


3. A regular twelve-a-gon ("dodecagon") has a perimeter of 72. 

a. What is its apothem?

b. What is its area?

4. In the diagram below, AD=8 and DB=5 and the area of ΔACD=20. What is area of ΔACB?

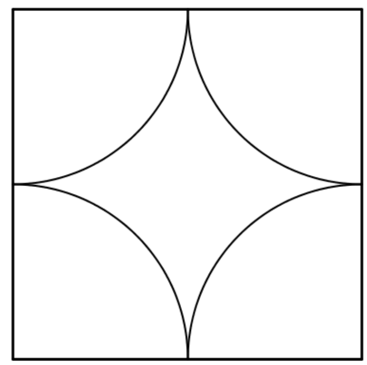


5. A parallelogram has adjacent sides of 10 and 15 meeting at a 45° angle. What is its area?

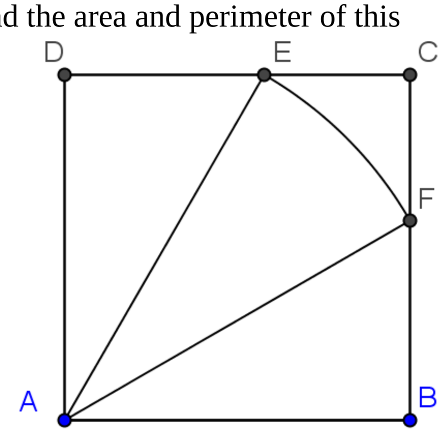
6. A slice of pizza with a central angle of 20° has an area of 30 square inches. What is the radius of the pizza? Decimal answer OK.

7. A square has a diagonal of 10. What is its area?

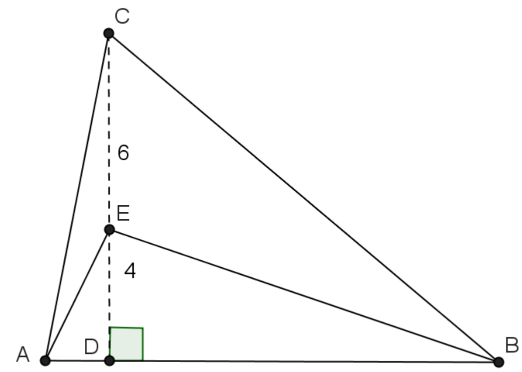
8. The square at the right has a side length of 8. Congruent ircular arcs are drawn from each corner. What is the perimeter and area of the shape in the center?



9. The diagram below shows a square with side length of 8 where angle BAD has been trisected by segments $\overline{AF}$ *and* $\overline{AE}$. AEF is a sector of a circle with center A. Find the area and perimeter of this sector.



10. In the diagram below, $\overline{AD}$ is one fourth as long as $\overline{BD}$.



a. What is the ratio of the area of ΔDAE to the area of ΔDBE?

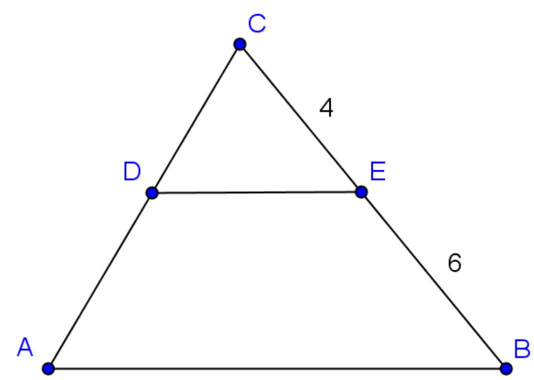
b. What is the ratio of the area of ΔDAE to the area of ΔDAC?

c. What is the ratio of the area of ΔDAE to the area of ΔABC?

d. What is the ratio of the area of ΔDAE to the area of ΔEBC?

11. Triangle ABC has an area of 50. $\overline{AB} // \overline{DE}$.

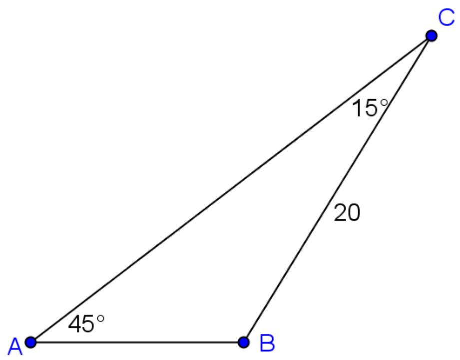
a. What is the area of trapezoid ABED?



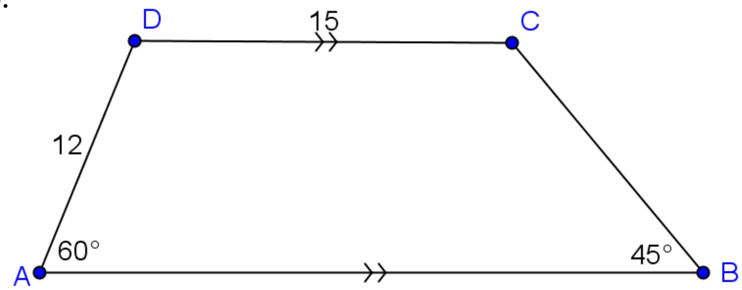
b. Draw a line segment $\overline{FG}$ parallel to $\overline{AB}$, where F is on side $\overline{AC}$ and G is on side $\overline{BC}$. If triangle CFG has an area of 25, then what is the length of $\overline{GC}$?

12. Find the area of each shape below:

a.



b.

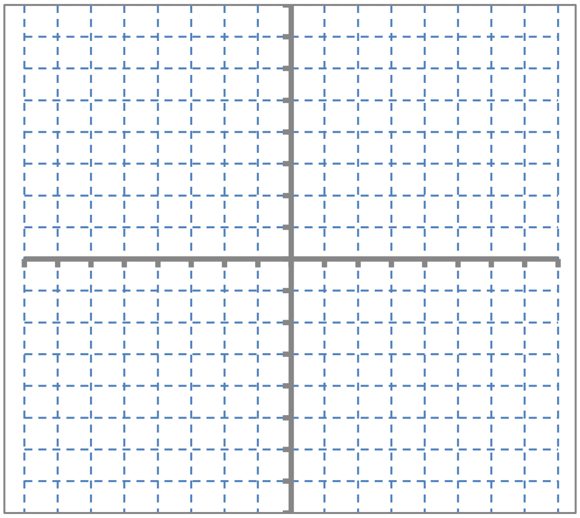


13. You have 40 inches of wire. You will either bend it to make the perimeter of a square or a circle. Which one has larger area? Support your answer with calculations.

14. In a circle with radius 12, a chord of length 15 divides the circle into two regions. Find the area and perimeter of the smaller region. Express your answer as a decimal. 

15. The ratio of a rectangle's length to width is 8:5. If its area is 200, then find the length of its diagonal.

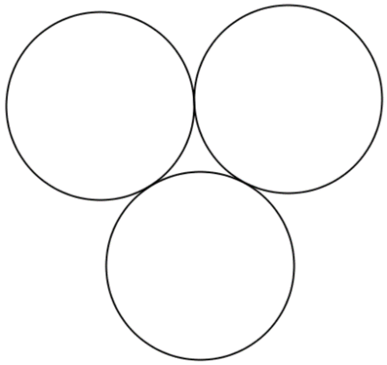
16. A triangle is formed by the lines $x+2y=12,\ y=2x+6,\ and\ y=0.5x.$



a. What is its area?

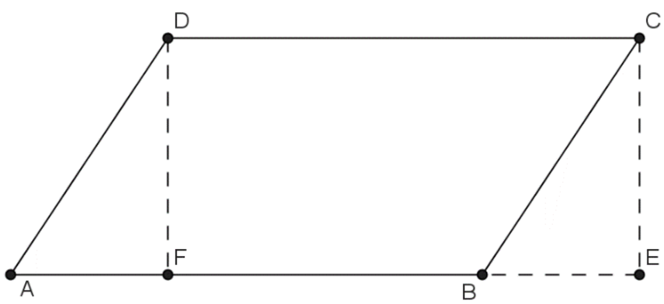
b. Write the equation of any line that splits the triangle into two triangles of equal area.

17. Three circles with radii 6 are tangent to each other. Find the area of the region enclosed between them.

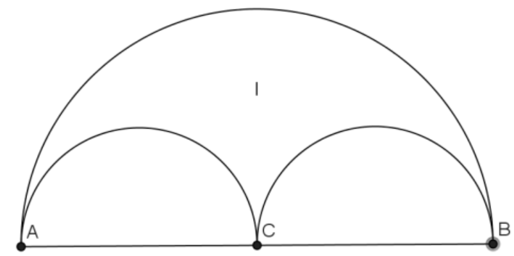


18. A rectangle has a perimeter of 38 and an area of 84. What are its dimensions?

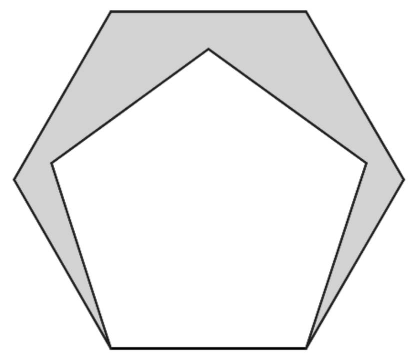
19. Explain why the area of parallelogram ABCD is equal to the area of rectangle FECD.



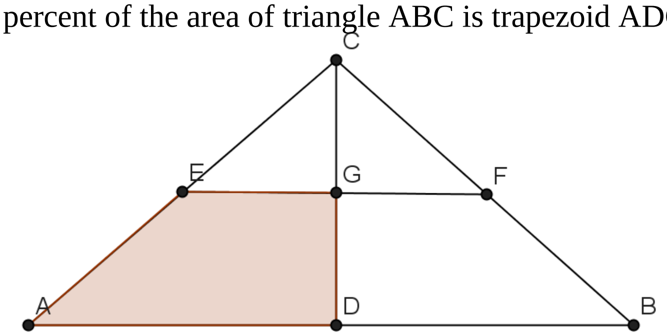
20. Find the area and perimeter of region I in the diagram below, given that C is the center of the larger semi-circle and $\overline{AC}$ *and* $\overline{CB}$ are congruent diameters of the smaller semi-circles. The larger semi-circle has a radius of 6.



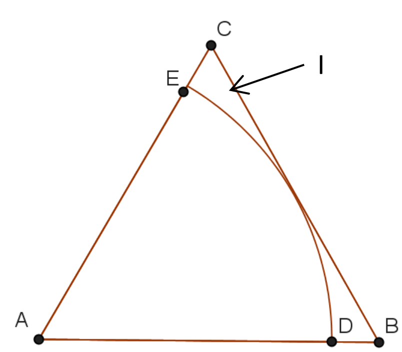
21. The regular hexagon and regular pentagon share a side of length 10. What is the area of the shaded region? Use a decimal approximation.



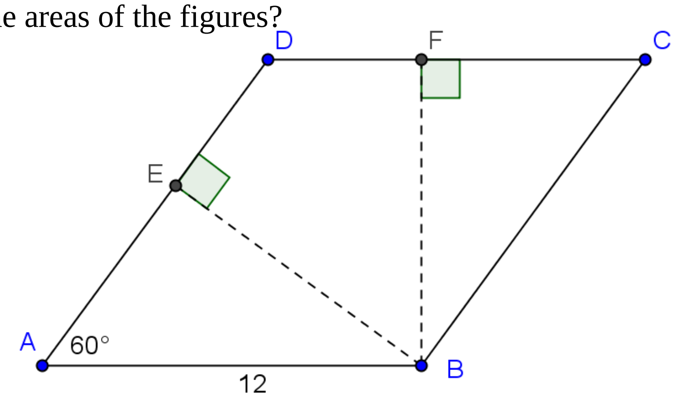
22. Points E and F are the midpoints of the sides of isosceles triangle ABC on the left below. What percent of the area of triangle ABC is trapezoid ADGE?



23. In the diagram on the right, an equilateral triangle of side 6 has a circular sector inscribed in it. What is the area and perimeter of region I?

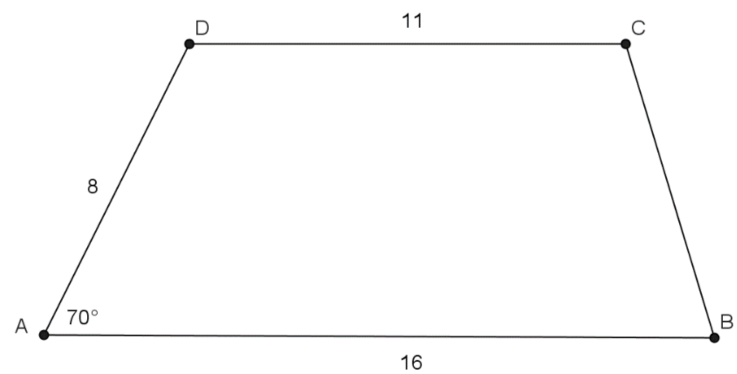


24. The rhombus below has side lengths of 12. Altitudes are drawn from one vertex to opposite sides, breaking the rhombus into three figures. What are the areas of the figures?

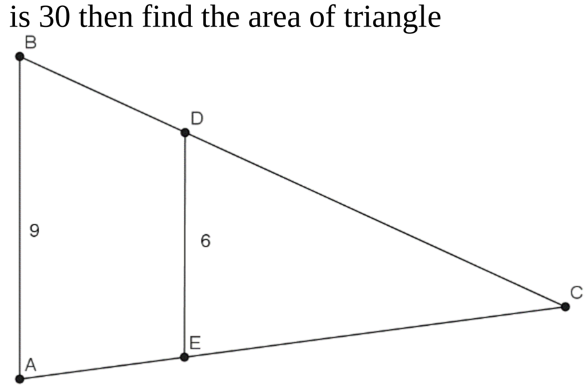


25. How many equilateral triangles of side 1 can fit inside an equilateral triangle of side 4?

26. Find the area of trapezoid ABCD below. 



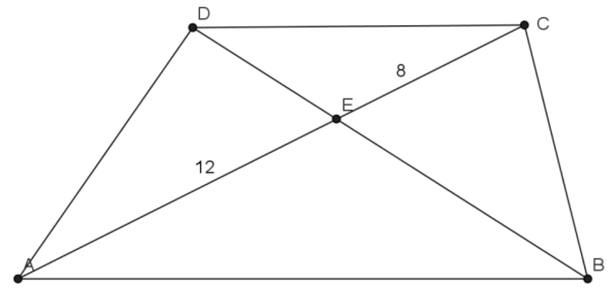
27a. $\overline{AB} // \overline{DE}$ in the diagram below. If the area of trapezoid AEDB is 30 then find the area of triangle CED.



b. Instead of being 30, now assume that the area of trapezoid AEDB exceeds the area of triangle CDE by 10 units. What is the trapezoid's area?

28. A rectangle's length exceeds its width by three units. If its length was doubled and its width was decreased by four units, then its area would fall by 10 square units. What are its dimensions?

29. Find the following using trapezoid ABCD below:



a. The ratio of the area of ΔCDE to that of ΔABE.

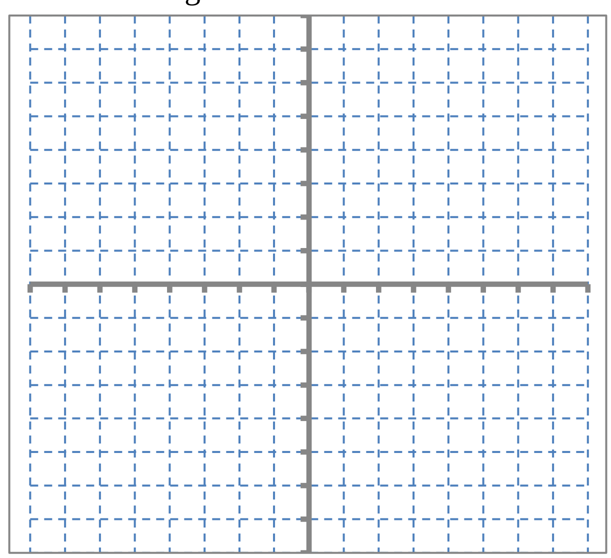
b. The ratio of the area of ΔCDB to that of ΔABC.

c. The ratio of the area of ΔABC to that trapezoid ABCD.

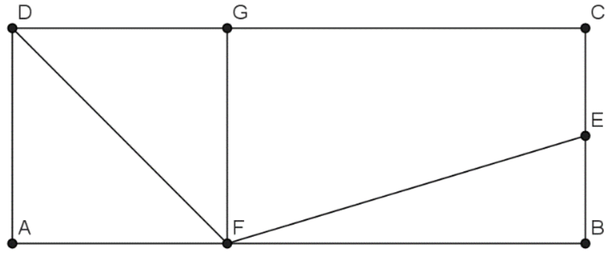
30. Given the points A(-3,4), B(1,4), C(6,-2) and D(2,-2), do the following:

a. What shape is ABCD and what is its area?

b. Triangle ABE has an area of 12 and E is somewhere on the line $y = 2x - 5$. Where could it be?



31. In rectangle ABCD below, E is the midpoint of side $\overline{BC}$. If the rectangle's area is 100 and the area of triangle FEB is 15, then find the following:

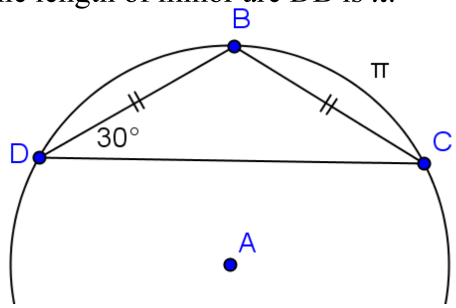


a. The area of BCGF.

b. The ratio of $\overline{FB}$ to $\overline{AF}$.

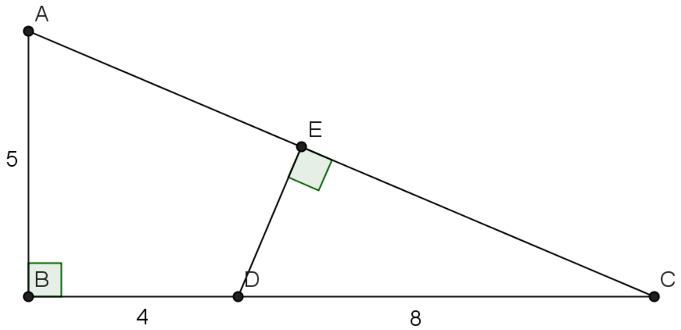
c. The area of triangle FAD.

32. What is the area of ΔBCD below? A is the center of the circle and the length of minor arc DB is π.

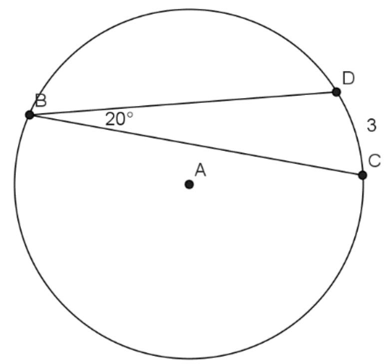


33. An equilateral triangle and a hexagon have same perimeter. What is the ratio of the area of the hexagon to the area of the triangle? Hint: pick any side length for either one!

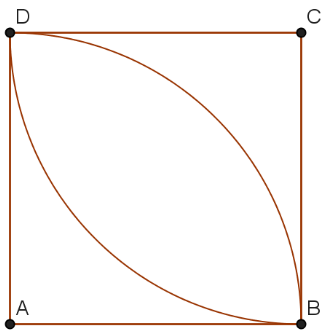
34. Find the area of triangle CED below. Hint: it is similar to something that you know the area of!



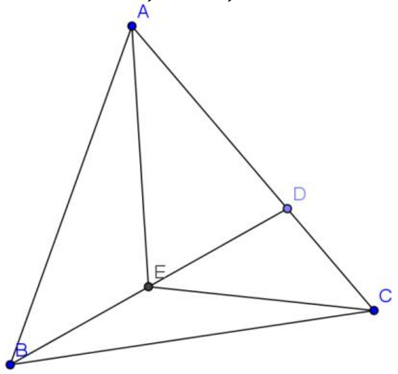
35. In circle A on the right above, inscribed angle B intercepts an arc whose length is 3 cm. What is the area of circle A?



36. The square below has a side length of 4. Circular arcs are drawn from opposite corners. What is the area inside both of the quarter circles?

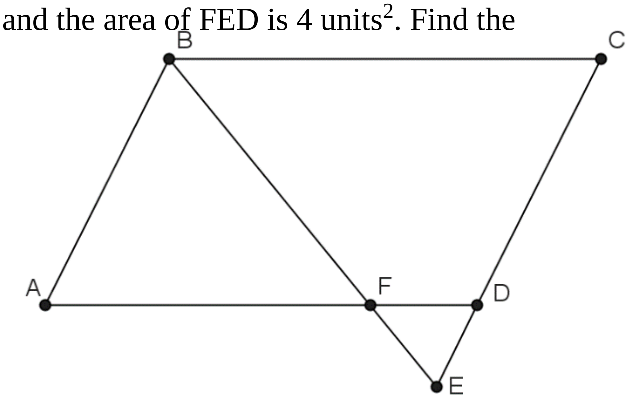


37. Given scalene triangle ABC where CD=2 and DA=3 and E is the midpoint of $\overline{BD}$, find the ratio of areas of CBE, CDE, and ABE.

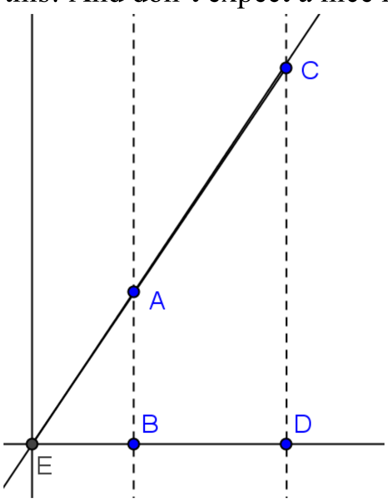


38. In the diagram on the right, side $\overline{CD}$ of parallelogram ABCD has been extended to point E, which is connected to point B. The area of triangle ABF is 64 units^2 and the area of FED is 4 units^2. Find the following: [from Art Of Problem Solving]

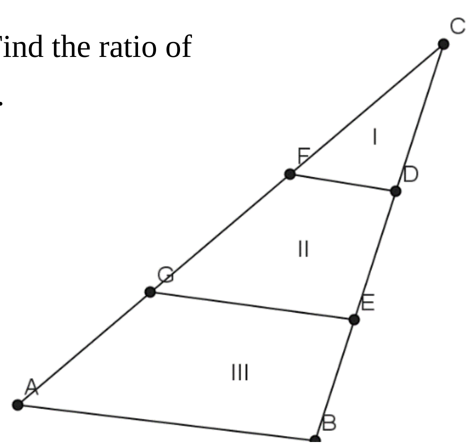
a. The ratio of the length of $\overline{FD}$ to $\overline{AF}$.
b. The area of triangle CFD.
c. The area of parallelogram ABCD.



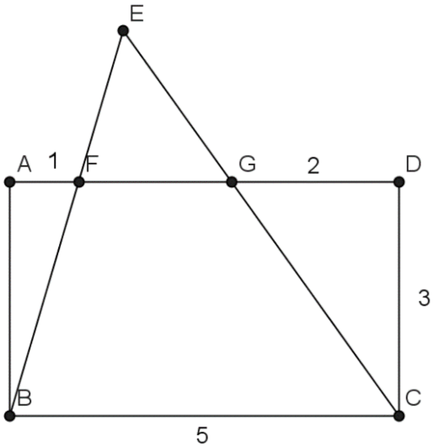
39. In the diagram below, point E is the origin and points E, A, and C are collinear. Lines x=2 and x=5 are drawn. If the area of ABDC is 20, then find the y-coordinate of point C. There are a few ways to approach this! And don't expect a nice round number!



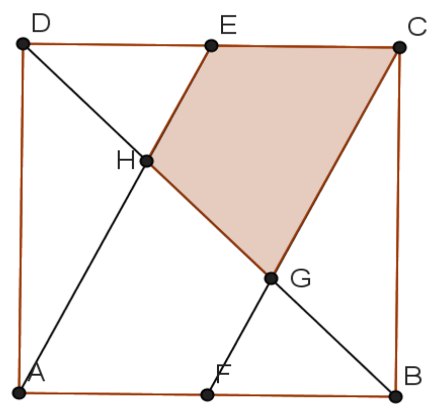
40. Sides $\overline{AC}$ *and* $\overline{BC}$ are trisected by the points G, F, D, and E. Find the ratio of the area of region I to the area of region II to the area of region III.



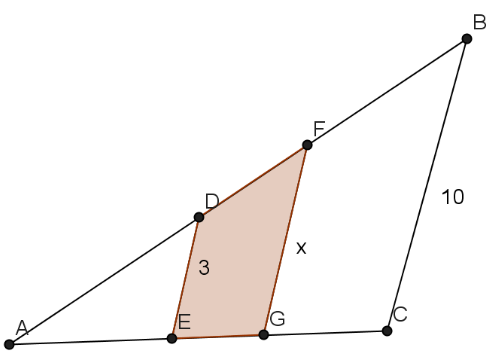
41. In rectangle ABCD segments have lengths indicated. What is the area of: (from AOPS)
a. triangle EFG b. triangle BEC



42. ABCD on the right is a square of side two. E and F are the midpoints of sides. Find area and perimeter of quadrilateral CEHG.



43. $\overline{DE} \,//\, \overline{FG} \,//\, \overline{BC}$. The shaded region represents one fifth of the area of triangle ABC. Find *x*.



44. In the right triangle ABC below, D is the midpoint of $\overline{BC}$ and $\overline{AB}$ is one half the length of $\overline{BC}$. Answer the following: (no calculators or trig tables, but using trigonometry to organize the information about similar triangles may be helpful.)

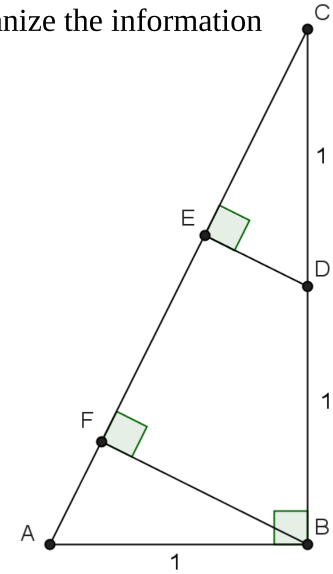
a. Name three triangles similar to ABC.

b. Find the length of $\overline{EC}$.

c. Find the area of triangle CED.

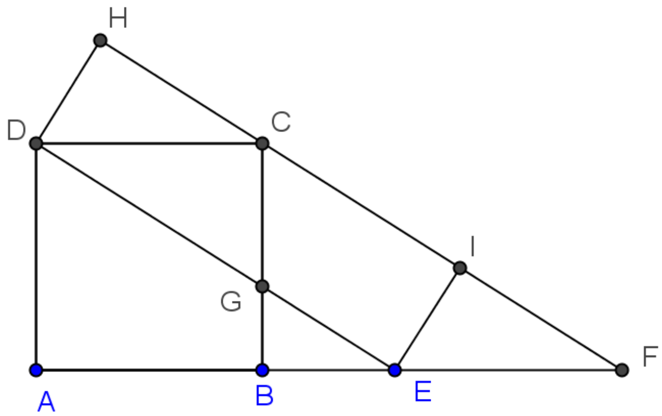
d. Find the area of triangle ABF.

e. Find the area of BDEF.

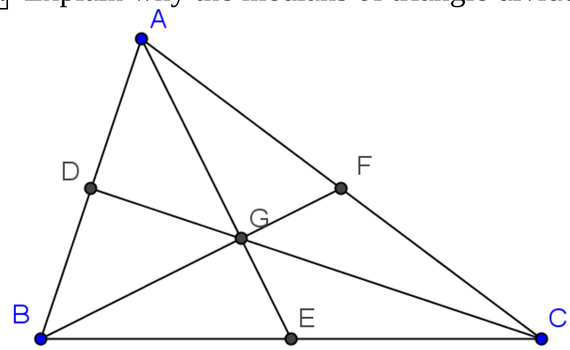


45. A rope is cut into two pieces. One piece forms the circumference of a circle and the other the perimeter of a square. For the square and the circle to have equal areas, what percent of the length of the rope needs to be used to form the square?

46. In the diagram below, ABCD is a square and DEIH is a rectangle. Points A, B, E, and F are collinear, as are points H, C, I, and F. Point G is not necessarily the midpoint of CB. Prove that the area of ABCD is equal to the area of DEIH. (from Weeks & Adkins, *A Course in Geometry*)



47. Explain why the medians of triangle divide it into six smaller triangles of equal area.



Answers

1a. 14 b. 3π c. 4π-8 d. $9\sqrt{3}$ e. $32\sqrt{5}$ 2a. 96 b. 144 3a. 11.2 b. 403.1 4. 32.5

5. $\frac{150}{\sqrt{2}} = 75\sqrt{2}$ 6. $\pi r^2 = 540$ *so* $r \approx 13.1$ inches 7. 50 8. Perim=8π area=64-16π

9. area= 4π perim= $8\sqrt{3} + 2\pi\sqrt{3}/3$ 10a. 1:4 b. 2:5 c. 2:25 d. 1:6 11a. 42 b. $10/\sqrt{2} = 5\sqrt{2}$

12a. $\frac{1}{2}\left(10\sqrt{3} - 10\right)\left(10\sqrt{3}\right) = 150 - 50\sqrt{3}$ b. $\frac{1}{2}\left(36 + 6\sqrt{3}\right)(6\sqrt{3}) = 54 + 108\sqrt{3}$

13. Square=100 circle=127.3 14. Area: 27.0 perim: 31.2 15. $\sqrt{445}$

16a. 30 b. any median: one is $y = \frac{1}{8}x + \frac{9}{4}$ 17. $36\sqrt{3} - 18\pi$ 18. 7-by-12

19. triangle ADF is congruent to BCE by HL 20. Area=9π perim=12π 21. 87.8 22. 3/8

23. area is $\left(9\sqrt{3} - 27\pi/6\right)/2 \approx 0.726$ perim = $\pi\sqrt{3}/2 + 3 + \left(6 - 3\sqrt{3}\right) \approx 6.52$

24. area of ΔABE and FBC are both $18\sqrt{3}$; BEDF is $36\sqrt{3}$ 25. 16 26. 101.52 27a. 24 b. 50

28. 7-by-10 29a. 4:9 b. 2:3 c. 3:5 30a. parallelogram: 24 b. y is 10 or -2 so (7.5,10) or (1.5,-2)

31a. 60 b. 6:4 c. 20 32. Radius is 3 so area is $9\sqrt{3}/4$ 33. 3:2

34. (8/13)^2* (30)= 1920/169 35. Circumf=27 so area=58.0

36. draw a diagonal from B to D; the region is the sum of 2 segments: each one is 4π-8 so 8π-16

37. 2:2:3 38a. 1:4 b. 16 c. 160

39. 200/21 : let the length of AB be *w*; then the length of CD is 2.5*w* (using similar triangles). Using the area of trapezoid, *w*=80/21 so the y-coordinate of C is 2.5*w*, or 200/21.

40. 1:3:5 41a. 2 b. 12.5

42. Area=1 perim= $1 + \sqrt{5} + 2\sqrt{2}/3$ 43. if A is whole area then $\left(\frac{x}{10}\right)^2 A - \left(\frac{3}{10}\right)^2 A = \frac{A}{5}$ so $x = \sqrt{29}$

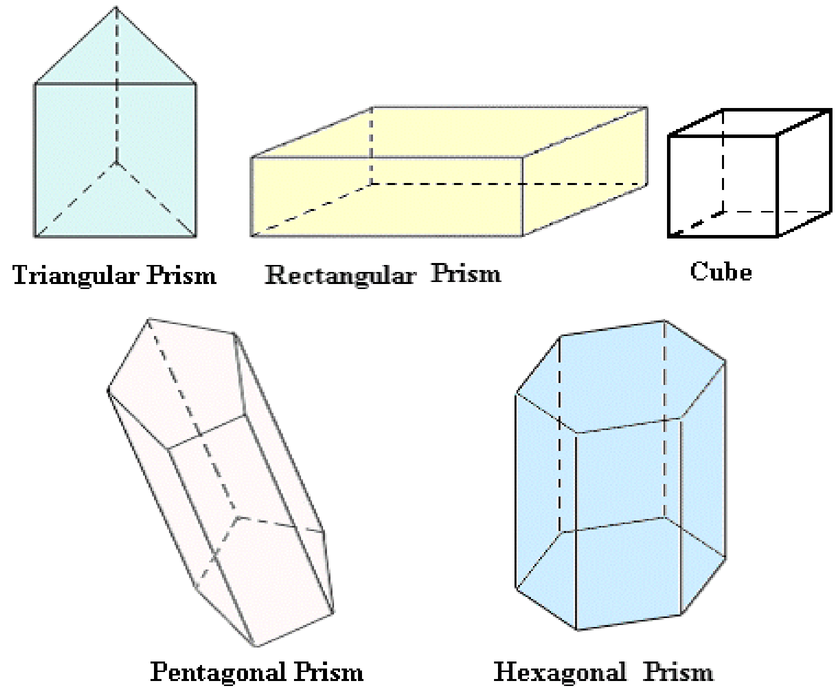
44a. DEC, AFB, BFC b. $2/\sqrt{5}$ c. 1/5 d. 1/5 e. 3/5 45. $\frac{2\sqrt{\pi}}{2\sqrt{\pi} + 4} \approx 47\%$

46. the area of ABCD is equal to the area of CDEF since both have the same base and the same height. And triangles HDC and IEF are congruent by ASA so they have the same area.

47. hmmm… areas of ABE and AEC are equal… as are GBE and GEC thus ABG and ACG have equal areas too. Since ADG and DBG are each ½ of ABG, and AGF and CGF are each ½ of AGC, those four small triangles must all have the same area. Now do a similar thing with another median…

Unit 9 Handout #1: Surface Area of Prisms

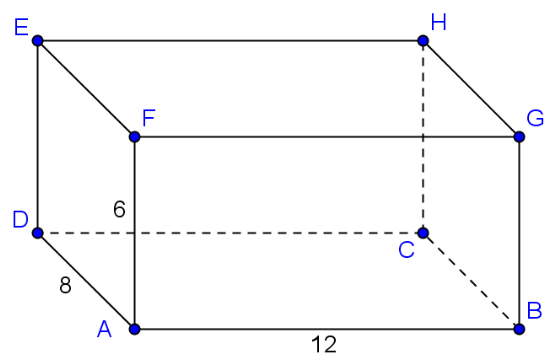
A ***prism*** is a three-dimensional solid whose ends are congruent and in parallel planes and whose sides are all parallelograms. Prisms tend to be names for the shapes at the ends (the top and bottom of those in the diagram on the right).



A prism is considered a ***"right prism"*** if the ends are directly above each other, making the sides all rectangles. Otherwise it may be called ***"oblique"***.

The ***total surface area*** of a prism includes the ends and sides; the ***lateral surface area*** is the area of the sides only.

Example #1: Given the right rectangular prism below, find the surface area, the length of diagonal $\overline{AH}$ and the measure of angle AHC.



The area of the top and bottom are both 8*12=96; the front and back are each 12*6=72. And the left and right sides are each 6*8=48. So the total surface area is 216.

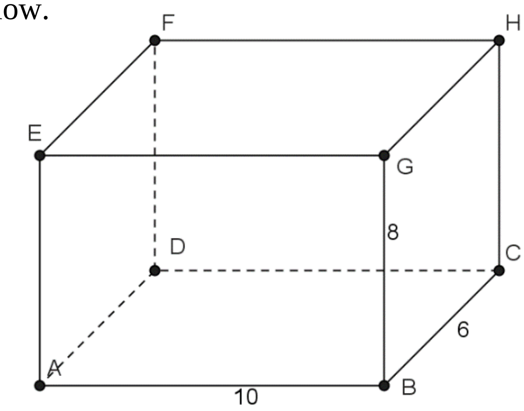
To find the length of $\overline{AH}$, look at triangle ACH. C must be a right angle since this is a right prism. Thus we can use the Pythagorean Theorem and $AH^2 = CH^2 + AC^2$. We can also use the Pythagorean Theorem to find AC: $AC^2 = AB^2 + BC^2$. Thus $AC^2 = 208$ and $AH^2 = CH^2 + AC^2 = 6^2 + 208 = 244$. So AH= $\sqrt{244} = 2\sqrt{61}$.

To find the measure of angle AHC, continue to use triangle AHC.

$$\tan AHC = \frac{AC}{CH} = \frac{\sqrt{208}}{6} \quad \text{so} \quad \angle AHC = \tan^{-1}\left(\frac{\sqrt{208}}{6}\right)$$

A calculator reveals that angle AHC measures approximately 67.4°. Note: you could also use sine or cosine to find this angle.

1. Answer the following questions about the rectangular prism below.

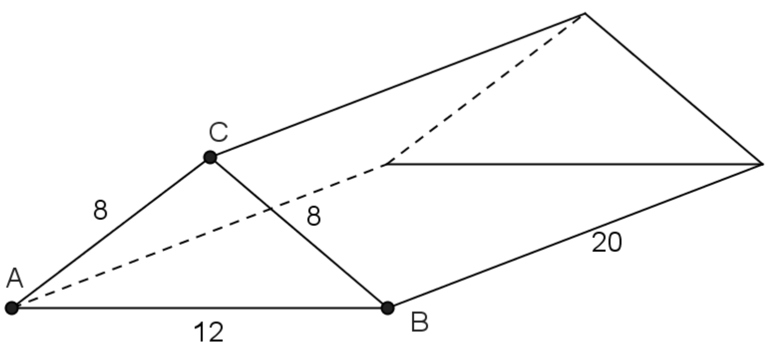


a. What is the total surface area?

b. What is the length of diagonal $\overline{DG}$?

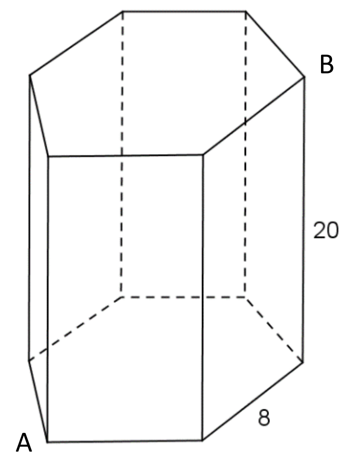
c. What is the measure of angle BDG? 

2. The right triangular prism below has a base that is an isosceles triangle. Find its total surface area.



3. The right prism below has bases that are regular hexagons.

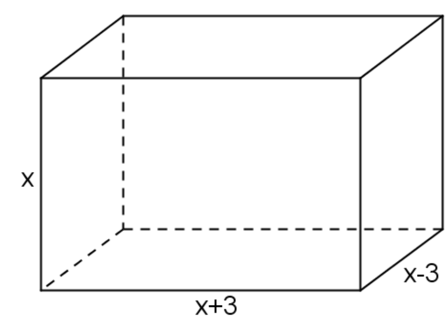
a. Find its total surface area.



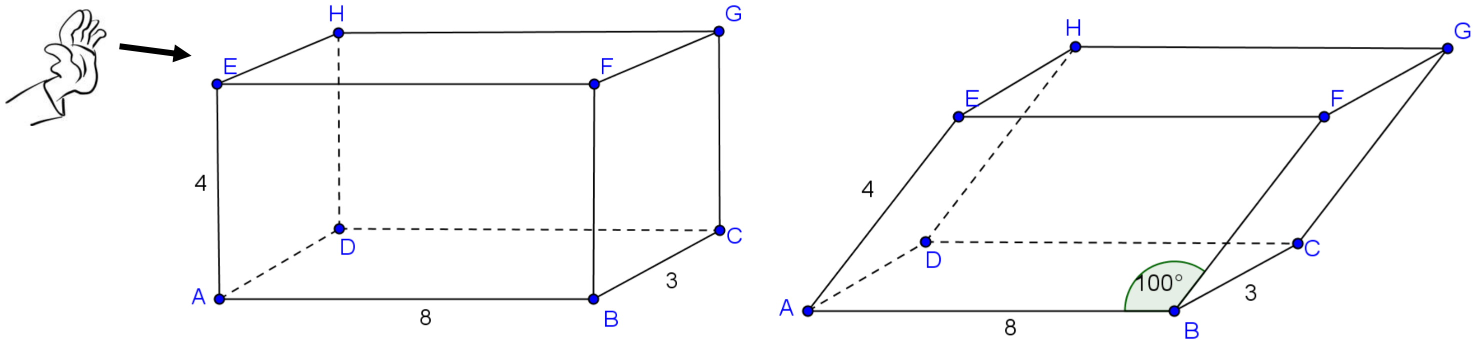
b. Find the length of $\overline{AB}$.

4. A right rectangular prism has a square base. Its height exceeds the side of its base by 6 units. Its surface area is 360. What is the area of its base?

5. Find the value of *x* if the right rectangular prism below has a surface area of 372.



6. Answer the following questions about the prism on the right below. Think of it as a right rectangular prism (on the left) whose top has been pushed, turning the front and back into parallelograms while the other four sides remain rectangles.

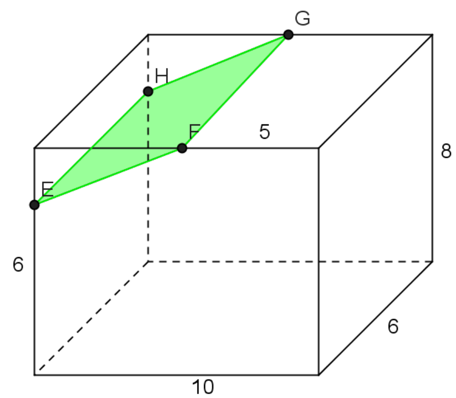


a. Angela says the shape at the right is an oblique rectangular prism. Brett calls is a right parallelogram prism. Who is correct?

b. Find its total surface area. 

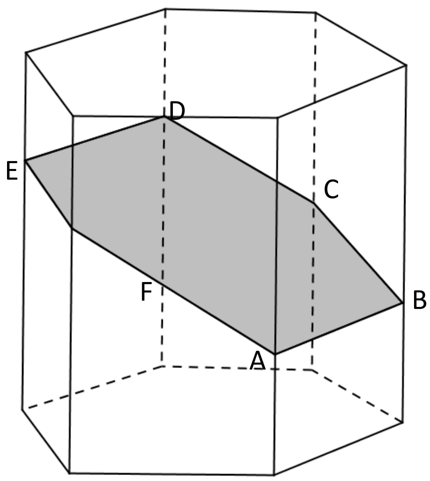
7. Continuation of #6. Instead of a 100° angle, what obtuse angle must it be tilted at in order to have a total surface area of 120? 

8. The right rectangular prism below is sliced into two pieces by rectangle EFGH. Two of the rectangle's corners are the midpoints of the top and the other two are one-fourth of the way down the left side. What is the surface area of the larger piece? Note: one could call it a right pentagonal prism, looking at the front and back as its ends.



9. The diagonal of a cube (from corner to corner through the center of the cube) is 10. What is the cube's surface area?

10. A right hexagonal prism has a base that is a regular hexagon with side 12 and a height 40. It is cut by a plane at 45° angle (this means that the plane of the shaded hexagon meets the planes containing the bases at a 45° angle). Find the lengths of the sides of the shaded hexagon, then find its area.



Answers

1a. 376 b. $10\sqrt{2}$ c. 34.4° 2. $560+24\sqrt{7}$ 3a. $960+192\sqrt{3}$ b. $4\sqrt{37}$ 4. 36

5. $\sqrt{65}$ 6a. both are: depends what you consider the base! b. 135.03

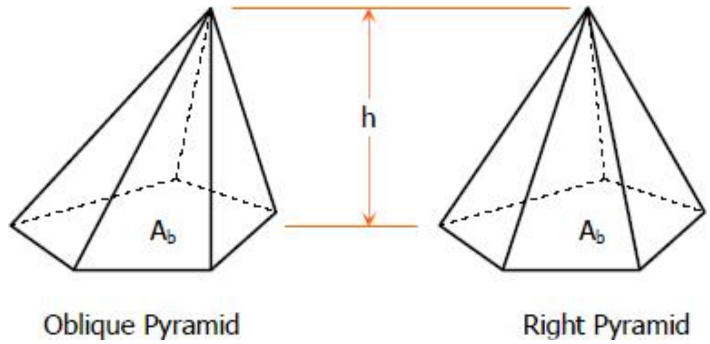
7. 131.4° 8. $324+6\sqrt{29}$ 9. 200

10. long way: AB=12 segment FC is $6\sqrt{3}$ higher than segment AB (halfway across the hexagon) so AF is $6\sqrt{7}$, as are EF, CD, & BC. AE and BD are $12\sqrt{6}$ b/c FC is 24. So the area of ABCDEF is $216\sqrt{6}$

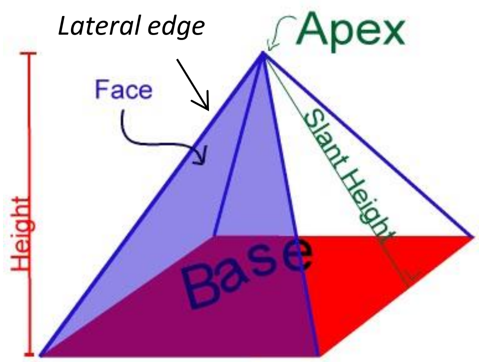
Short way: we are basically stretching a regular hexagon perp to sides AB & DE by a factor of $\sqrt{2}$ (since it was a 1-1-$\sqrt{2}$ right triangle).. so the area should be $\sqrt{2}$ times as large and $216\sqrt{3}\sqrt{2}=216\sqrt{6}$

Unit 9 Handout #2: Surface Area of Pyramids

A ***pyramid*** is a three-dimension shape with a polygon as a base and a single point (called the ***apex***) where its sides converge at the top. If the apex is directly above the center of the base polygon then it is called a ***right pyramid***, otherwise it is called an ***oblique pyramid***.

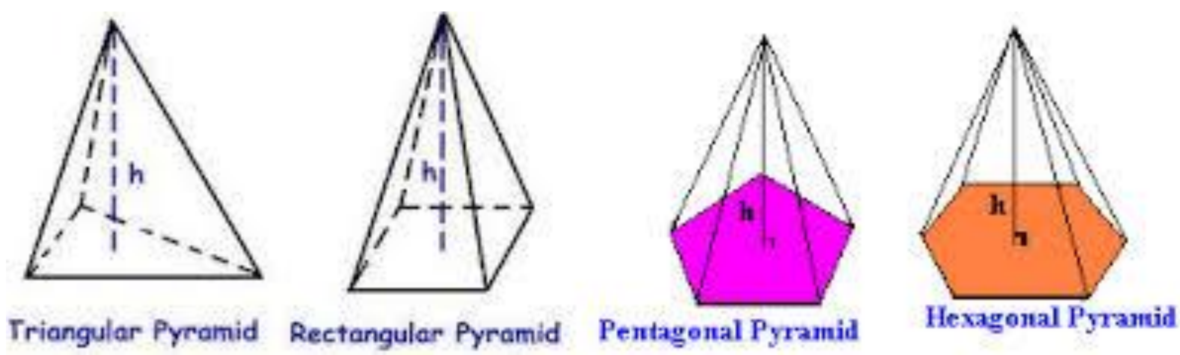


The ***height*** of a pyramid is the distance from the apex to the base, measured along a segment perpendicular to the base.



The pyramid's sides are called ***faces*** and are all triangular. The ***lateral edge*** joins a vertex of the base polygon to the apex. And the ***slant height*** is the distance from the apex to the midpoint of a side of the base polygon (it is most commonly used when the base is a regular polygon).

Like prisms, pyramids are named for the shape of the base.

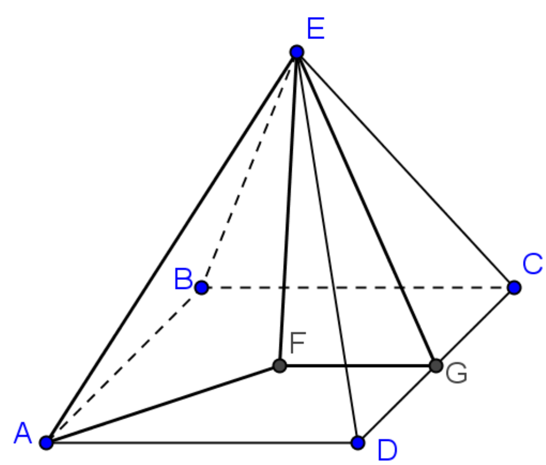


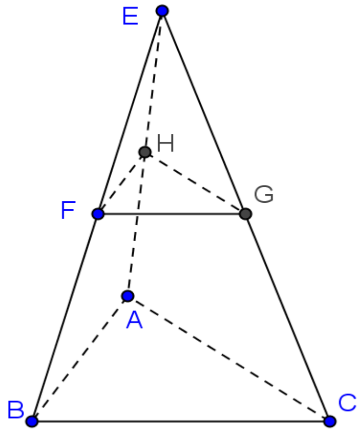
When working with a right pyramid with a regular polygon base, it can be useful to draw a ***three right triangles***. They relate the base edge to the slant height to the height to the lateral edge.

-One has vertices of the apex, the center of the base, and a vertex of the base (ΔEFA on the right).

-Another has vertices of the apex, the center of the base, and the midpoint of one of the base's edge (ΔEFG). This latter one's hypotenuse is the slant height and $\angle EGF$ is the angle at which the face meets the base.

-A third triangle has vertices E, G, and D. It is a right triangle because a right pyramid with a regular base has faces that are isosceles triangles, so the median $\overline{EG}$ is also an altitude.





When a pyramid is cut by a plane parallel to the pyramid's base, it is divided into two shapes. The top part is a pyramid while the lower part is a ***frustum***.

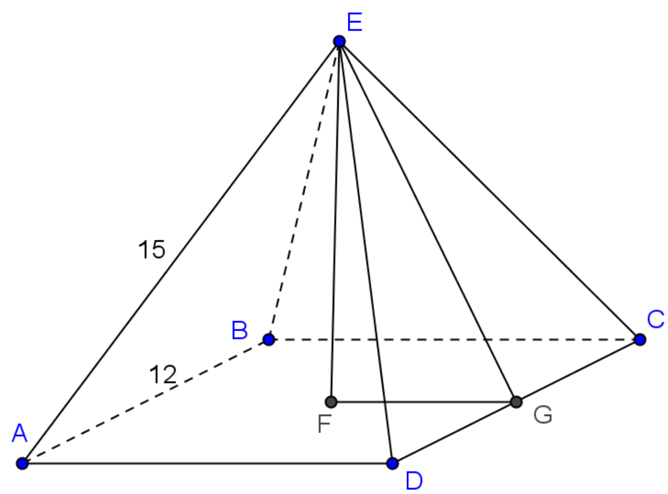
The frustum on the left has vertices A, B, C, F, G, and H. Each face is a trapezoid and the top and bottom are similar triangles.

Example #1: The right pyramid below has a square base with side length of 12. Each lateral edge measures 15. Find the following:

a. The slant height. **b. The total surface area**

c. The height. **d. The angle each face makes with the base.**

a. The slant height: look at triangle EDG. Angle G is a right angle since ΔEDC is isosceles with EC=ED. Thus median EG is also an altitude. In ΔEDG, GD=6 and hypotenuse DE=15. Therefore slant height EG can be found using the Pythagorean Theorem: $ED^2 = DG^2 + EG^2$. Plugging the other two sides in, $EG = \sqrt{189} = 3\sqrt{21}$.



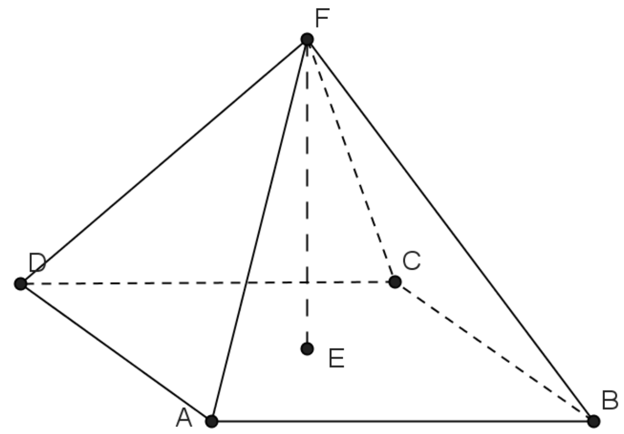
b. The area of each face is one half of the base times height, or $\frac{1}{2} \cdot 12 \cdot 3\sqrt{21} = 18\sqrt{21}$. So the total of the four faces is $72\sqrt{21}$; adding the area of the square base (144) yields a total surface area of $144 + 72\sqrt{21}$.

c. The height EF can be found using ΔEFG, where F is a right angle since we have a right pyramid. FG is half the length of the square base's side, or 6. So, using the Pythagorean Theorem, $EG^2 = FG^2 + EF^2$. Plugging the lengths FG and EG in yields $\left(\sqrt{189}\right)^2 = 6^2 + EF^2$ which means $EF = \sqrt{153} = 3\sqrt{17}$.

d. This is asking up for the measure of angle EGF. We can use any trigonometric function in ΔEFG: $\cos EGF = \frac{FG}{EG} = \frac{6}{3\sqrt{21}}$. Thus angle EGF measures approximately 64.1°.

1. A right pyramid has a square base with side 10. Its height is 12. Find the following:

a. Its slant height.



c. The length of lateral edge $\overline{AF}$.

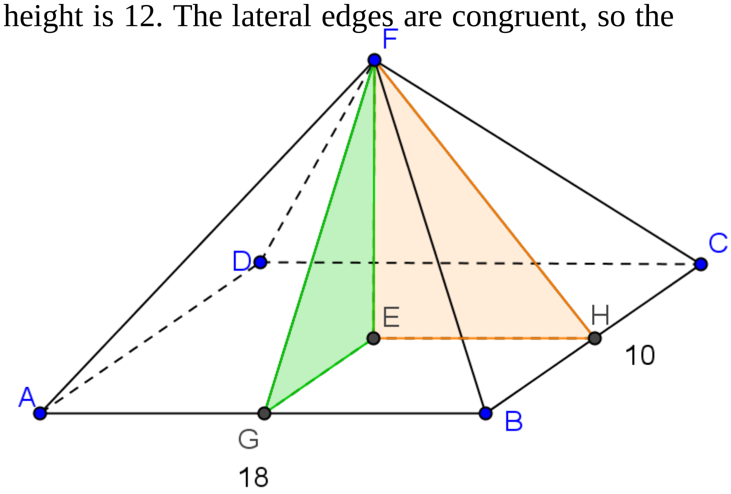
c. Its lateral area.

d. Its total area.

e. The angle at which a face makes with the base. Hint: draw a right triangle with vertices of E, F, and the midpoint of $\overline{AD}$. 

2. The base of a rectangular pyramid is 10 by 18. Its height is 12. The lateral edges are congruent, so the apex is directly above the center of the rectangle.

a. Find its total surface area.



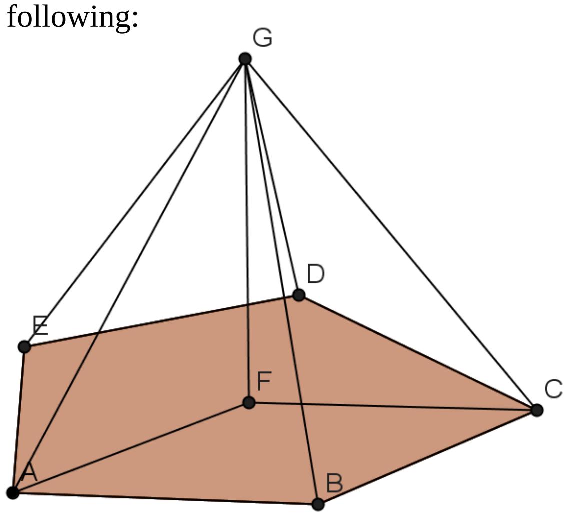
b. What is the length of each lateral edge?

c. At what angle does the face meet the side of the base whose length is 18 (angle FGE)? 

d. At what angle does the face meet the side of the base whose length is 10? 

e. How far is point G from the midpoint of $\overline{CF}$?

3. ABCDE is a regular pentagon with side 10. With apex G, it forms a right pentagonal pyramid, so $\overline{FG}$ is perpendicular to the pentagon. Altitude FG has length 12. Find the following:



a. The area of the base.

b. Its slant height.

c. The length of the lateral edges.

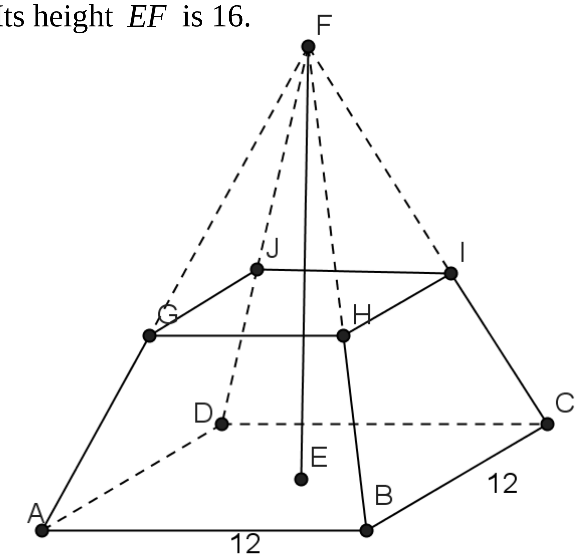
d. Its lateral surface area.

e. The angle at which $\overline{CG}$ meets $\overline{CF}$ in triangle CFG.

f. The angle at which the apothem of the pentagon meets one of the faces.

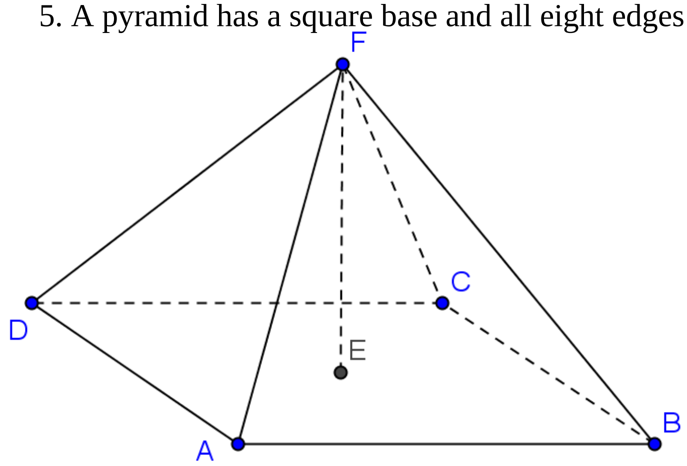
4. The right pyramid below has a square base (ABCD) of side 12. Its height $\overline{EF}$ is 16.

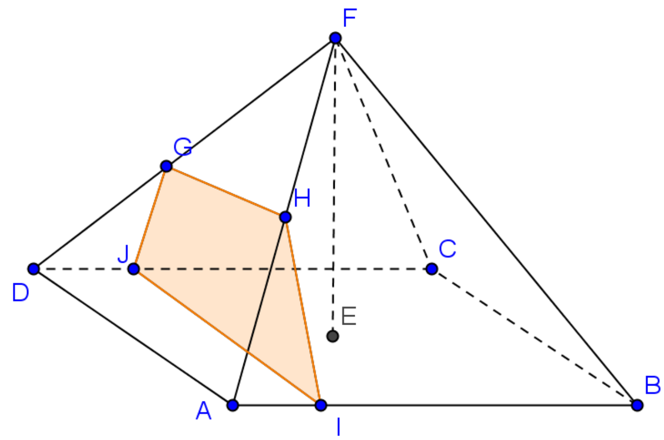
a. Find its total surface area.



b. A frustum is created by slicing it by square GHIJ, which is parallel to ABCD. Edge $\overline{HI}$ is 8 units long. What is the total surface area of frustum ABCDJIHG?

5. A pyramid has a square base and all eight edges are 4 units long.



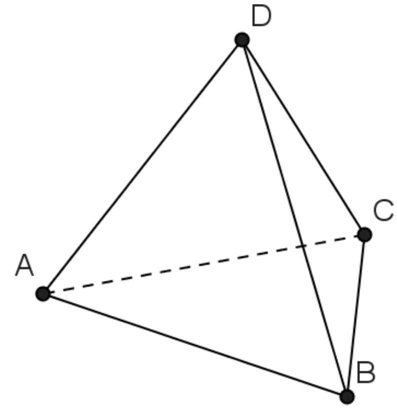


a. Find its total surface area.

b. Find the height $\overline{EF}$ (from the center of the square base to the apex)

c. The pyramid is sliced by a plane parallel to $\overline{EF}$ and perpendicular to $\overline{CD}$ *and* $\overline{AB}$, creating quadrilateral GHIJ. It bisects edges $\overline{AF}$ *and* $\overline{DF}$. What shape is GHIJ and what is its area? What is its perimeter?

6. ABCD is a right regular triangular pyramid. Its four sides are all equilateral triangles. It is also called a tetrahedron. Its height is 12. Find its surface area. Hint: since it is a right pyramid, D is directly above the "center" of triangle ABC (whatever that means!).



answers

1a. 13 b. $\sqrt{194}$ c. 260 d. 360 e. 67.4°

2a. 180+150+234=564 b. $5\sqrt{10}$ c. 67.4° d. 53.1° e. $\sqrt{13.5^2+7.5^2+6^2}=1.5\sqrt{122}$

3a. 172 b. 13.8 c. 14.7 d. 345 e. 54.7° f. 60.2°

4a. $144+48\sqrt{73}$ b. 208 + $\frac{5}{9}\cdot 48\sqrt{73}$ (using scale and area—FHI is 2/3 perim of FBC so 4/9 area so HIBC has 5/9th the area of FBC)

5a. $16+16\sqrt{3}$ b. $2\sqrt{2}$ c. It is a trapezoid. area is $3\sqrt{2}$: AD=4; GH=2 and its altitude is ½ of EF so $\sqrt{2}$; HI and JG are $\sqrt{3}$, using triangle AHI. So perimeter is $6+2\sqrt{3}$

6. the center is the center of the inscribed circle. Call the length AB 2*x;* the distance from the center of ABC to the edge is $x/\sqrt{3}$; the slant height is $x\sqrt{3}$ so by Pythag $x=3\sqrt{6}$ and the surface area is $216\sqrt{3}$

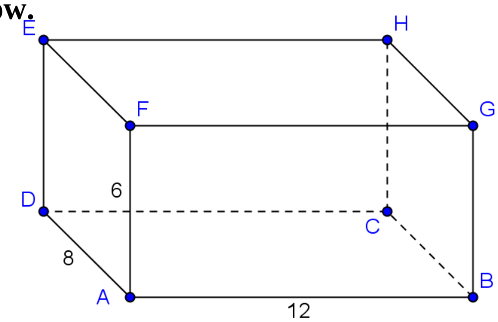
Unit 9 Handout #3: Volumes of Prisms and Pyramids

The ***volume of a prism*** is $V = bh$, where b is the area of the base and h is the height (measured perpendicular to the base).

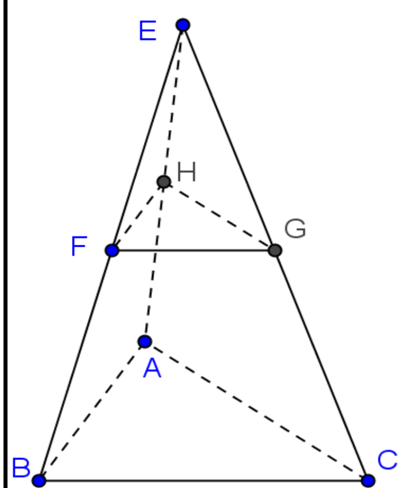
The volume of a pyramid is $V = \frac{1}{3}bh$, again where b is the area of the base and h is the height measured perpendicular to the base.

Example #1: Find the volume of the right rectangular prism below.

For a prism, volume is the product of base and height. The area of the base is 8 times 12, or 96. And the height is 6. So the volume is 6*96 or 576 cubic units.



Example #2: Right pyramid ABCE below has an equilateral triangle base with side 10 and a height of 16. Then find the volume of frustum ABCGFH given that FG=4.



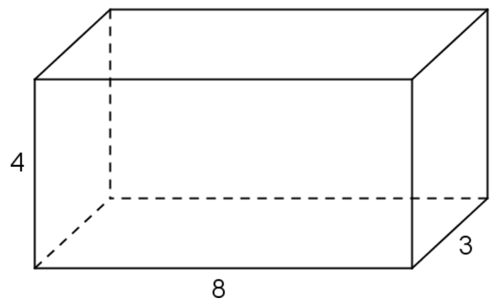
The volume of pyramid ABCE is one-third the product of the base area and the height. To find the area of the base, one can either use the formula for the area of an equilateral triangle, $A = \frac{s^2\sqrt{3}}{4}$, or draw an altitude from one vertex and use 30/60/90 triangles to find its length. Doing the latter, the length of the altitude from A to BC is $5\sqrt{3}$ so the area of the base is $\frac{1}{2}\cdot 10\cdot 5\sqrt{3}$ or $25\sqrt{3}$. Then the area of the pyramid is $\frac{1}{3}bh = \frac{1}{3}\left(25\sqrt{3}\right)(16) = \frac{400\sqrt{3}}{3}$.

To find the volume of the frustum, we can subtract the volume of the smaller pyramid EFGH from the volume of ABCE. The area of equilateral triangle FGH can be found using similar triangles. Since its sides are 4/10 as long as those of ABC, its area is $\left(\frac{4}{10}\right)^2 = \frac{16}{100}$ as large so $4\sqrt{3}$. Its height can be found using similar triangles. Its height will be 4/10 as large as the height of ABCE, so (4/10)(16)=(64/10). Thus the volume of pyramid EFGH is $\frac{1}{3}\left(4\sqrt{3}\right)\left(\frac{64}{10}\right) = \frac{256\sqrt{3}}{30} = \frac{128\sqrt{3}}{15}$. Thus the volume of the frustum is

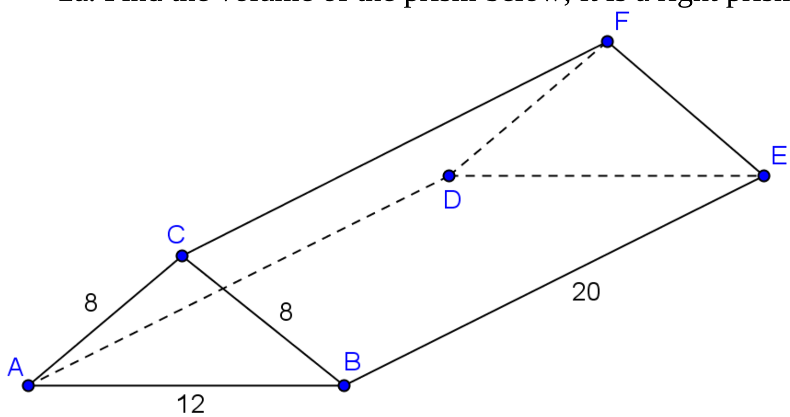
$$\frac{400\sqrt{3}}{3} - \frac{128\sqrt{3}}{15} = \frac{1872\sqrt{3}}{15}.$$

Note: one could have used volume of similar figures as a short cut!

1. Find the volume and surface area of the rectangular prism below.

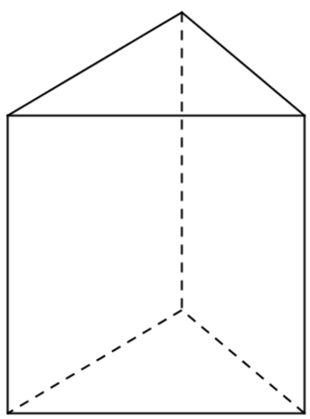


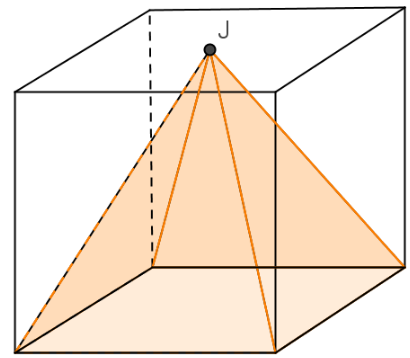
2a. Find the volume of the prism below; it is a right prism with a triangular base.



b. Christine looks at the shape above and considers rectangle ABED to be the base. With this as the base, is it a prism? A pyramid?

3. The bases of the right triangular prism below are equilateral triangles with side length 8 cm. Its height is 15 cm. A total of 300 cubic cm of liquid is poured into it. How far from the top is the level of the liquid?

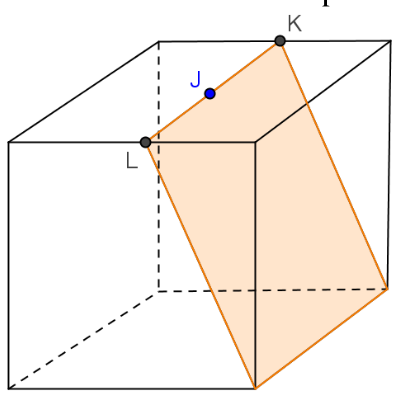


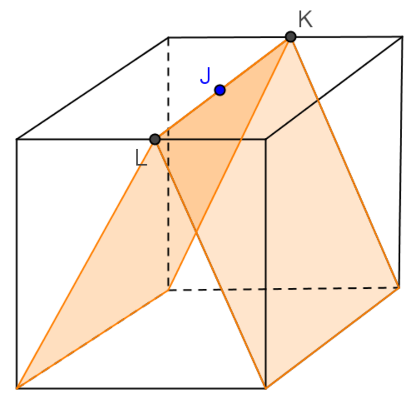


4. ***The volume of a pyramid***: The goal of this extended problem is to find a formula for the volume of a pyramid in terms of its height and the area of its base. We will use the pyramid at the right, which is in a unit cube. It has a square base of side length one and its apex J is one unit above the center of the square base.

a. What is the volume of the cube?

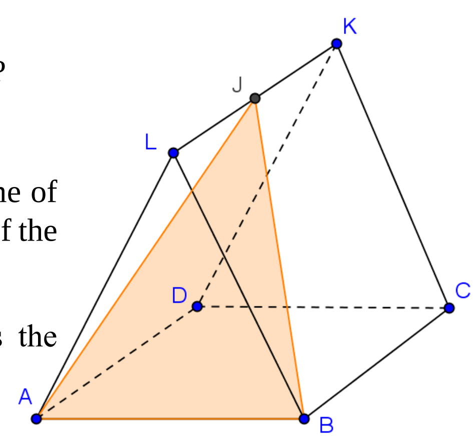
b. To turn the cube into the desired pyramid, we need to cut some pieces away. Make the cut in the diagram on the left below, from $\overline{LK}$ to the bottom edge of the cube. What shape piece is removed and what is the volume of the removed piece?





c. The next piece cut off is the same; from $\overline{LK}$ to the opposite edge of the base (see diagram at right above). What is the volume of the part cut off and what is the volume of the remaining triangular prism?

d. Is the remaining piece larger that the pyramid? How do you know?



e. Now we have the triangular prism on the right. To find the volume of the pyramid, we'll assume the volume is proportional to the product of the base area and the height, so it is in the form $V = k \cdot base \cdot height$.

We'll split the triangular prism into three parts, one of which is the pyramid we want.

Fill in the table below. The last column should be in terms of *k*.

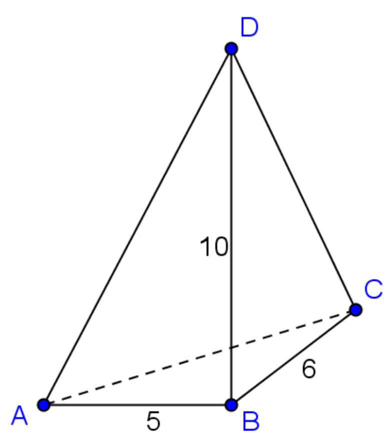
Piece	Base area	Height	Volume $V = k \cdot base \cdot height$
Right Pyramid with base ABCD and apex J			
Oblique pyramid with base LAB and altitude JL			
Oblique pyramid with base KCD and altitude JK			

f. Add up the right column to get the total volume in terms of *k*. Then solve for *k*, using the total volume you came up with in part *c* above.

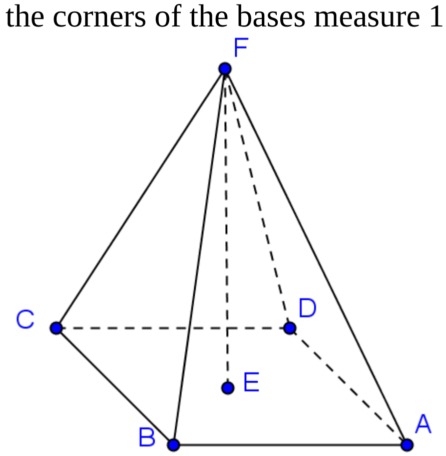
g. Conjecture: how does the volume of a pyramid relate to its base area and height?

5. Find the volume of each shape below:

a. $m\angle ABC = m\angle ABD = m\angle DBC = 90^\circ$

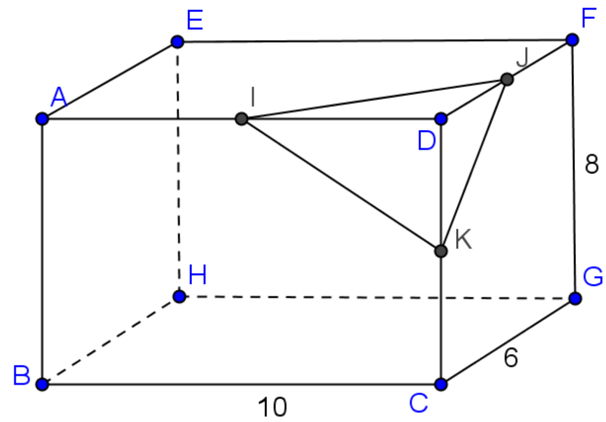


b. ABCD is a square base with side 10. Edges from F to the corners of the bases measure 15.



6. Answer the following questions about the rectangular prism below. Points I, J, and K are the midpoints of three edges.

a. What is the volume and surface area of the prism?

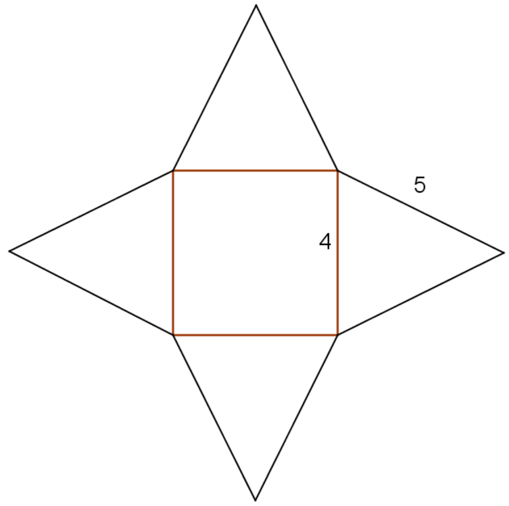


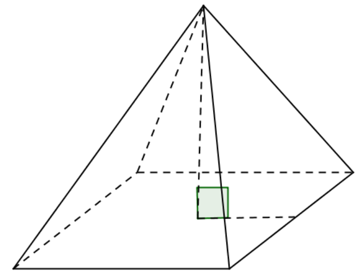
b. What is the volume of pyramid KIDJ?

c. What percentage of the volume of the prism does KIDJ represent? Are you surprised?

d. A pyramid is formed whose base is BCGH. Its apex is A. Find its volume.

7. To make a pyramid, Andrea starts with a square of side 4. She then adds an isosceles triangle with leg 5 to each side. Finally, she folds the triangles up until their points meet directly above the center of the pyramid. Find the following:



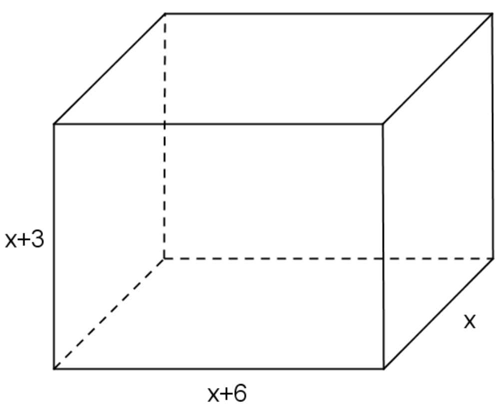


a. The volume of the pyramid.

b. The total surface area of the pyramid.

c. The angle that each face makes with the base. 

8. The rectangular prism has a surface area of 366; find its volume.



9. A small cube has a side of *w*. A larger cube has a side of 2*w*.

a. Find the lengths of the diagonals of both cubes and then find their ratio.

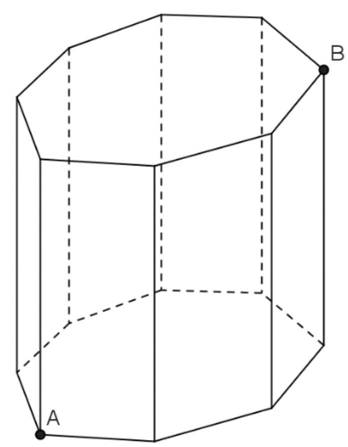
b. Find the surface area of both cubes and the find their ratio.

c. Find the volume of both cubes and the find their ratio.

d. Explain why the ratios in parts *a*-*c* above are not all the same.

10. The regular right octagonal prism below has a base perimeter of 80 and a height of 30. Find the following:

a. Its total surface area. 



b. Its volume.

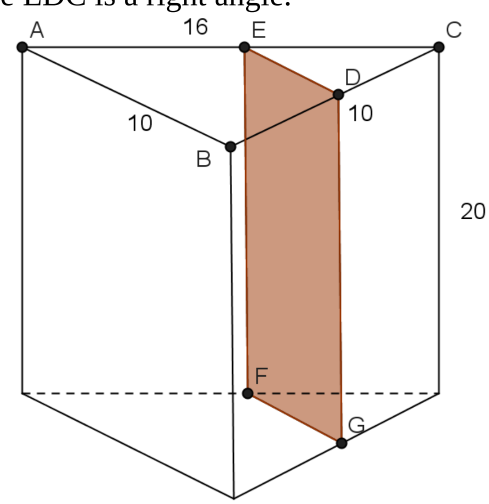
c. The distance from A to B (both through the inside of the prism and shortest distance on the outside.

11. The length of the side of a cube is increased by one unit.

a. If the surface area increases by 90 square units, then how long was the cube's side originally?

b. Instead, if the volume increases by 127 cubic units, then how long was the cube's side originally?

12. Triangle ABC is the top of a right triangular prism where AB=BC=10 and AC=16. [Note: this means it is a right prism with a triangular base—not that the base is a right triangle!] The prism is sliced by rectangle DEFG where D and G are the midpoints of the sides of the bases and angle EDC is a right angle.



a. Find the volume and total surface area of the entire prism.

b. How should the sum of the volumes of the two pieces created by the slice compare to the volume of the entire prism?

c. How should the sum of the surface areas of the two pieces compare to the surface area of the entire prism?

d. How long is $\overline{CE}$? Note: use similar triangles- but ABC is not one!

e. Now find the volume and surface area of the smaller piece.

13. The right pyramid ABCDE below has a square base of length 12 and a height of 10 (from A to the center of the base). Points F, H, I, and J are the midpoints of the edges from A to the base vertices.

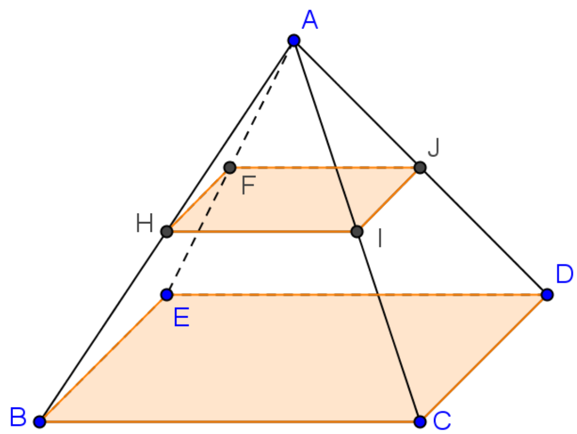
a. What is the ratio of the area of ΔAHI to the area of ΔABC?

b. The top of the pyramid is sliced off by a plane that contains points F, H, I, and J. This creates a small pyramid and a larger frustum. Find the volume of both pyramids.

c. Find the ratio of the volume of the smaller pyramid to the volume of the original pyramid. Are you surprised by the answer? Explain.

d. Find the total surface area of frustum BCDEFHIJ.

e. What is the length of $\overline{JB}$?

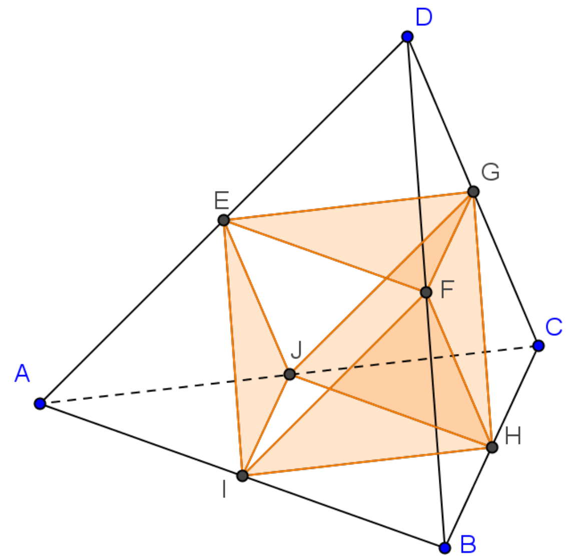


14. ABCD is a tetrahedron, meaning it has four faces that are all equilateral triangles. Assume its volume is 120. Points E, F, G, H, I, and J are midpoints of its edges.

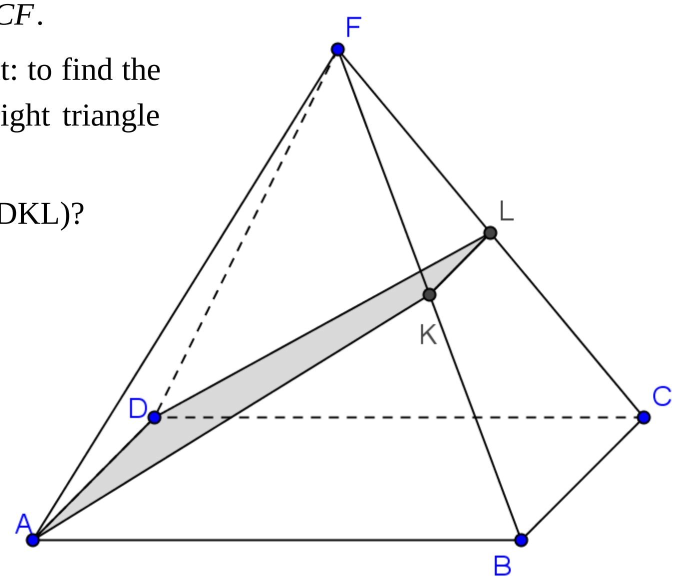
a. What is the volume of tetrahedron DEFG? Hint: use similar figures, don't try to compute lengths!

b. Tetrahedrons DEFG, AEIJ. FBIH, and CGHJ are all sliced off the original tetrahedron ABCD. What is the volume of the remaining shape? How would you describe the remaining shape?

c. The surface area of the remaining shape is what percent of the surface area of the tetrahedron ABCD?



15. ABCD is a square with side 12 and the height of the right pyramid (with apex F) is 16. A plane slices it into 2 pieces where K and L are the midpoints of $\overline{BF}$ *and* $\overline{CF}$.

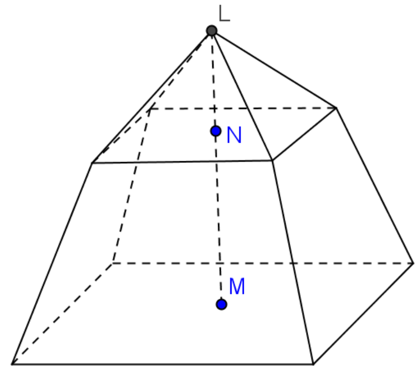


a. Find the area and perimeter of trapezoid ADLK. (Hint: to find the altitude from K to $\overline{AB}$, make it the hypotenuse of a right triangle where one vertex is directly below K in square ABCD)

b. What is the surface area of lower solid (vertices ABCDKL)?

c. What is the volume of the lower solid?

16. The tent below is a frustum of one square-based pyramid with another pyramid on top. The side of the lowest square is 10 feet and the side of the upper square is 6 feet. ML measures 8 feet. And the sides of the lower pyramid meet the base at a 70° angle. Find the volume of the tent as well as its lateral surface area.



Answers

1. V=96 SA=136 2a. $240\sqrt{7}$ b. neither 3. 4.17; area base = $16\sqrt{3}$ so of liquid is 10.83 cm from base
4a. 1 b. a triangular prism with base ¼ and height 1 so volume ¼ c. cut off ¼ so volume remaining is ½
d. It must be larger since it includes the pyramid and some extra pieces.
e. 1; 1; k; ½ ; ½ ; k/4; ½ ; ½ ; k/4 f. 1.5k = ½ so k=1/3 g. $V = base \cdot height/3$

5a. 50 b. $\frac{1}{3} \cdot 100 \cdot \sqrt{175} = \frac{500\sqrt{7}}{3}$ 6a. V=480; SA=376 b. 10 c. no; base is 1/8 of ABCD and hgt is ½, so that's 1/16 → and since pyramid's V=(1/3)bh, its area should be 1/48th, which it is! d. 160

7a. hgt is $\sqrt{17}$ so volume is $\frac{16\sqrt{17}}{3}$ b. $16 + 8\sqrt{21}$ c. $\tan^{-1}(\sqrt{17}/2)$ =64.1°

8. 440 since $x(x+3) + x(x+6) + (x+3)(x+6) = 183$ *thus* $x = 5$

9a. $w\sqrt{3}$ *and* $2w\sqrt{3}$, so 2. b. $6w^2$ *and* $24w^2$, so 4.c. w^3 *and* $8w^3$, so 8.
d. the dimensions: linear items are scaled by 2; areas by 4; and volumes by 8.

10a. apoth = 12.1 so SA≈3366 b. 14,485 c. thru is $x^2 = 30^2 + \left(10 + 10\sqrt{2}\right)^2$ so 38.5; around is $30\sqrt{2}$

11a. $6(x+1)^2 - 6x^2 = 90$ so x=7 b. $(x+1)^3 - x^3 = 127$ *so* $3x^2 + 3x - 126 = 0$ *or* $x^2 + x - 42 = 0$ so x=6

12a. V=960; SA=816 b. the same c. larger by 2 times the area of DEFG.
d. ΔCDE~ΔCEB so EC=6.25 e. small: V=187.5 and SA = 318.75

13a.¼ b.big 480; small 60 c. 1/8; no-sim shapes with lengths 2:1 so vols are 8:1 d.$180 + 36\sqrt{34}$ e. $\sqrt{187}$

14a. similar volume with scale of ½ means volume is 1/8th as large, so 15.
b. 120-4(15)=60… ½ has large as original. Shape has 8 congruent equilateral triangles. It is like two pyramids with base EFHJ built back-to-back. It is called an octahedron. c. ½

15a. AD=12 KL=6; Altitude from K to AD… down 8 and across 9 so $\sqrt{145}$. Length of AK (use altit) 3^2 + 145=AK^2 so $\sqrt{154}$. Thus perimeter is $18 + 2\sqrt{154}$ and area is $9\sqrt{145}$

15b. ABCD is 144 ADLK is $18 + 2\sqrt{154}$; CBKL is ¾ of FBC (alt from F to BC is $\sqrt{292}$) so ¾ of $6\sqrt{292}$ which is $4.5\sqrt{292} = 9\sqrt{73}$

area of AKB: altit from K to AB… 8 down and 3 across so $\sqrt{73}$ so (1/2)*12 $\sqrt{73}$ same on other side

So surface area is 144 + $18 + 2\sqrt{154}$ + $9\sqrt{73}$ +12 $\sqrt{73}$ so 162+ $2\sqrt{154} + 21\sqrt{73}$

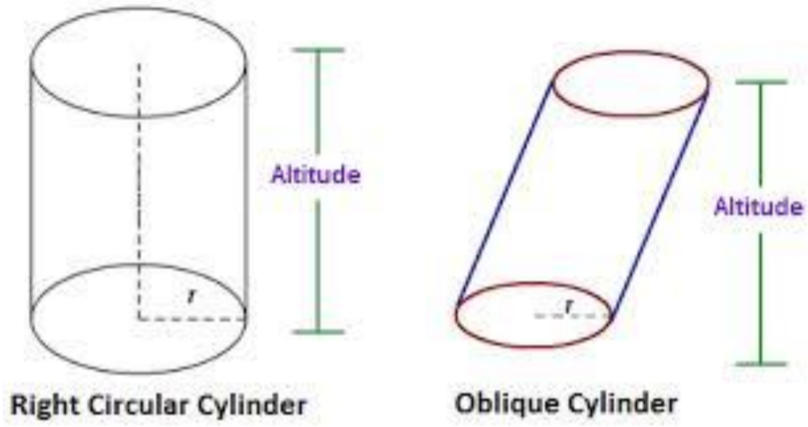
15c. trim front and back (perp from K and L to ABCD)… each has base 36 and hgt 8 so vol of each is 96
What's left is a triangular prism→ front and back are Δ's with base 12 and hgt 8 and depth is 6 so 288
so total volume = 288+192 = 480

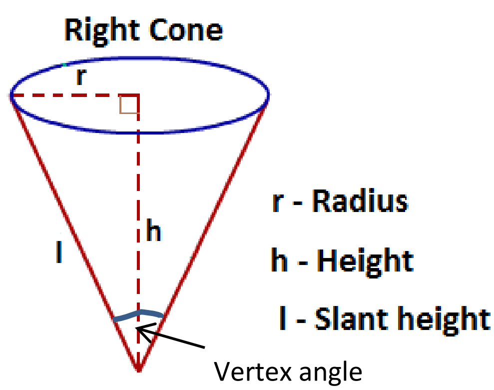
16. MN is 5.5 feet (2*tan70) and LN=2.5 feet
Volume of frustum: vol of full pyr would be (1/3)(100)(13.74) or 458 cubic feet: take away $(6/10)^3$ so 359 cubic feet ; volume of small pyr is (1/3)(36)(2.5)=30 so 389 cubic feet total
Surface area of small pyr is 4*(1/2)(6)(3.91)=46.9; surface area of frus (4)(1/2)(10)*11.7* 0.64=149.7 so 196.6 square feet

Unit 9 Handout #4: Volumes of Curved Solids

Cylinders are like prisms, in that each cross-section parallel to the base is congruent to the base. There are right cylinders (where one base is directly above others) and oblique cylinders. '

Like prisms, the volume of a cylinder is the product of the area of the base and the height (where the altitude is measured perpendicular to the base). Thus $V = \pi r^2 h$, where *r* is the base radius and *h* is the altitude.





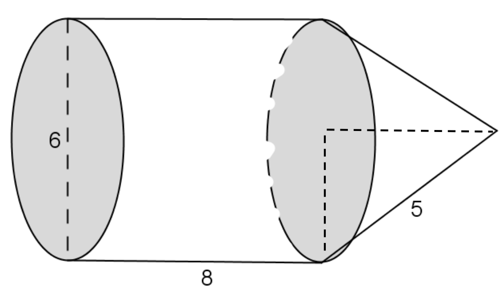
Cones are like pyramids, as a circular base tapers to a point at one end. A cone may be right or oblique; in a right cone, the point at the end is directly above or below the center of the base.

Like a pyramid, the volume of a cone is one third of the product of the base area and the height, or $V = \frac{1}{3}\pi r^2 h$.

The diagram shows that the Pythagorean Theorem relates the slant height, the radius, and the height, so $l^2 = r^2 + h^2$.

A ***sphere*** is the set of points in three dimensions that are equidistant from a fixed point. The volume of a sphere with radius *r* is given by the formula $V = \frac{4}{3}\pi r^3$.

Example #1: The shape below is a cylinder with a cone on one end. Find the volume of shape and the vertex angle of the cone.



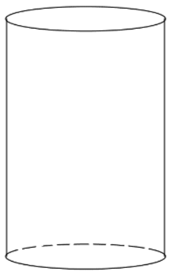
The volume of the cylinder is the area of the base times the height, which is $V = \pi r^2 h = \pi(3^2)\cdot 8 = 72\pi$.

To find the volume of the cone we need the height. The slant height is 5. Using the Pythagorean Theorem to find *h*, we get $3^2 + h^2 = 5^2$ so *h* is 4.

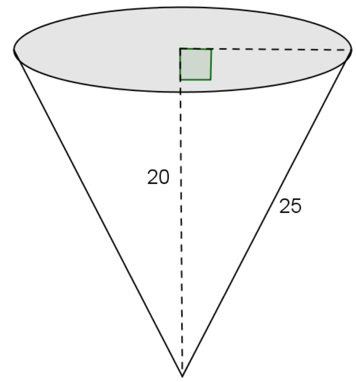
The volume in then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 3^2 4 = 12\pi$. So the object's volume is 72π+12π=84π.

The vertex angle. The sine of one half the vertex angle is 3/5, so one-half the angle is $\sin^{-1}(3/5) = 36.9^\circ$. So the vertex angle is twice this, or 73.8°.

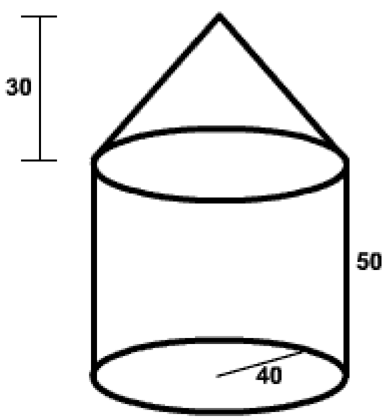
1. The right cylinder below has a base radius of 5 and a height of 10. What is its volume?



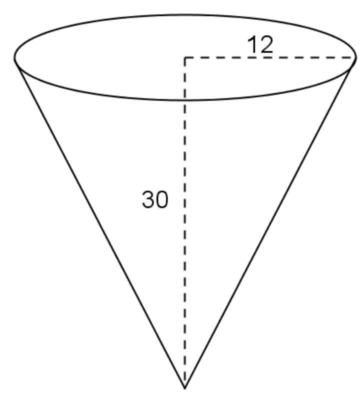
2. Find the volume of the cone below. Then find its vertex angle.



3. What is the volume of the object below?



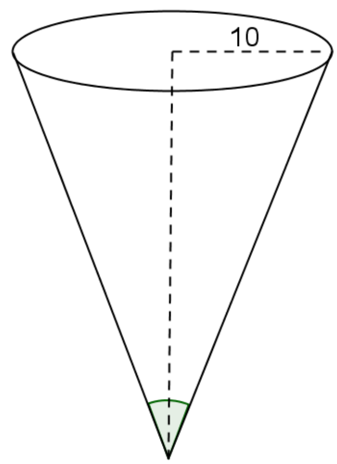
4. A conical cup filled to a height of 7.5 cm with liquid.
   a. What is the radius of the circle at the water's surface?
   b. What is the volume of liquid in the cone?



5. The mailbox below is a rectangular prism with a half cylinder on top (disregard the flag!). The rectangle at the base is 20 inches by 6 inches and the height of the mailbox (to the top of the half cylinder) is 10 inches. Find the volume of the mailbox.

6. Given that the vertex angle of the cone below is 40°, do the following:

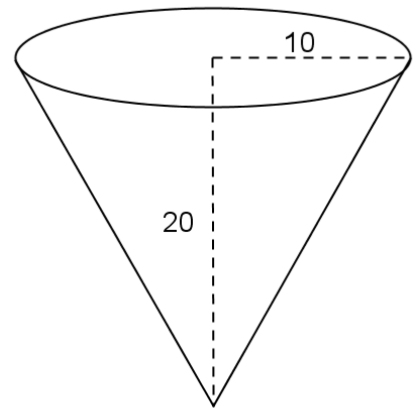
a. Find its volume. 



b. It is filled with water. The water is then poured into a cylindrical cup with a base radius of 7. How high from the bottom of the cup is the water?

7. Answer the following questions about the conical cup below.

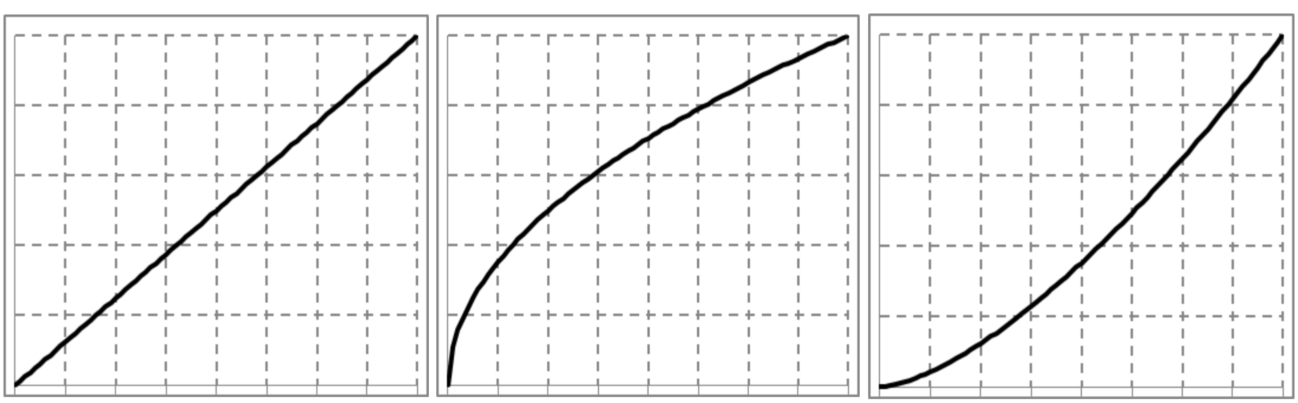
a. If it is filled with liquid up to a height of 8 cm from the bottom, what is the radius of the circle at the liquid's surface?



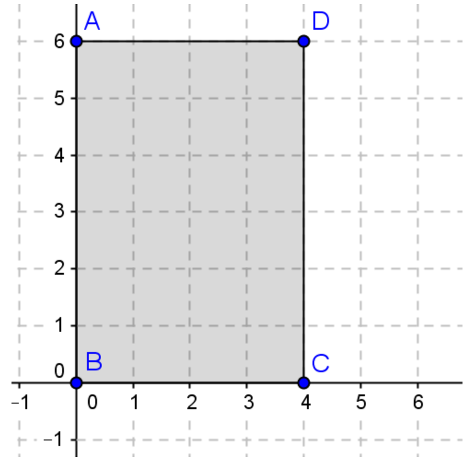
b. If it is filled with liquid up to a height of *x* cm from the bottom, what is the radius of the circle at the liquid's surface (in terms of *x*)?

c. The cup is filled with a liquid whose volume is one fifth of the cup's volume. Find the height of the liquid's surface.

d. Now suppose water is added to the empty cup at a constant rate (in cubic units per second). Which graph below best shows the height of liquid in the cup (*y*) as a function of time (*x*)? Explain.



8. The rectangle below is rotated about the *y*-axis, creating a solid. What shape is created and what is its volume? In other words, imagine that $\overline{AB}$ is held fixed while $\overline{BC}$, $\overline{CD}$, *and* $\overline{AD}$ spin 360° around the *y*-axis. All points on the inside of that form a shape.

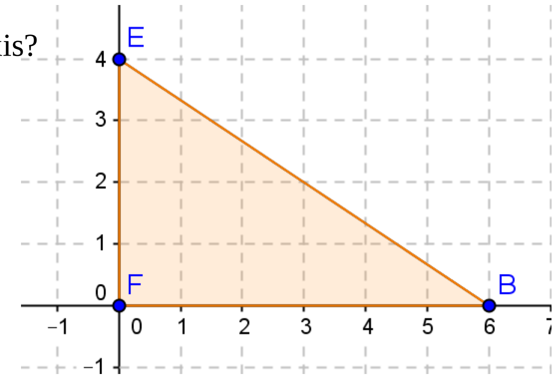


9a. Instead, the rectangle from the prior problem is rotated around the *x*-axis. What is the volume of the solid created?

b. Instead it is rotated around the line *x*=6. What is created and what is its volume?

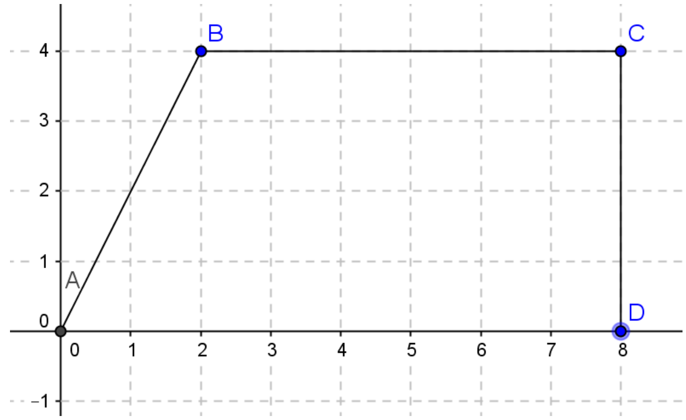
10. Right triangle FEB is revolved around the *y*-axis.

a. What solid is created as the triangle is swept around the *y*-axis?



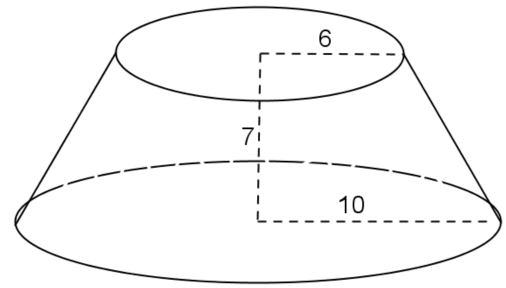
b. What is the volume of the solid created?

11. What is the volume of the solid created when ABCD is revolved around the *x*-axis?



12. What is the volume of the solid created when ABCD (from the prior question) is revolved around the line $y = -2$? Be careful!

13. Find the volume of this frustum of a cone. Think of it as a big cone minus a little cone. You'll probably want to use similar triangles to find the height of the big cone (or you could use trig…).

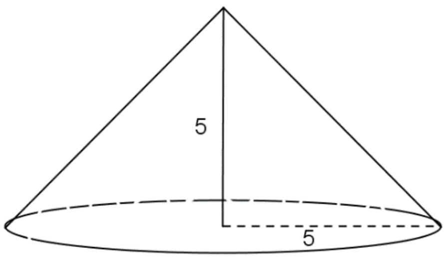


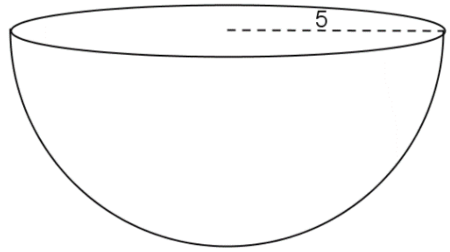
14. A cylinder with a radius of 4 inches and a height of 10 inches has 1.5 inches of water in it. Cubes with sides of 2 inches are placed it the cylinder, sinking to the bottom.

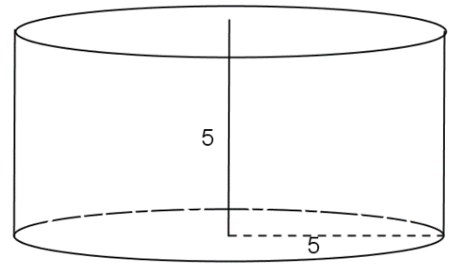
a. What is the water level if one cube is placed in the cylinder?

b. What is the water level if five cubes are placed in the cylinder?

Class activity on volume of a sphere: ***a cone plus a hemisphere equals a cylinder!***



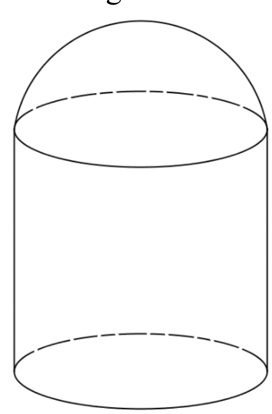




	Area of cone slice	Area of hemisphere slice	Area of cylinder slice
Slice at the bottom:	25π	0	25π
Slice at the top			
Slice 1 unit from top			
Slice 2 units from top			
Slice 3 units from top			
Slice *w* units from top			

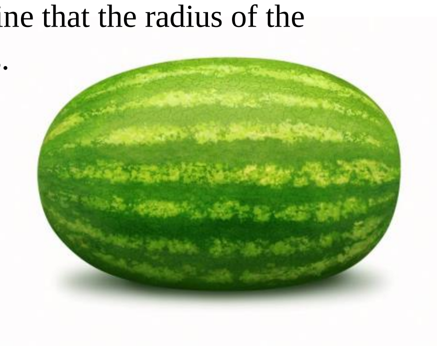
15. Find the volume of a sphere with a radius of 5.

16. The shape below is a hemisphere (half sphere) on top of a cylinder. If the cylinder's height is 15 and base radius is 6, then find the volume of the shape.



17. What is the radius of a sphere if its volume is 200π?

18. Think of a watermelon as a cylinder with a hemisphere at each end. Imagine that the radius of the cylinder is 6 inches and the length of the melon, from end to end, is 24 inches.



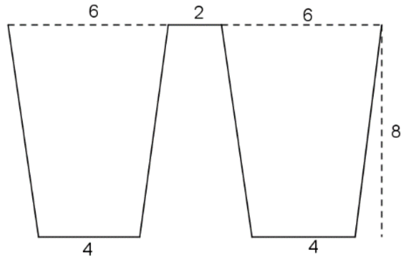
a. What is the total volume of the watermelon?

b. Now assume that the outermost inch all around is the rind, and is not edible. What is the volume of the edible part of the melon? Hint: think of it as a smaller cylinder and two smaller hemispheres.

c. What percent of the melon is edible?

d. Jill is picky and considers the outermost 1.5 inches inedible. To her, what percent of the melon is edible?

19. When the cake pan (on the left below) is sliced vertically through its center, you get the shape on the right. The two pieces are isosceles trapezoids with bases of 4 and 6 and heights of 8. What is the volume of the cake pan?



Answers

1. 250π 2. Volume=1500π vertex angle=73.7° 3. $96{,}000\pi$ 4a. 3 b. 22.5π 5. $840+90\pi$

6a. about 916π b. about 18.7 units 7a. 4 (similar triangles) b. 0.5x c. $\frac{1}{3}\pi(0.5x)^2 x = \frac{400\pi}{3}$ so x=11.7

d. the middle one since it grows quickly at first and slowly over time 8. a cylinder with volume 96π

9a. 144π b. a big cylinder (r=6) with a small cylindrical hole in the center (r=2); its volume is 192π

10a. a cone with vertex E b. 48π 11. $320\pi/3$

12. the part from x=2 to x=8 has volume 192π; the other part is $36\pi-4\pi/3 - 8\pi$ or $80\pi/3$ so total is $656\pi/3$

13. "the big cone": hgt: $\frac{h-7}{h} = \frac{6}{10}$ so h=17.5 and V=$1750\pi/3$

Small cone: hgt =10.5 so V=$378\pi/3$ so frustum volume is $1372\pi/3$

14a. water amount is 24π cubic inches; cross sectional area is ($16\pi-4$) square inches so 1.63 inches

b. water amount is 24π cubic inches; cross sectional area is ($16\pi-20$) square inches up to 2 inches high and 16π after that….uses 60.53 inches^3 to cover cubes; 14.87 left so 14.87/(16π)=0.296 so 2.296 inches

15. $500\pi/3$ 16. 684π 17. $\sqrt[3]{150} \approx 5.31$

18a. $\frac{4}{3}\pi \cdot 6^3 + \pi \cdot 6^2 \cdot 12 = 720\pi$ cubic inches b. $\frac{4}{3}\pi \cdot 5^3 + \pi \cdot 5^2 \cdot 12 = 466.67\pi$ c. 64.8% d. only 50.6%

19. 320π cubic inches: Big conic frustum minus inside conic frustum:

Big one: radius of cone is 7 and radius of bottom of frus is 6: hgt of "full" cone: $\frac{6}{7} = \frac{x-8}{x}$ so x=56

So full cone is $\frac{1}{3}\pi(7^2)(56)$ and missing part is $\frac{1}{3}\pi(6^2)(48)$ so big frustum is $\frac{1016\pi}{3}$

Small one: full cone has base radius of 2 height of 16; missing part has base radius of 1 and hgt of 8

So small frustum is $\frac{1}{3}\pi(2^2)(16) - \frac{1}{3}\pi \cdot 1^2 \cdot 8 = \frac{56\pi}{3}$ so cake pan volume is $\frac{960\pi}{3} = 320\pi\ inch^3$

Unit 9 Handout #5: Surface Area of Curved Solids

The ***surface area of a cylinder*** can be thought of in two parts:

1. The area of the circles at the top and bottom: each is πr^2.
2. The lateral surface area, which is the circumference of a circle with height, or $2\pi rh$.

So the total surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$

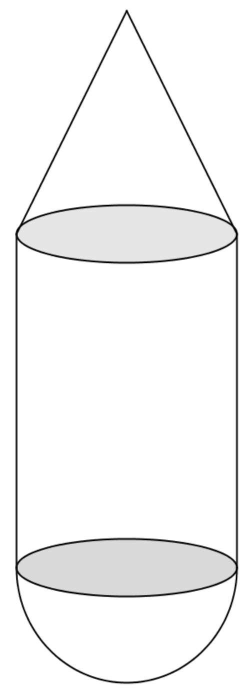
The ***surface area of a cone*** can be thought of in two parts:

1. The area of the circular base is πr^2.
2. The lateral surface area, which is πrl, where l is the slant height.

Thus the total surface area of a cone is $SA = \pi r^2 + \pi rl$

The ***surface area of a sphere*** with radius r is $4\pi r^2$.

All formulas will be derived in exercises in this section.



Example #1: The object on the right consists of a cylinder with a hemisphere on one end and a right cone on the other. The diameter of the cylinder is 12 cm and the height of the object, from tip to the very bottom is 32 cm, with 8 cm representing the height of the cone. Find its surface area.

Starting with the cylinder. Its base radius is 6 and its height is 32 cm minus 8 cm (for the cone) minus 6 cm (for the hemisphere), so 18 cm. Thus its lateral surface area is $2\pi rh = 2\pi(6)(18) = 216\pi$. Since its circular bases are not part of the surface area of the object, we can disregard them.

The cone's lateral surface area is πrl. We can find l using the Pythagorean Theorem: $l^2 = r^2 + h^2 = 6^2 + 8^2 = 100$, so l is 10 cm. The lateral surface area is thus $\pi rl = \pi(6)(10) = 60\pi$. Again, the circular base can be disregarded since it is not part of the objects surface.

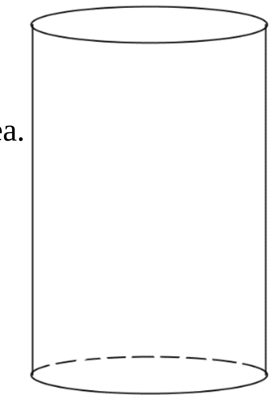
The hemisphere: since the surface area of a sphere is $4\pi r^2$, the surface area of a hemisphere must be $2\pi r^2 = 2\pi(6)^2 = 72\pi$.

So the object's total surface area is 216π + 60π + 72π, or 348π square centimeters.

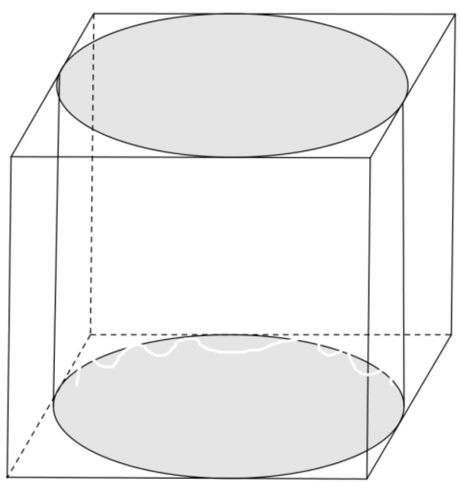
1. ***Surface area of a cylinder.*** The total surface area of a cylinder is the area of the top and bottom plus the lateral surface area. One way to think about the lateral surface area is to "unroll" the cylinder; the lateral part becomes a rectangle. Alternatively, you can think of it as a circumference with height.

a. Find the total surface area of the cylinder below. Its bases have radii of 4 and its height is 10.

b. Now imagine the bases have radii of r and its height is h. Find its total surface area.

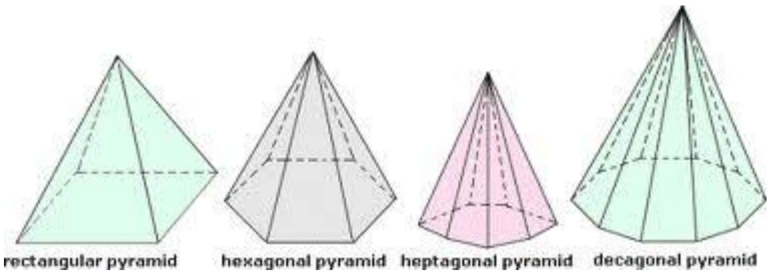


2. A cylinder is inscribed in a cube of side length 10. Find the volume and total surface area of the cylinder.



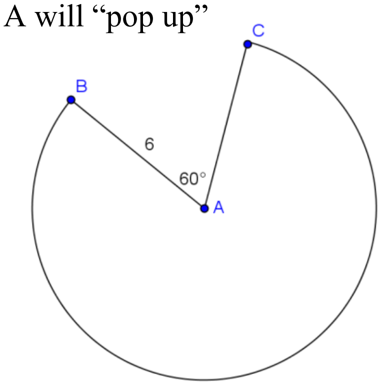
3. ***Surface area of a cone, method 1***.

a. Show that the lateral surface area of any right pyramid with a regular base is equal to one-half the product of the perimeter of the base and the distance from the apex to the midpoint of any base.



b. Think of a cone as a right pyramid whose base is a regular polygon with many many sides. Show that applying the formula from the part *a* above results in a lateral surface area of πrl.

4. ***Surface area of a cone, method 2***. One way to create a cone is to cut a sector out of a circle and connect the two exposed radii. In the diagram below, connect $\overline{AB}$ *and* $\overline{AC}$. Point A will "pop up" creating a cone!

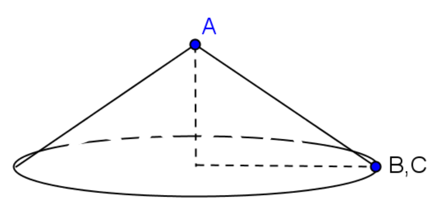


a. What is the slant height of the cone?

b. What is the circumference of the circular base of the cone?

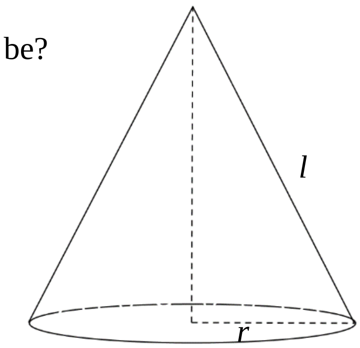
c. What is the radius of the cone's circular base?

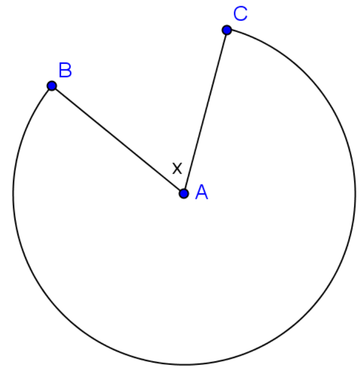
d. Looking at the "flat cone", what is the lateral surface area of the cone?



5. ***Surface area of a cone, method 2 (continued)***. Now let's do it backwards: given a cone, what size circle was it created from and what size sector was "cut out". The cone on the left has a base radius of *r* and a slant height of *l*.

a. Based on this, what must the radius of circle A be?

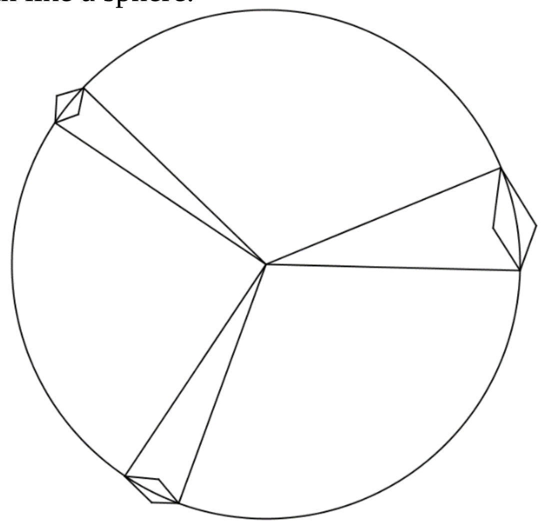




b. What must the length of major arc BC be?

c. The area of a sector and length of an intercepted arc are both related to the central angle by the formula: $\frac{central\_angle}{360} = \frac{\sec tor\_area}{\pi r^2} = \frac{arc\_length}{2\pi r}$. Given that we know the length of the major arc BC and the radius of circle A, use the last two parts of this equation to find the area of the sector that forms the outside of the cone. Be careful with your use of letters, as this problem defined r as the base radius of the cone, not the radius of the circle it came from!

6. ***Surface area of a sphere.*** Imagine you created a huge sphere-like form by taking lots of congruent pyramids with square bases of side one and large heights and putting their apexes together, to form the form's center. It is "dimpled", like a golf-ball, rather than perfect smooth like a sphere.



a. Each pyramid's height is approximately the radius of the form, but not quite. Why?

b. What is the volume of each pyramid, assuming the height is the form's radius *r*?

c. The volume of a sphere is $\frac{4}{3}\pi r^3$. So, in terms of *r*, how many pyramids do you need to make the sphere-like form?

d. What is the area of the base of each square?

e. Use your answers to parts *c* and *d* above to get the surface area of your sphere-like form.

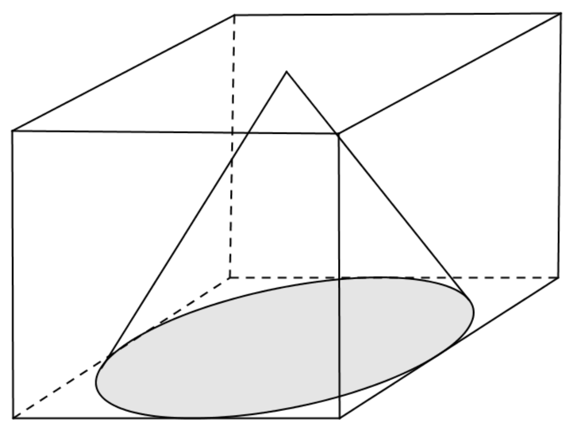
Note: what we created was a "dimpled" sphere-like object, and not exactly a sphere. But if we used more and more pyramids with smaller and smaller bases, it would eventually approach a sphere. Just like a regular polygon with more and more sides eventually becomes indistinguishable from a circle.

7a. A sphere has a volume of 288π What is its surface area?

b. What radius does a sphere have if its surface area (in square units) is equal to its volume (in cubic units)?

8. A right cone is inscribed in a right rectangular prism whose base is a square of side 10 and whose height is 8.

a. What is the total surface area of the cone?

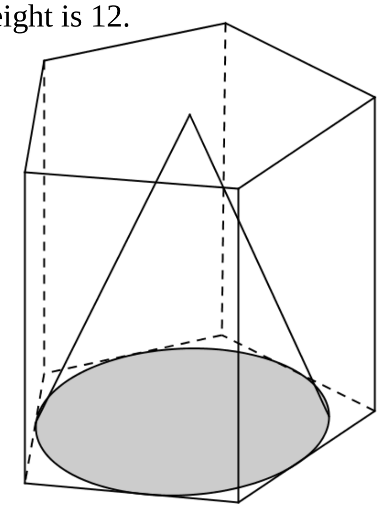


b. What percent of the volume of the prism does the cone's volume represent?

9. A right pentagonal prism has a base that is a regular pentagon of side 8. Its height is 12.

a. Find its volume and surface area. 

b. What is the volume of the largest cone that will fit inside of it?

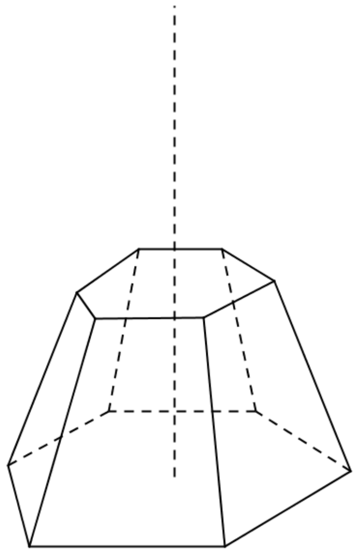


10. A right cone has a vertex angle of 60° and a height of 10.

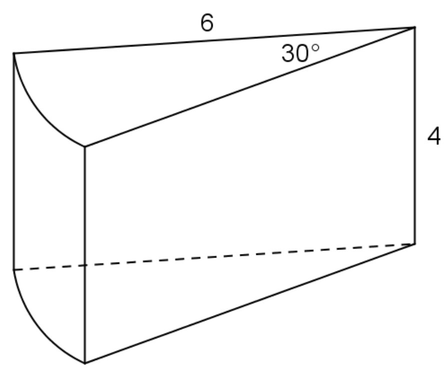
a. Find its volume.

b. Find its total surface area.

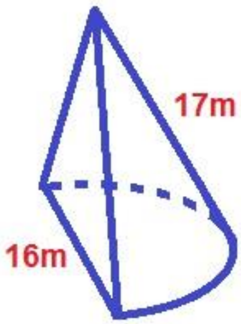
11. A frustum is created like this: start with a right pyramid whose base is a regular hexagon of side 8 and height is 10. Then cut it by a plane half way to its apex parallel to its base and remove the upper part. What remains is shown below. Find its volume and surface area.



12. A wedge of cheese is cut from a cylinder with radius 6 and height 4. What is the volume and total surface area of the wedge?



13. A cone is sliced in two parts—thru the vertex and a diameter of its circular base. Given that the diameter's base is 16 and the slant height is 17, find the volume and total surface area of the solid.



14. A soda can is a cylinder with a base radius of 4 cm and a height of 15 cm.

a. Find its volume and total surface area.

b. To save money on packaging, the soda company wants to make a different shaped cylindrical can. It has the same volume, but its height is equal to the diameter of the base. What is the radius of the base that makes the volume the same?

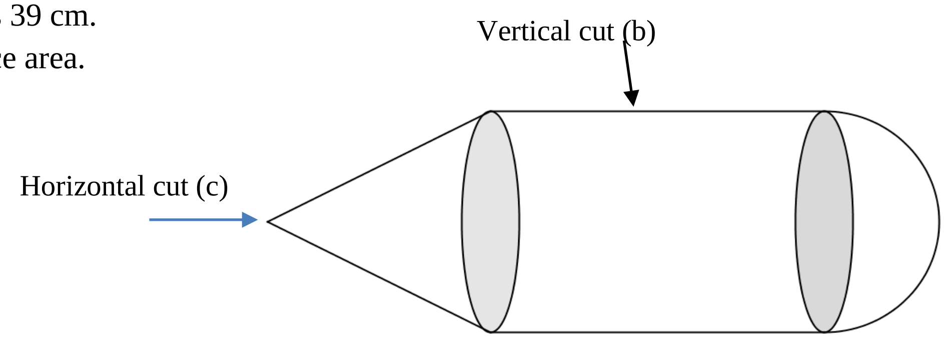
c. What percent less metal is needed to make this different-shaped can? (ie, its total surface area is some percentage less than the surface area of the original can).

15. A propane tank is a cylinder on its side with a hemisphere on each end. If the radius of the cylinder is 1 meter and the length of the cylinder is 8 meters, then find the volume and surface area of the tank. (Ignore the bumps on the top!)

16. (continuation). If a propane tank was spherical instead, then what radius would it need to have the same volume? And how would the surface area of the spherical one compare to that of the original one?

17. The object below is a cylinder with a cone attached to one end and a hemisphere attached to the other end. The cylinder is 18 cm long and has a diameter of 12 cm. The object's length, from the tip of the cone to the pole of the hemisphere, is 39 cm.

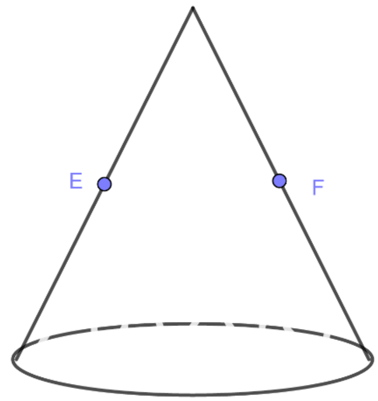
a. Find its volume and surface area.



b. It is sliced vertically by a plane (parallel to the circular bases of the cylinder) into two pieces of equal volume. How far from the left endpoint must it be sliced?

c. Instead of being sliced vertically, it is sliced horizontally. A knife enters the solid at the tip of the cone and works horizontally across the center of the object, making a cut all 39 cm long. The object is thus cut into two congruent pieces. What is the area of the newly exposed region on one piece?

18. The cone on the right has a base radius of 6 and a height of 8. Points E and F are on the outside of the cone exactly one half way up the side. They are on opposite sides of the cone—given that they are both one half way up the cone, they are as far from each other as possible.



a. How far apart are E and F if one can go directly through the cone?

b. How far apart are E and F is one walks along the outside of the cone, staying one half way between the vertex and the base?

c. What is the shortest distance from E to F staying along the outside of the cone? Get a real cone and a piece of string to get a sense of what the shortest path is! Hint: "unfold" the cone!

Answers

1a. 112π b. $2\pi r^2 + 2\pi rh$ 2. its radius is 5 so volume is 250π and its surface area is 150π

3. make a triangle with two consecutive vertices of the base and the apex. Its area is (1/2)*(base side)*(distance from apex to midpoint of side). If the base is a regular n-sided polygon, then there are n such triangles. So the lateral surface area is (1/2)(n)(base side)(distance from apex to midpoint) → multiply the middle two terms to get base perimeter. So lateral area = (1/2)(base perim)(dist from apex..)
b. To use this formula for a cone, the base perimeter is the circumference, or 2πr. And the distance from the apex to the midpoint of a "side" of the base is the slant height l. So (1/2)(2πr)(l)= πrl.

4a. the radius, so 6 b. major arc BC, so 10π c. 2πr=10π so 5 d. $(300/360)\pi(6^2)=30\pi$

5a. the slant height, l b. the circum of the cone's base, so 2πr c. $\frac{\sec tor\_area}{\pi l^2} = \frac{2\pi r}{2\pi l}$ so area is πrl.

6a. because the base is flat, the surface is like a golf ball and the height of the pyramid is slightly less than the sphere's radius. (but the sphere will be so large, it should not make much of a difference!)

b. $(1/3)(1)^2(r) = r/3$ c. $\frac{4}{3}\pi r^3 \Big/ \frac{r}{3} = 4\pi r^2$ d. 1 e. $4\pi r^2$

7a. r= 6 so SA is 144π b. 3 8a. 25π + $5\pi\sqrt{89}$ b.V of cone is $200\pi/3$; V of prism is 800 so ≈26.2%

9a. apothem is 5.51 so vol is (0.5)*(8)*(5.51)*(5)*12= 1321.3; SA = 2(110.1)+5*8*12=700.2 b. 380.90

10a. 1000π/9 b. 100π 11. volume = $280\sqrt{3}$ surface area = $120\sqrt{3}+36\sqrt{37}$

12. V is 1/12th of cylinder: 12π; SA = 10π+48 13. V=$\frac{1}{2}\cdot\left(\frac{1}{3}\pi r^2 h\right)$ and h=15 by Pythag…so V=160π

SA=triangle + ½ lateral area + semi-circle so 120+68π+32π or 120+100π

14a. volume = 240π surface area=152π b. radius= x; height = 2x so $V = \pi x^2(2x) = 240\pi$ and x≈ 4.93

15. volume = 28π/3 cubic meters ; surface area = 20π

16. $V = 28\pi/3 = (4/3)\pi r^3$ so r ≈1.91 and surface area is 45.98 square meters, which is much less than the 62.8 square meters for the original. (but the tank might roll away since it is spherical!)

17a. V=972π and SA=$288\pi + 18\pi\sqrt{29}$ b. 23.5 cm → need part on left to be 486π so cylinder part is 306π → 306π=36π*x and x=8.5 so 15 + 8.5

c. triangle plus rectangle plus semi-circle → 306 + 18π

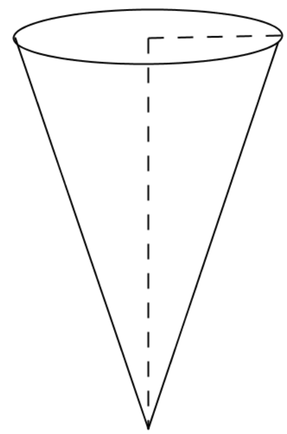
18a. 6 b. 3π c. 10sin54 = 8.09 – unfold it; it is a chord on a circle with radius 5 and central angle of 108°

Unit 9 Handout #6: Mixed Problems on Solids

Formulas:	
$V = base \cdot height$	$Lateral\ \ A = 2\pi rh$
$V = \frac{1}{3} \cdot base \cdot height$	$Lateral\ \ A = \pi rl$
$V = \frac{4}{3}\pi r^3$	$A = 4\pi r^2$

1. A cup is shaped like a cone. It has a base radius of 10 cm and a height of 30 cm.

a. Find its capacity (ie, the volume of liquid that can fit inside it).



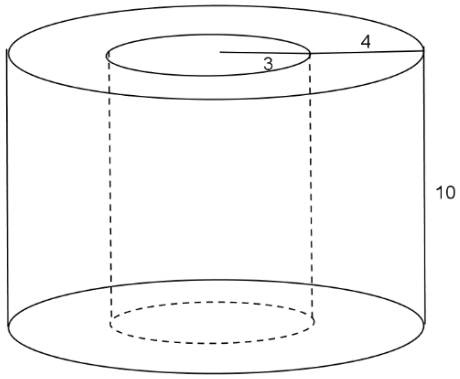
b. If it is filled up to a height of 15 cm, what volume of water is in the cup?

c. What fraction of the total volume of the cup does this half-filled cup represent? Are you surprised by this answer? Could you have known without doing the computations?

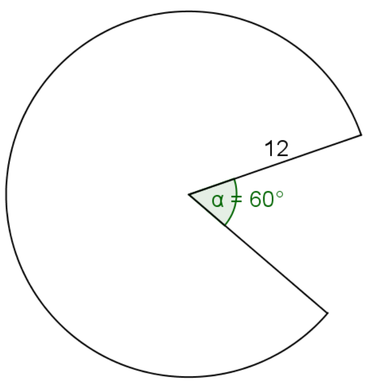
d. Assume the height of water from the bottom of the cone is *h* units (measured vertically). Use similar triangles to find the radius of the exposed circle of water in terms of *h*.

e. The cup is emptied and then 400π cubic cm of water is poured in. How high is the water from the bottom? Use the result from part *d* above.

2. The washer shown below is like a large cylinder with a smaller cylindrical hole cut out from its center. The radius of the central hole is 3 cm and the width of the washer is 4 cm. Its height is 10 cm. Find its volume and total surface area.

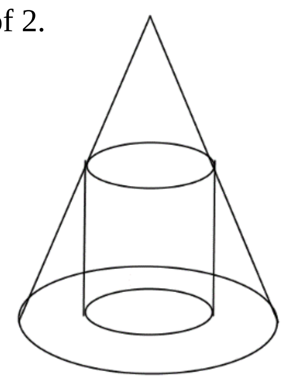


3. A circle with radius 12 has a 60° sector cut out of it. It is then folded so that the two "exposed" radii are joined, creating a cone. Find the total surface area and volume of the cone, and then find its vertex angle. 



4. A cone has a height of 12 and a base radius of 5. A cylinder is inscribed in the cone, such that its circular base is concentric ("has the same center") as the cone's base and has a radius of 2.

a. What is the volume of the cylinder?



b. Now, inscribed a cylinder whose base radius is 4 cm instead of 2 cm. Will its volume be larger or smaller? Support your answer with calculations!

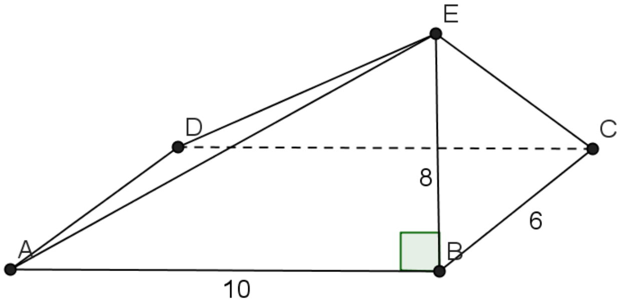
5. A spherical balloon is initially empty and is then inflated at a constant rate (of cubic cm of air per second). Ten seconds after its inflation started, it has a radius of 6 cm.

a. What rate is air being added to the balloon?

b. What will its radius be after another ten seconds? Andy thinks it will be 12 cm. Is he close?

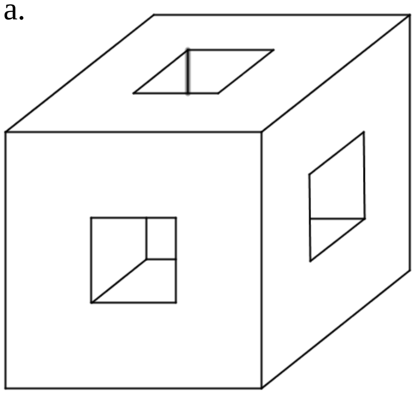
c. When will its radius reach 12 cm? Could you have known this based on similarity?

6. This pyramid below has a rectangular base (6-by-10). And point E is 8 units directly above point B. Find its volume and total surface area. Will the four faces be right triangles?

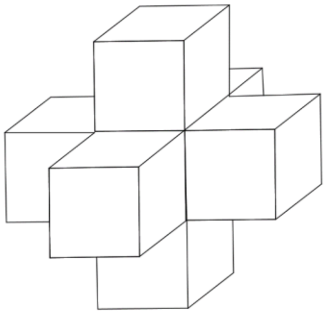


7. Find the volume and total surface area of each shape. The shape in part *a* is a 3-unit cube with holes through the center from each face whose cross-sections are 1-unit squares. The shape is part *b* is assembled of cubes with 1-unit sides—it is solid in the middle. Note: you can think of the shape in part *b* as what was cut out of the cube to end up with the shape in part *a*. (don't ask how they cut it out!)

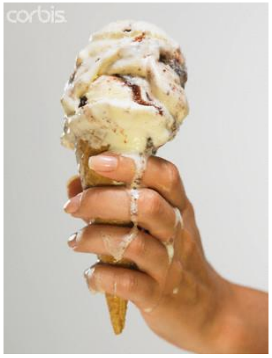
a.

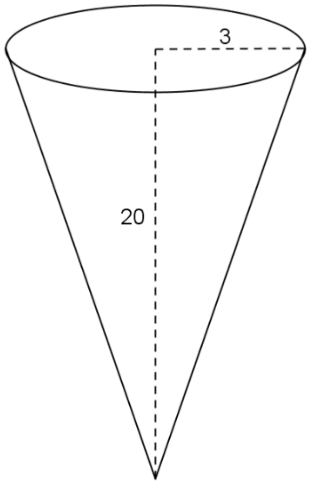


b.

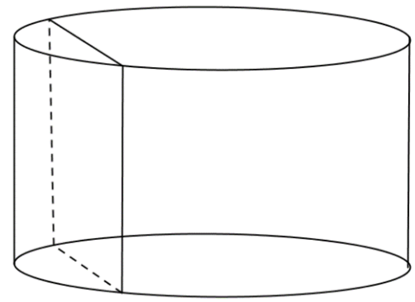


8. An ice cream cone has a radius of 3 cm and a height of 20 cm. A spherical scoop of ice cream with radius 4 cm is placed on the cone. But it melts before any gets eaten. Assume that 75% of the melted ice cream ends up in the cone, with the other 25% making a mess of the outside. Will it fill the cone? If not, how high from the bottom of the cone will the level of the melted ice cream be? You may assume that the volume of melted ice cream is the same as the volume of frozen ice cream, though this is probably not exactly the case!





9. *This one is literally a piece of cake!* A cake is shaped like a cylinder with a base radius of 12 cm and a height of 10 cm. Bobby takes a knife and slices himself a piece. But he does not cut if from the center. He cuts a chord 12 cm long on the top and slices straight down from there.

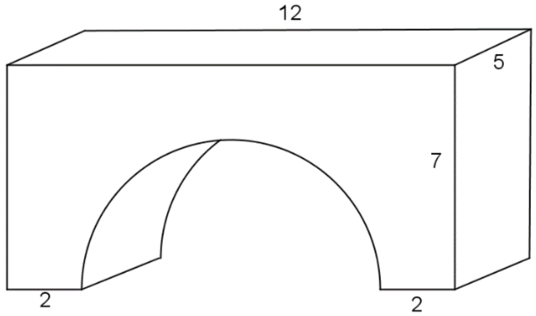


a. Find the volume and surface area of Bobby's piece of cake.

b. Bobby's sister Alexa takes a more conventional approach, slicing a wedge from the center (so a sector of the top circle and then straight down). For Alexa slice to have the same volume as Bobby's, what should the central angle of her slice be?

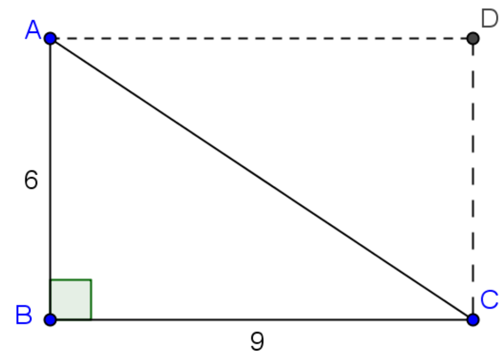
c. Assume the cake was frosted on the top and sides. Who got more frosting?

10. A block in a building set is shaped like a rectangular prism with a semi-circular (semi-cylindrical?) hole. Find its volume and total surface area.



11. Imagine a spherical fruit with a big spherical pit in the center. Three cuts are made at right angles to each other to remove the one-eighth of the fruit, exposing part of the pit as in the picture below. If the radius of the fruit is 5 cm and the radius of the pit is 2 cm, then find the volume and surface area of the remaining shape.

12. Answer the questions below:



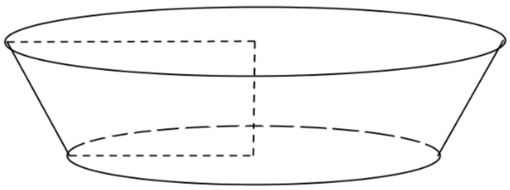
a. Rectangle ABCD is revolved around side $\overline{AB}$. What shape is created? What is its volume? What is its total surface area?

b. Triangle ABC is revolved around side $\overline{AB}$. What shape is created? What is its volume? What is its total surface area?

c. What is the volume of the shape created by revolving ΔADC around $\overline{AB}$?

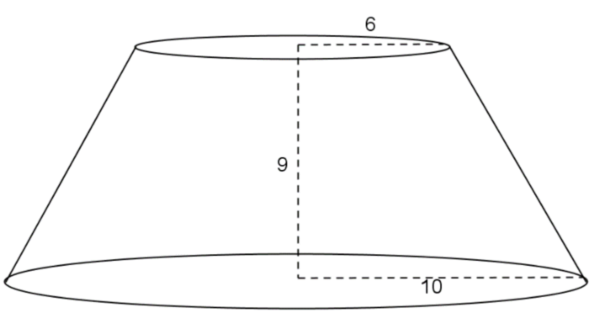
d. Triangles ABC and ADC are congruent. Why is the volume created in *c* greater than the volume created in *b*?

13. Peach Pie! Brandon makes a peach pie. The pie dish is 3 inches high, with a radius of 6 inches at the bottom and eight inches at the top. Each peach is a sphere with radius 2 inches, having a spherical pit with radius ½ inch. How many peaches are needed to fill the pie dish?



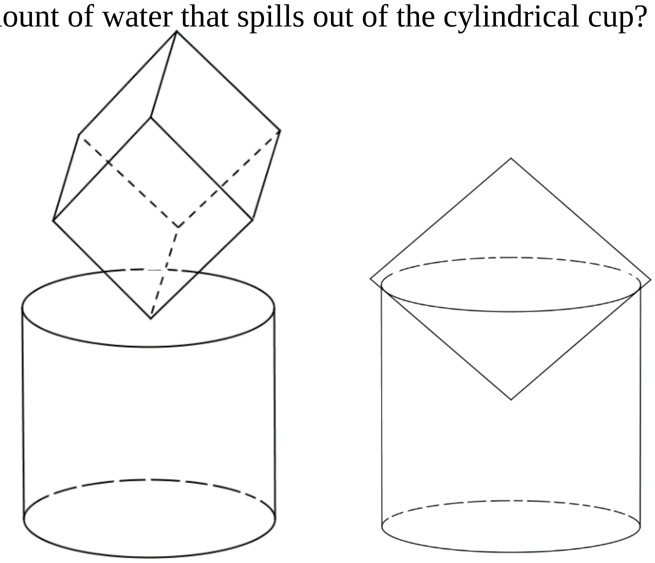
14. A lampshade is a frustum of a cone. Steve wants to paint the outside.

a. What is the area of the outside? (This does not include the top or bottom)



b. Steve accidently sliced the lampshade open along a line segment from the top circle to the corresponding point in the lower circle. He unrolls it so that it is flat. What shape does he have?

15. A cylinder with radius 4 and height 10 is filled with water. A solid cube with side 6 is dunked into the cylinder in a way such that a diagonal of the cube is vertical and lines up with the center of the cylinder. The cube is dunked until it stops, as its sides hit against the edges of the cylindrical cup. What is the amount of water that spills out of the cylindrical cup? This one is very tough!



Answers

1a. 1000π cubic cm b. 125 π cubic cm c. 1/8th – no since it is ½ as wide, ½ as deep, and ½ as tall
d. 400π = (1/3)* π *(h/3)^2 * (h) so h^3 = 10800 and h is about 22.1 cm

2. volume = 400π cubic cm surface area = 40π + 60π + 140π +40π = 280π

3. lateral surface area= 120π; base radius = 10 so total surface area = 220π
Height = 2*sqrt(11) so volume = (1/3)*π*100*2*sqrt(11) which is about 694.6
Vertex angle : sinx = 10/12 so x=56.44° and vertex angle is twice this, or about 113°

4a. hgt = 7.2 (since h/3 = 12/5) so vol is ≈28.8π or 90.5 b. hgt=2.4 so vol = 38.4π or ≈120.6 so bigger

5a. volume at 10 seconds (perhaps V(10) is notation you've seen!) = (4/3)π*6^3 = 288π
So 28.8πcubic cm per second – or about 90.5
b. he's way off… volume will be 576π so 576π = (4/3)*π*r^3 and r^3=432 so r is about 7.56 cm
c. volume = (4/3)*π*12^3 = 2304π → 2304/28.8 = 80 seconds.
yes, since double radius means balloon is 2^3 times as large—so 8 times volume and thus 80 sec.

6. V=160 and SA=$174+6\sqrt{41}$ 7a. V=20 SA=72 b. V=7 SA=30

8. Volume of ice cream is 64π; volume of cone is 60π; almost fits but not quite.

9a. area of segment on top is about 13 sq cm so V= 130 cc and SA=272 cm sq b. 10.3°
c. Bobby's frosting area = 13+40π=138.7 sq cm ; Alexa's is 34.7 → wow!

10. area of front = rect minus ½-circle = 84-(0.5)π*(4^2) = 84-8π or ≈58.9 so vol= 58.9*5 = 294.5
Surface area: front+back = 117.8; sides are each 35 so 70; top is 60;
Bottom is 2 rectangles (each 2-by-5) plus lateral surface area of ½ cylinder = 20π
So total surface area is about 330.6

11. volume = 7/8th of original plus 1/8th of pit = $\frac{7}{8}\cdot\frac{4}{3}\pi\cdot 5^3+\frac{1}{8}\cdot\frac{4}{3}\pi\cdot 2^3\approx 462.34$

SA: 7/8th of original fruit + 1/8th pit + 3 "rainbows"=$\frac{7}{8}\cdot 4\pi\cdot 5^2+\frac{1}{8}\cdot 4\pi\cdot 2^2+3\left(\frac{\pi 5^2-\pi 2^2}{4}\right)\approx 330.65$

12a. a cylinder with volume 486π and surface area 270π b. cone; V=162π and SA=81π+$27\pi\sqrt{13}$
c. 324π d. more of the area of ABC is closer to AB and creates less volume when revolved

13. V=$\frac{1}{3}\pi\cdot 8^2\cdot 12-\frac{1}{3}\pi\cdot 6^2\cdot 9=148\pi$; each peach is $\frac{4}{3}\pi\cdot 2^3-\frac{4}{3}\pi\cdot(0.5)^3=10.5\pi$ so 296/21 or 14.1

14a. height of original cone is 22.5 by similar triangles: x/6=(x+9)/10 so x=13.5..
Lateral surface area of original cone = $\pi rl=\pi\cdot 10\cdot\sqrt{10^2+22.5^2}\approx 773.53$
Part of cone missing has height = 13.5 and base radius=6 so slant height is 14.77
So subtract $\pi\cdot 6\cdot 14.77=278.47$ and get about 495.1

14b. something like the letter C… a washer with a sector cut out of it

15. The water that comes out is the volume of the pyramid that sticks into the cylinder. It has an equilateral triangle base and three faces that are isosceles right triangles. (to visualize, hold a cube so one corner is down and imagine cutting it with a plane… the plane makes equal cuts on three sides).
How big is the pyramid? The largest equilateral triangle that fits into a circle with radius 4 has a side of $4\sqrt{3}$. The faces have hypotenuse of $4\sqrt{3}$ and legs of $4\sqrt{1.5}$. The height of the pyramid is $2\sqrt{2}$ so its volume is $\frac{1}{3}\cdot 12\sqrt{3}\cdot 2\sqrt{2}=8\sqrt{6}$

Unit 9 Handout #7: Review Problems

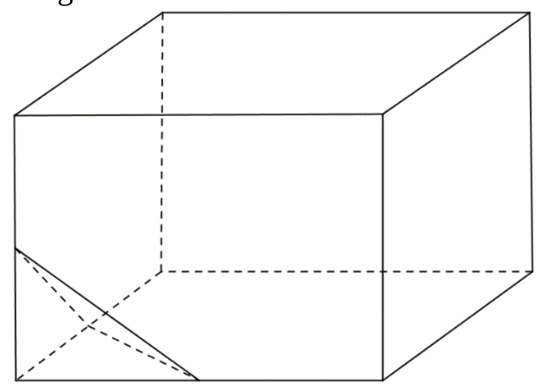
Formulas:	Volumes:	$base \cdot height$ $\frac{1}{3} \cdot base \cdot height$ $\frac{4}{3}\pi r^3$	surface areas:	$2\pi rh$ πrl $4\pi r^2$

1. A rectangular prism is 10 units wide, 6 units deep, and 8 units high.

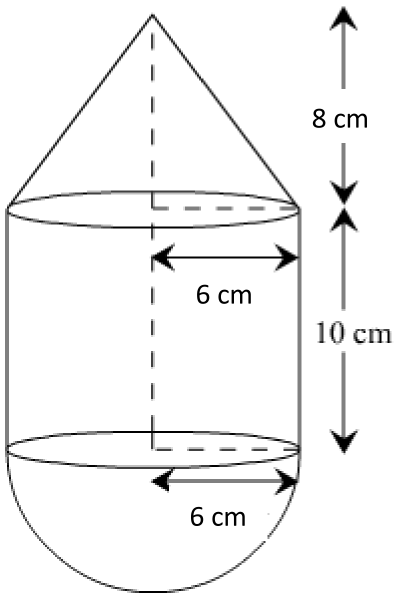
   a. Find its volume and total surface area.

   b. Find the length of its diagonal (from one corner, through the center, to the opposite corner).

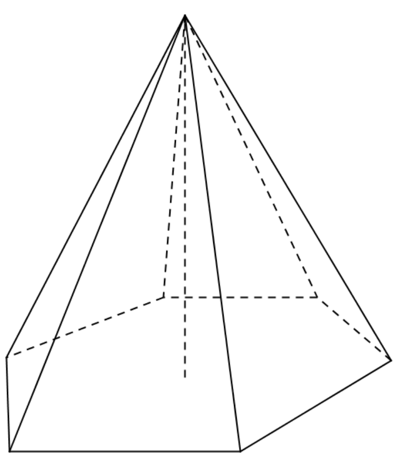
   c. The midpoints of three edges near one corner are connected, forming a pyramid with a triangular base. Find its volume.



2. Find the volume and total surface area of the figure below. It consists of a cone, a cylinder, and a hemisphere.



3. The base of a right pyramid is a regular hexagon with side length of 8. The height is 10. Find its total surface area and volume. (no calculator needed)

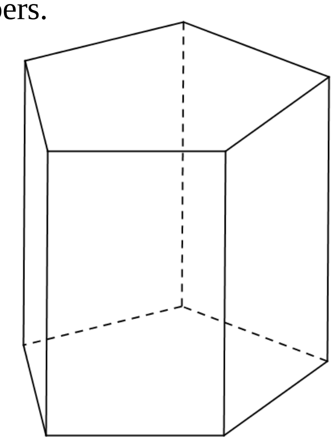


4. A right prism's base is a regular pentagon of side 8 and its height is 10. It is on the right above.

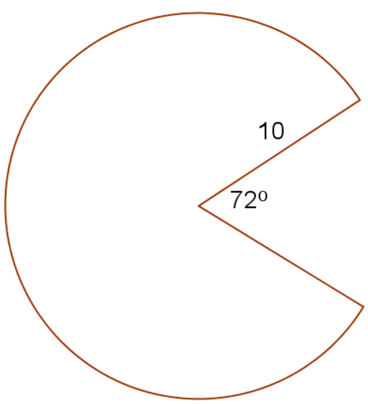
a. Find the volume and total surface area of the prism.

b. A cube has the same volume; find the length of the cube's side.

c. Which shape has a smaller total surface area? Support your answer with numbers.

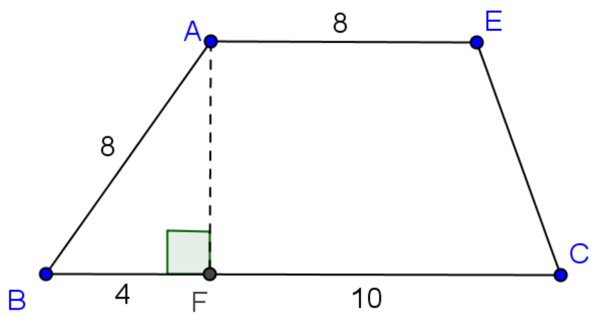


5. The sector of a circle on the left below is folded into a cone by joining the two exposed radii. Find the volume and vertex angle of the cone.



6. A birthday cake is a cylinder with a radius of 12 cm and a height of 10 cm. Julie takes a slice with a 30 degree central angle. What is the volume and total surface area of her slice?

7. Find the volume of the solid created when trapezoid ABCE below is revolved around $\overline{BC}$.

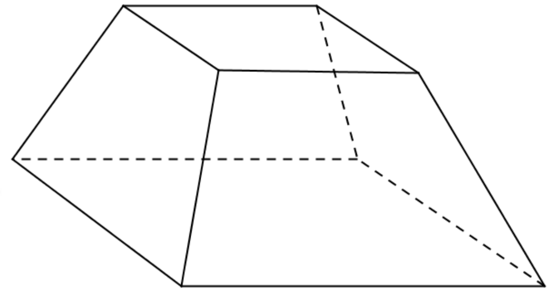


8. A right pyramid with a square base of side 10 is sliced by a plane parallel to the base, creating a frustum. The sides of the upper square base are each 6 units and the height (measured perpendicular to the bases) is 5.

a. How tall was the original pyramid?

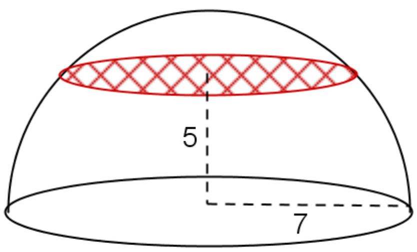
b. Find the frustum's volume and total surface area.

c. Find the angle that one of the faces makes with the lower base.

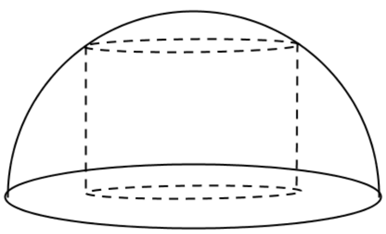


9. A hemisphere has a radius of 7.

a. A circle is parallel to the base, 5 units away. What is the area of the circle?



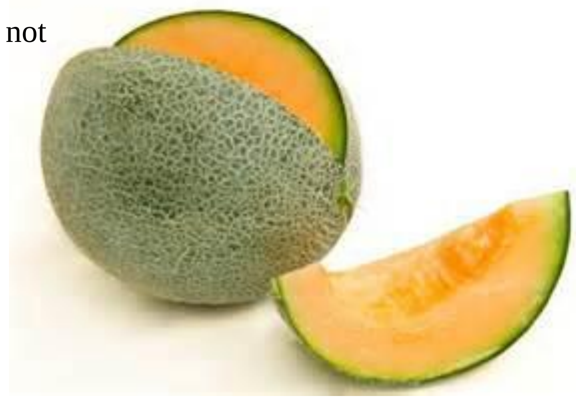
b. A cylinder with a base radius of 4 is inscribed in the hemisphere. What is its volume?



10. A balloon is inflated at a constant rate. As it inflates, it is always a sphere. After 10 seconds its radius is 5 cm. How long does it take until its radius is 8 cm?

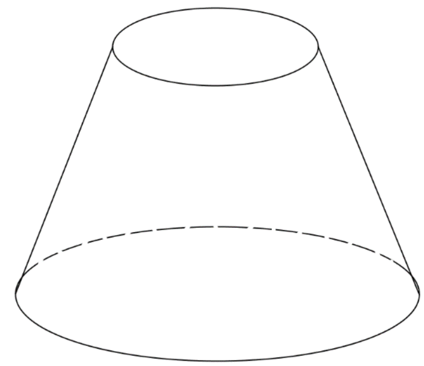
11. A cantaloupe has a diameter of 8 inches; the seed chamber in the center has a diameter of 4 inches; its rind is ¼ inch thick. A slice is taken from the cantaloupe that represents one-eighth of the melon.

a. What is the volume of the edible part of the entire melon (do not include the rind or the seed chamber)?

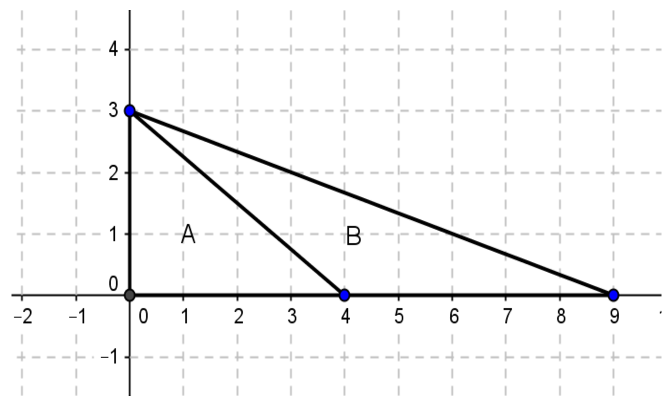


b. What is the surface area of the slice (include the rind and the melon)?

12. The frustum of a cone is shown on the right. The radius of its top circle is 4 cm and the radius of its lower circle is 6 cm. Its slant height is 8. Find its volume and total surface area. Hint: use similar triangles to find the height of the cone before it was cut!



13. Region A is the right triangle whose vertices are (0,0), (0,3), and (4,0). Region B's vertices are (4,0), (9,0), and (0.3). Find the volume of solids created by…



a. Revolving A around the *x*-axis.

b. Revolving A around the line *x*=4.

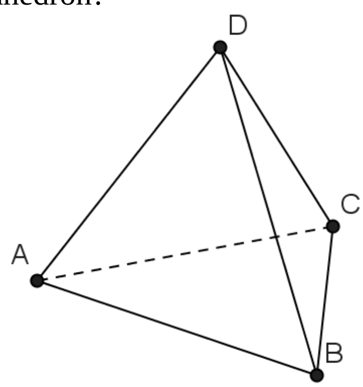
c. Revolving B around the *x*-axis.

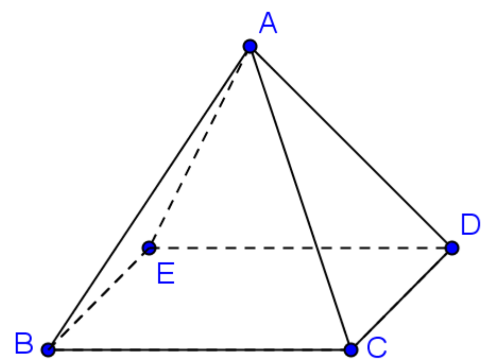
d. Revolving B around the *y*-axis.

e. Revolving B around the line x=9.

f. Revolving A around the line x=-2.

14. A tetrahedron is a right pyramid with an equilateral triangle base (see left below). A tetrahedron's six edges are all 8 units long. A right pyramid with a square base has eight edges—all are 8 units long. Find the volume of both. What is the ratio of the volume of the square-based pyramid to the volume of the tetrahedron?





Answers

1a. volume = 480 cubic units; surface area = 376 square units b. $10\sqrt{2}$ c. 10

2. volume = $\frac{1}{3}\pi \cdot 6^2 \cdot 8 + \pi \cdot 6^2 \cdot 10 + \frac{2}{3}\pi \cdot 6^3 = 600\pi$ SA = $\pi \cdot 6 \cdot 10 + 2\pi \cdot 6 \cdot 10 + 2\pi \cdot 6^2 = 252\pi$

3. area of base = $96\sqrt{3}$ so volume = $320\sqrt{3}$ surface area= $96\sqrt{3} + 48\sqrt{37}$
(these numbers seem familiar; have we done this one before?)

4a. base apothem = 5.51 (4/tan36) so area of base=110.2 and V = 1102 so SA= 2(110.2) + 5(80)=620.4

b. $s^3 = 1102$ so s is about 10.33 c. cube's SA = 6*(10.33)^2 = 640.1 so pentagonal prism has small SA

5. circumf of cone's base circle = (288/360)*(20π)=16π so cone's radius = 8 and its height is then 6, using the Pythagorean Theorem. So volume = $\frac{1}{3}\pi 8^2 \cdot 6 = 128\pi$; vertex angle is $2 \cdot \sin^{-1}(8/10)$ =106.3°

6. she takes 1/12th of the cake so volume = $\frac{1}{12} \cdot \pi \cdot 12^2 \cdot 10 = 120\pi$

SA: top & bottom are sectors of circle and each is $\frac{1}{12} \cdot \pi \cdot 12^2 = 12\pi$; sides are rectangles: each is 120

curved edge is 1/12 lateral SA of cylinder = $\frac{1}{12} \cdot 2\pi \cdot 12 \cdot 10 = 20\pi$ so total surface area= 44π+240

7. Cone-cylinder-cone; all have radii of $4\sqrt{3} = \sqrt{48}$ so V= $\frac{1}{3}\pi \cdot 48 \cdot 4 + \pi \cdot 48 \cdot 8 + \frac{1}{3}\pi \cdot 48 \cdot 2$ =480π

8. height of pyramid before it was cut… 12.5 using similarity (x/3)=(x+5)/5 Or (2/5)=(5/x)

So volume = $V_{big} - V_{small} = \frac{1}{3} \cdot 100 \cdot 12.5 - \frac{1}{3} \cdot 36 \cdot 7.5 = 326.67$

surface area: top=36; bottom=100; slant height of frustum = $\sqrt{29}$ so each face= $8\sqrt{29}$

the total surface area is $136 + 32\sqrt{29}$; Angle at which face makes with lower base: $\tan^{-1}(5/2) = 68.2°$

9a. 24π using Pythag b. height is $\sqrt{33}$ using Pythag so $16\pi\sqrt{33}$

10. short way: using similarity→ volume is based on radius cubed so 10*(8/5)^3 = 40.96 seconds

Long way: in 10 seconds volume is $\frac{4}{3}\pi \cdot 5^3 = 523.6$ cubic cm so inflates at 52.36 cc per second

Radius of 8 cm means volume is 2144.66 cc; so 2144.66/52.36 = 40.96 seconds

11a. $\frac{4}{3}\pi \cdot 3.75^3 - \frac{4}{3}\pi \cdot 2^3 \approx 187.4$ cubic inches b. 1/8 of big sphere + 1/8 of small sphere + 2 rainbows:

$\frac{1}{8} \cdot 4 \cdot \pi \cdot 4^2 + \frac{1}{8} \cdot 4 \cdot \pi \cdot 2^2 + \pi(4^2 - 2^2) = 22\pi$

12. V=616.478 and SA=132π

13a. 12π b. 32π c. 15π d. 65π e. frustum – cone = 70π f. frustum-cylinder = 40π

14. pyramid base=64 and hgt = $4\sqrt{2}$ so volume = $\frac{256\sqrt{2}}{3}$

Tetrahedron base= $16\sqrt{3}$ and hgt= $8\sqrt{\frac{2}{3}} = \frac{8\sqrt{6}}{3}$ so volume= $\frac{128\sqrt{2}}{3}$ → exactly ½ the other one!

Fall Geometry 2: Preparation for Final Exam

Unit 6: Right triangles
- Similar right triangles
- Pythagorean Theorem
- Special right triangles
- Trigonometry

Unit 7: Circles
- Tangent lines
- Inscribed and central angles
- Arc length

Unit 8: Area
- Triangles and rectangles
- Parallelograms and trapezoids
- Regular polygons (apothems etc)
- Sectors and segments of circles
- Scale and area

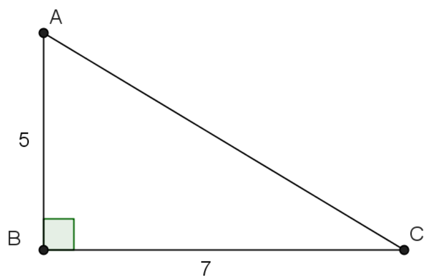
Unit 9: Volume and Surface Area

Overall themes
-algebra: solving equations (linear and quadratic); solving systems of linear equations
-word problems
-coordinate plane: graphing lines; distance formula; finding intersections

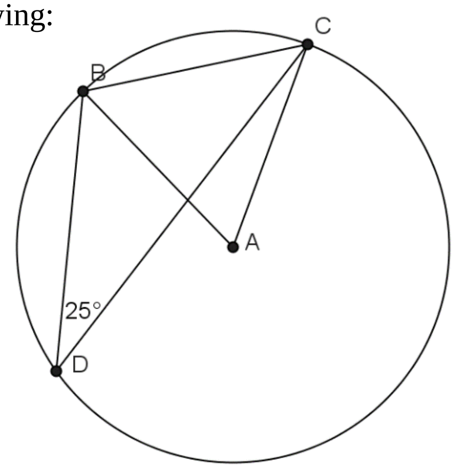
Basic questions

1. Find the surface area and volume of a cube where the diagonal of one of the faces is 6 units long.

2. Find the length of $\overline{AC}$ and the measure of angle C in the right triangle below. 



3. Circle A below has a radius of 8. Arc BD measures 90°. Find the following:



a. The measure of angle BAC.

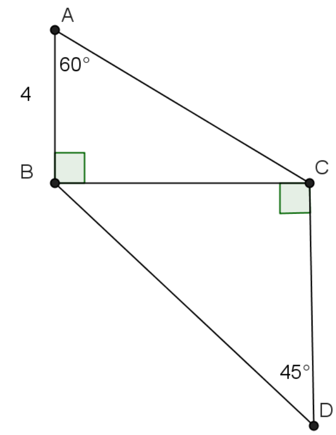
b. The measure of angle ABC.

c. The length of arc BC.

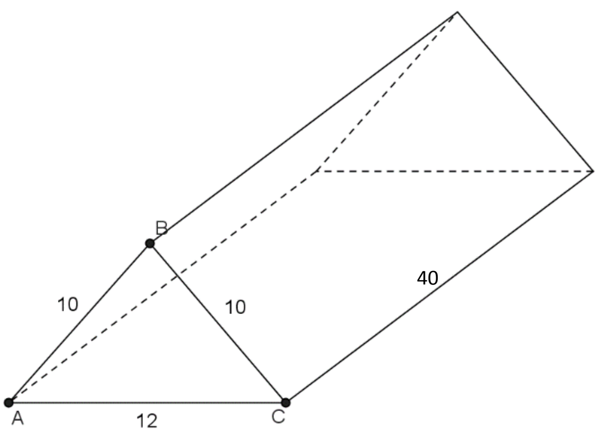
d. The area of sector ABC.

e. The measure of angle DCA.

4. Find the area and perimeter of the trapezoid below.

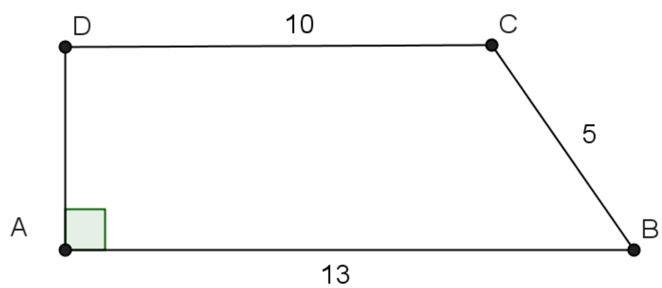


5. Find the volume and surface area of the right triangular prism below.



6. Find the following in the trapezoid below:

a. The area



b. The perimeter.

c. The length of diagonal $\overline{BD}$.

7. Given a regular 12-sided polygon with perimeter of 60, do the following. 

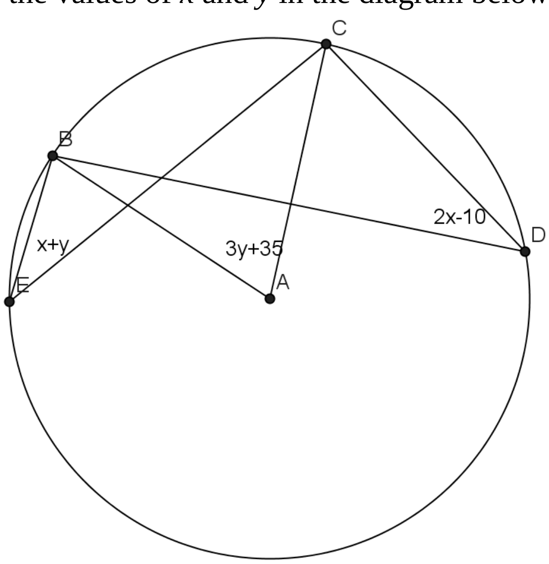
a. Find the apothem.

b. Find the radius

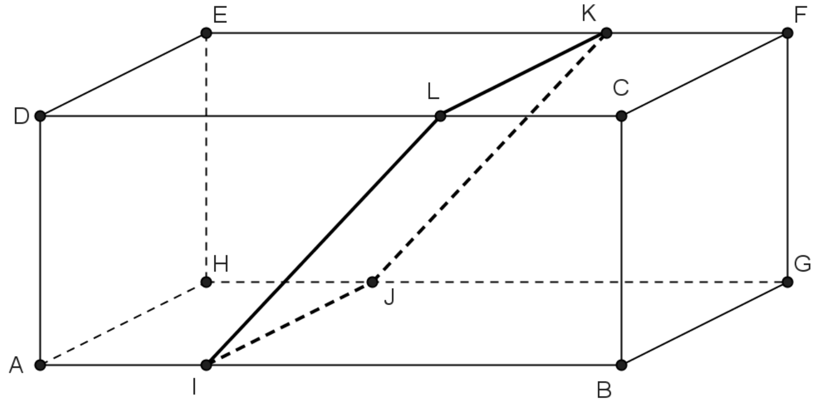
c. Find the area.

d. Assume that this polygon is the base of a right pyramid with height of 10. Find the volume and total surface area of the pyramid.

8. Find the values of *x* and *y* in the diagram below.



9. The rectangular prism below is sliced by a plane (IJKL). Given that AB=12, BG=6, BC=5, KF=LC=4, and IB=GJ=7, find the following:



a. The volume of the rectangular prism.

b. The length of diagonal $\overline{AF}$.

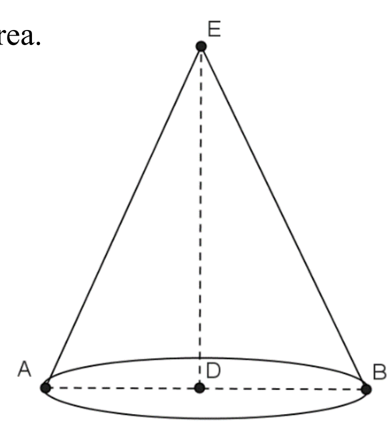
c. The measure of angle BAG. 

d. The volume of right trapezoidal prism IBGJCLKF.

e. The surface area of right trapezoidal prism IBGJCLKF.

f. The measure of angle BIL. 

10. The cone below has height 10 and volume 30π. Find its total surface area.

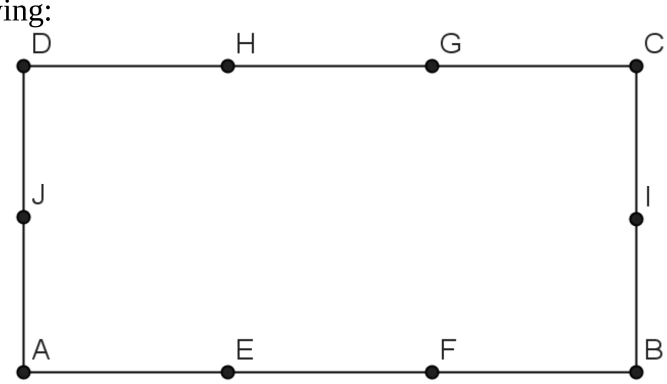


11. Find the following without your calculator.

a. $\sin 30^\circ$ b. $\cos 45^\circ$ c. $\tan 60^\circ$

12. In rectangle ABCD, points E, F, G, and H trisect sides $\overline{AB}$ and $\overline{CD}$. Points I and J bisect sides $\overline{BC}$ and $\overline{AD}$. Given that the area of triangle CEB is 30, the following:

a. The area of rectangle ABCD.



b. The area of triangle JEB.

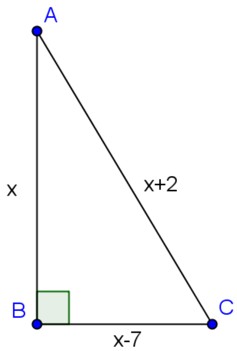
c. The area of trapezoid AGCF.

13. A solid three-dimensional object is cut into two pieces by a plane.

a. Is the sum of the volume of the two pieces equal to the volume of the original object? Explain.

b. Is the sum of the surface areas of the two pieces equal to the surface area of the original object? Explain.

14. Find the value of x in the right triangle below, and then find the measure of angle A. 



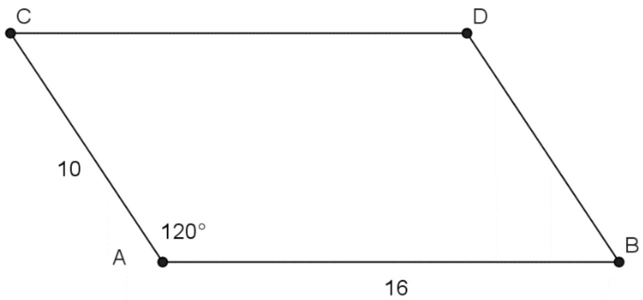
15. The propane tank below is a cylinder with hemispheres at both ends. The diameter of the cylinder is 4 feet and the length of the tank (from end to end) is 20 feet.

a. Find its volume and surface area.

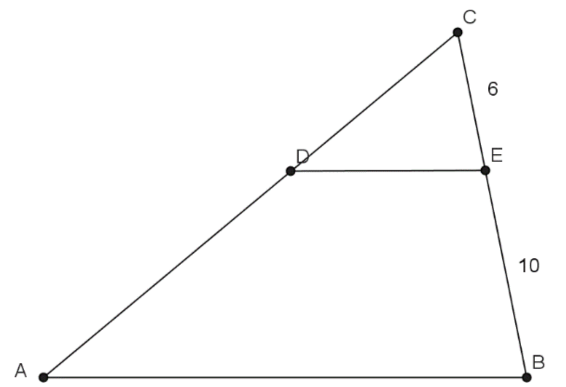
b. If a spherical container were to hold the same volume, then what would its radius be?

16. A rectangle's width exceeds its length by 5 units. When its width and length are each increased by 3 units, its area is increased by 34 units. Find its original area.

17. Find the area and perimeter of the parallelogram below: Then find the length of diagonal $\overline{DA}$.



18. In the triangle below, $\overline{DE} // \overline{AB}$. The area of ΔABC is 160.



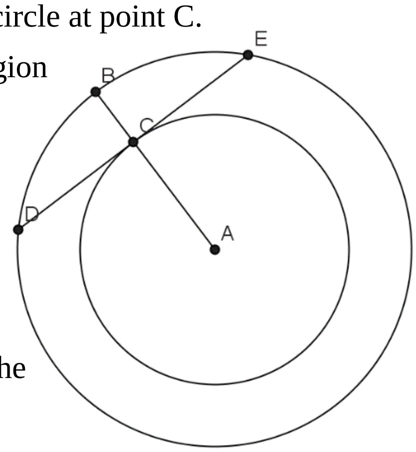
a. Find the area of ΔCDE.

b. Find the area of trapezoid ABED.

c. Find the length of the altitude from A to side $\overline{BC}$ in triangle ABC.

Harder Questions

19. Two circles are concentric, in that they have the same center (point A). The length of $\overline{BC}$ is 2. Segment $\overline{ED}$ is a chord of the larger circle and is tangent to the smaller circle at point C.

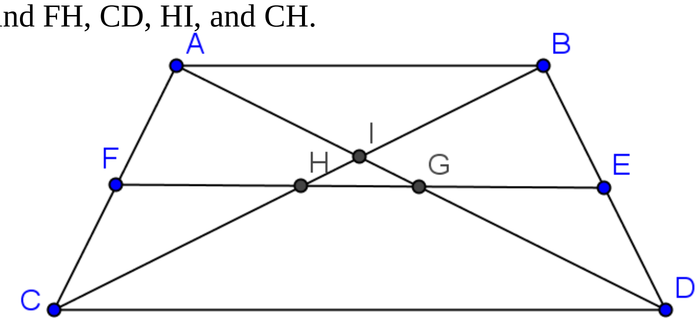


a. If the length of AC is 5, then find the area of the "washer"—the region inside the larger circle but outside the smaller circle.

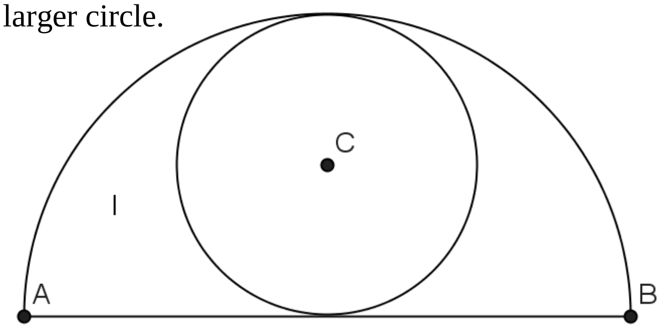
b. Instead, if the area of the washer is 30π, then find the length $\overline{AC}$ (the length $\overline{BC}$ is still 2).

c. Instead, if the length of chord $\overline{DE}$ is 10, then find the length of $\overline{AC}$ (the length of $\overline{BC}$ is still 2).

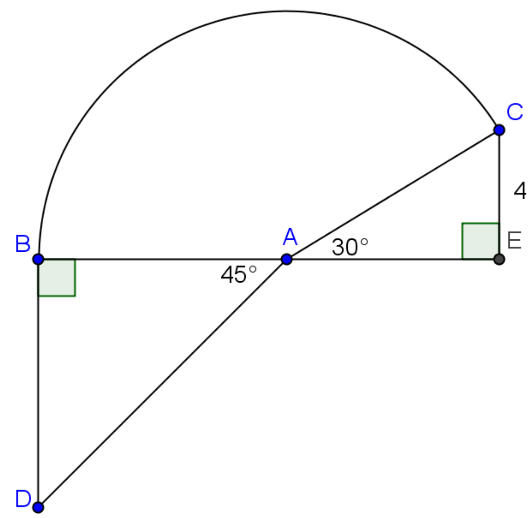
20. $\overline{EF}$ is the midsegment of trapezoid ABDC below (meaning it joins the midpoints of the opposite side). Given than AB=20, HG=5, DE=8, and BI=12, find FH, CD, HI, and CH.



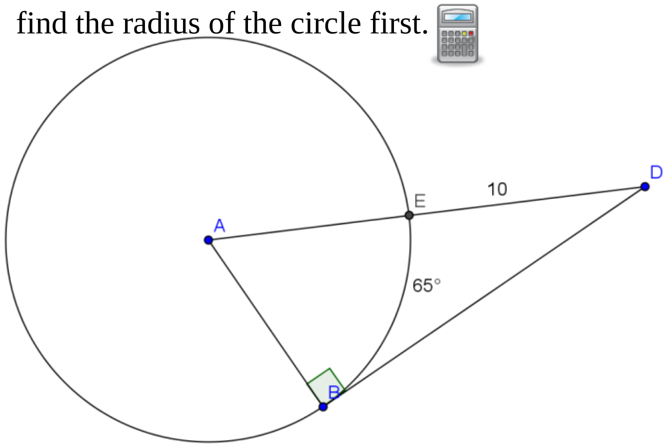
21. Find the area and perimeter of region I below, given that the radius of the small circle is 6. $\overline{AB}$ is a diameter, and the small circle is tangent to $\overline{AB}$ and to the larger circle.



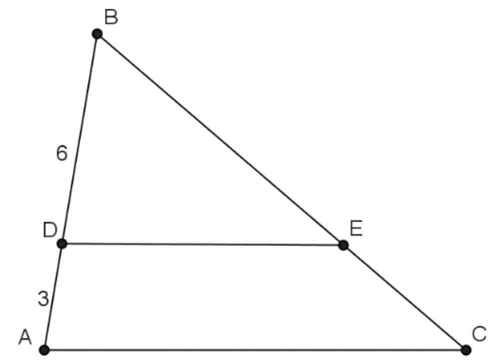
22. Find the area and perimeter of the shape below. The curved shape is the sector of a circle whose center is A.



23. Find the length of tangent segment $\overline{DB}$ given that $\overline{ED}$ is 10 and minor arc BE measures 65°. Hint: find the radius of the circle first. 



24. In the triangle below (not to scale), $\overline{DE} \parallel \overline{AC}$.



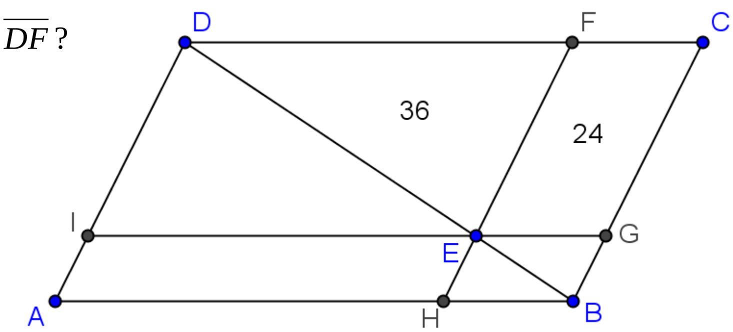
a. What is the ratio of the length of $\overline{DE}$ to the length of $\overline{AC}$?

b. What is the ratio of the area of ΔBDE to the area of ΔABC?

c. What is the ratio of the area of trapezoid ACED to the area of ΔBDE?

d. If the area of ΔABC exceeds the area of ΔBDE by 18 units, then find the area of ΔABC.

25. In the diagram below, $\overline{FH}$ *and* $\overline{GI}$ are parallel to the sides of parallelogram ABCD. The area of triangle DEF is 36 and the area of quadrilateral EFCG is 24. Answer the following:



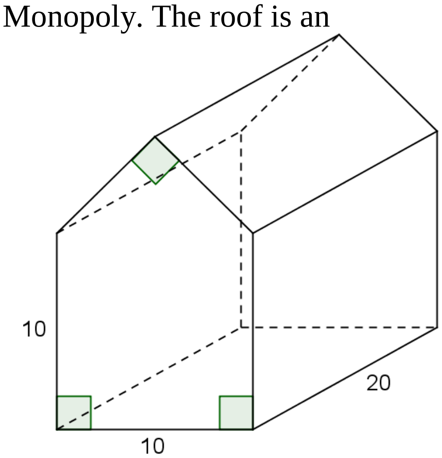
a. What is the ratio of length of $\overline{FC}$ to the length of $\overline{DF}$?

b. Must BGEH be similar to EFDI?

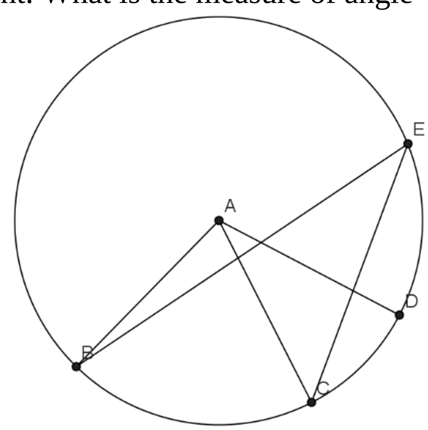
c. What is the area of ΔBEG?

d. What is the area of AIEH?

26. Find the volume and surface area of this hotel from the board game Monopoly. The roof is an isosceles right triangle.

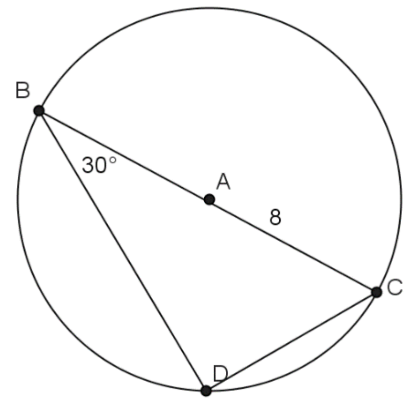


27. Minor arc BD measures 105° and angles CAD and BEC are congruent. What is the measure of angle BAC?

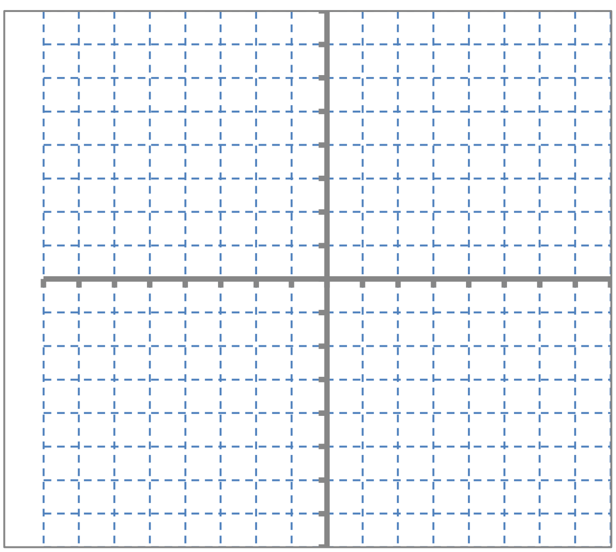


28. The nuts below are approximately right hexagonal prisms with cylindrical holes drilled though their centers (disregard the threading and the rounded edges). If each side of the hexagon is 12 mm, the depth of the nut is 10 mm, and the diameter of the hole is 16 mm, then find the volume and total surface area of one nut.

29. The radius of circle A below is 8 units. Find the area and perimeter of the small segment of the circle created by chord $\overline{CD}$.

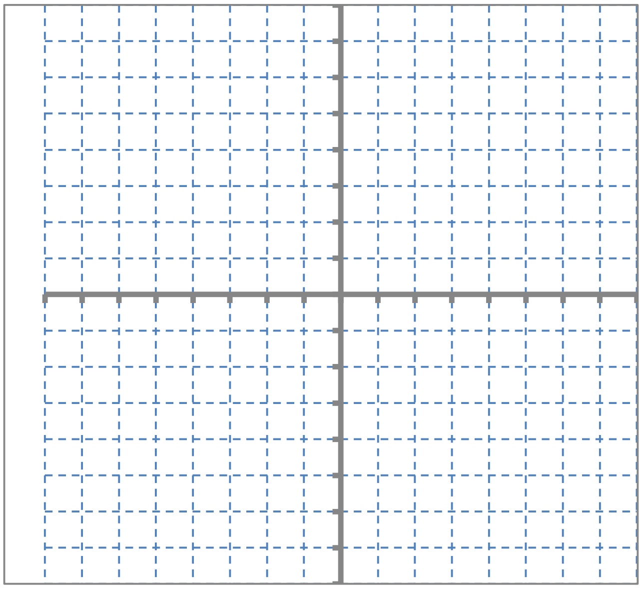


30. Find the area of the trapezoid bound by the lines $y = 1,\ y = 4,\ y = x + 3,\ and\ y = 5 - 0.5x$



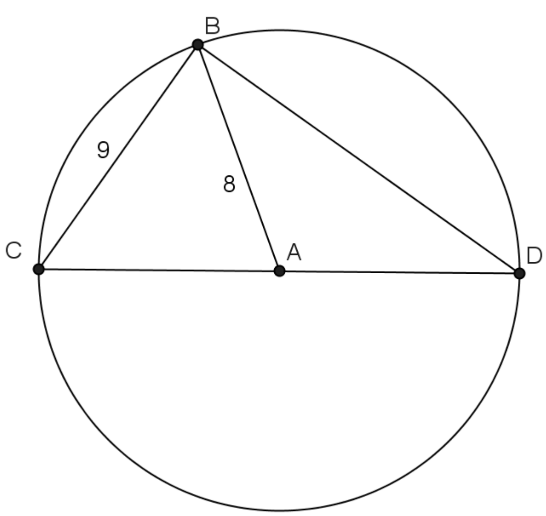
31. Find the area and perimeter of the triangle whose vertices are (0,0), (-3,4), and (-8,-5). Plot the points on the grid above.

32. A rectangle is bound by the lines y=1, y=6, x=1, and x=8. The line y=0.5x divides the rectangle into two pieces. What is the area of the larger piece?

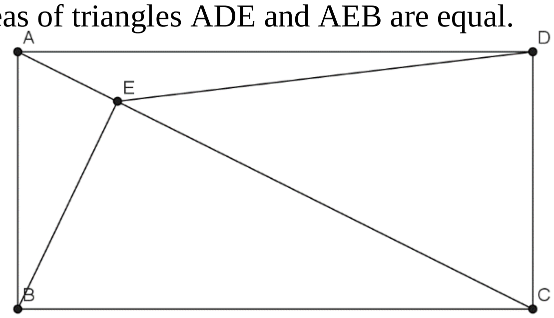


33. Triangle ABC's area is 12. Point A's coordinates are (-7,-2) and point B's coordinates are (-1,-2). Point C is somewhere on the line $y = -2x + 2$. Find the two possible coordinates of C. Use the grid above.

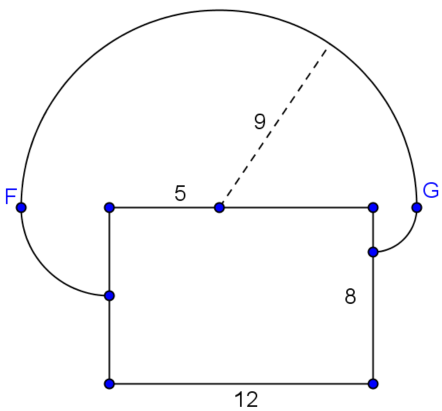
34. Find measure of angle D and the length of chord $\overline{DB}$ below.



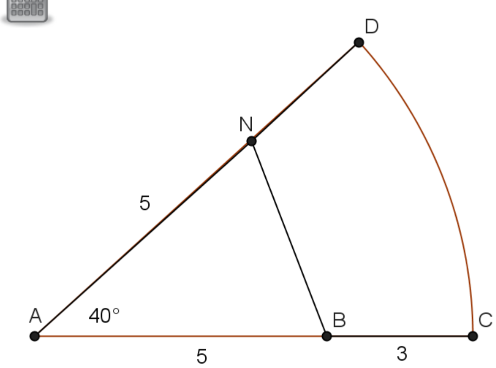
35. E is on diagonal $\overline{AC}$ of rectangle ABCD. Explain why the areas of triangles ADE and AEB are equal.



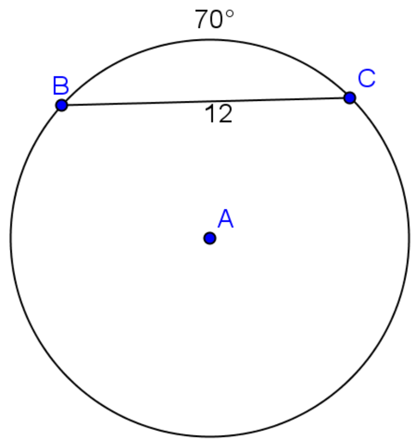
36. A dog is chained to a 12-by-8 meter rectangular building with a chain 9 meters long. The chain is attached to the long side of the house, 5 meters from one corner. What is the area of the region that the dog can get to (outside of the house)? Hint: draw $\overline{FG}$.



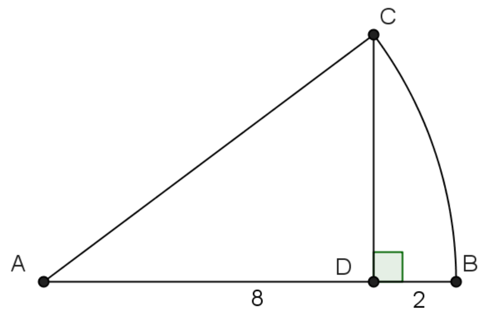
37. Given sector of circle ACD, find the area and perimeter of BCDN. 



38. In the circle below, the chord measures 12 and the arc is 70°. Find the length of the arc. Decimal approximation ok!



39. Given that ABC is a sector of a circle centered at A, find the area and perimeter of DBC.

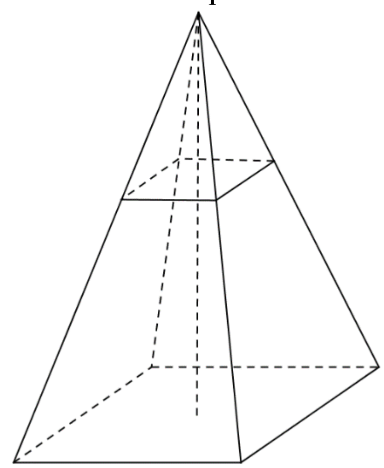


40. A pot is a cylinder with a base radius of 6 inches and a height of 7 inches. It currently has 180π cubic inches of water in it.

a. Ten spherical meatballs are placed in the pot. Each meatball has a radius of one inch. They sink to the bottom. How much does the water level of the pot rise? (assume they do not absorb any water or create any liquid).

b. How many ***more*** meatballs can be placed in before the pot overflows?

41. The pyramid below is a right pyramid with a square base of side 6 and a height of 12. A plane intersects the pyramid to create smaller square; it is parallel to the base and has a side length 2. The “large pyramid” refers to the whole figure, while the “small pyramid” refers to the one with the same apex and the base whose sides are 2 units long.



a. What is the ratio of the height of the large pyramid to that of the small pyramid?

b. What is the ratio of the area of one of the triangular faces of the large pyramid to the area of one of the faces of the small pyramid?

c. What is the ratio of the volume of the large pyramid to the volume of the small pyramid?

d. Find the total surface area and volume of the large pyramid (you didn’t need to do it to answer any of the above questions!)

e. Find the length of the edge of the large pyramid: the segment from the apex to one of the corners of the square base.

f. At what angle does the face of the large pyramid meet its base? 

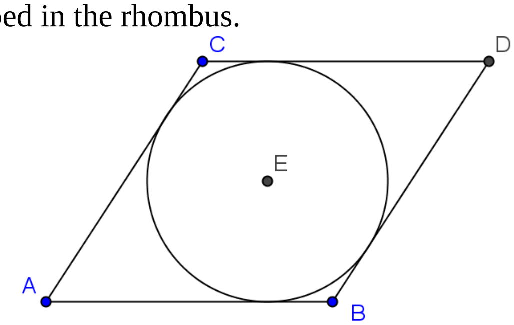
42. A square is transformed: its length is doubled and its width is reduced by three units.

a. If its area decreased by five square units, then what was the square's area?

b. Instead, if the sum of the areas of the original square and the new rectangle is 105, then what was the square's area?

43. The rhombus has sides of 4 and $\angle D$ measures 60°. A circle is inscribed in the rhombus.

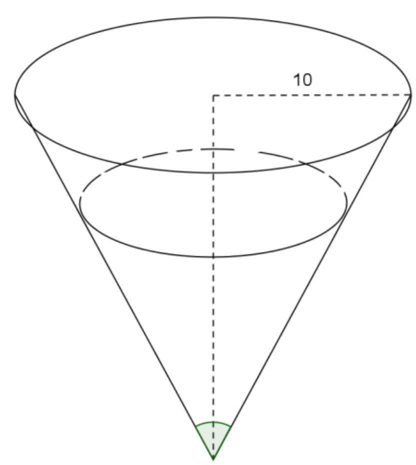
a. Find the radius of the circle.



b. Find the area of the region inside the rhombus but outside the circle.

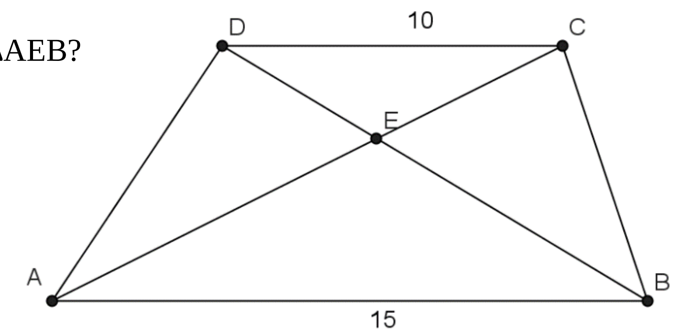
44. A conical cup has a radius at the top of 10cm and a vertex angle of 60°.

a. What is its volume?



b. Water is added to the cup until the water in the cup is exactly one-half of the cup's capacity. Find the radius of the exposed water. This can be done algebraically or with similarity and scale.

45. In trapezoid ABCD below, the length of $\overline{AC}$ is 15.



a. What is the ratio of the area of ΔDEC to the area of ΔAEB?

b. Why are triangles ACD and BCD equal in area?

c. What is the ratio of the area of ΔACB to the area of ABCD?

d. What is the length of $\overline{EC}$?

e. What is the ratio of the area of ΔBEC to the area of ΔAEB?

f. What is the ratio of the area of ΔAEB to the area of ABCD?

Answers

1. volume = $54\sqrt{2}$ and surface area = 108 2. AC=$\sqrt{74}$ C is about 35.5°

3a. 50° b. 65° c. 20π/9 d. 80π/9 e. 20° 4. Area= $24+8\sqrt{3}$ Perim= $12+4\sqrt{3}+4\sqrt{6}$

5. Vol=1920 surface area = 1376 6a. 46 b. 32 c. $\sqrt{185}$

7a. 9.3 b. 9.7 c. 280 d. volume = 933 surface area = 690.2

8. x=25; y=15 9a. 360 b. $\sqrt{205}$ c. 27° d. 165 e. $151+6\sqrt{34}$ f. 59°

10. $9\pi+3\pi\sqrt{109}$ 11a. ½ b. $\frac{1}{\sqrt{2}}$ c. $\sqrt{3}$ 12a. 90 b. 15 c. 45

13a. yes b. no (increases by twice the area of the cut) 14. x=15 and A=28°

15a. V=224π/3 SA=80π b. 3.83 16. 100/9

17. area = $80\sqrt{3}$ perimeter=52 AD=14 18a. 45/2 b. 137.5 c. 20

19a. 24π b. 6.5 c. 21/4 20. FH=10; CD=30; HI=3; CH=15

21. area = 18π perim = 12+12π 22. area = $32+\frac{80\pi}{3}+8\sqrt{3}$; perim = $12+\frac{20\pi}{3}+8\sqrt{2}+4\sqrt{3}$

23. r=7.32 so 15.7 24a. 2/3 b. 4/9 c. 5/4 d. 162/5

25a. 1:3 b. yes, both are similar to BCDA since angles are equal and sides are in proportion c. 4 d. 24

26. vol = 2500 ; surface area = $850+200\sqrt{2}$

27. 70° 28. volume= $2160\sqrt{3}-640\pi$; surface area = $432\sqrt{3}+720+32\pi$

29. Area = $\frac{32\pi}{3}-16\sqrt{3}$; perimeter = $\frac{8\pi}{3}+8$ 30. 33/2 31. P=$5+\sqrt{89}+\sqrt{106}$; A= 23.5

32. 26 33. altitude must be 4 so (0,2) or (4,-6) 34. BD=$5\sqrt{7}$ and angle D = 34°

35. same base AE and same height so same area 36. 45.5π square meters

37. area is about 14.3 and perim is about 15.0 38. r is about 10.46 so arc length is about 12.78

39. area is about 8.18 and perim is about 14.44 40a. 10/27 inch b. 44 is OK; 45th results in overflow

41a. 3:1 b. 9:1 c. 27:1 d. vol=144 and surface area = $36+12\sqrt{153}$ e. $9\sqrt{2}$ f. 76°

42a. 25 b. 49 43a. $\sqrt{3}$ b. $8\sqrt{3}-3\pi$

44a. h=$10\sqrt{3}$ so V=$\frac{1000\pi\sqrt{3}}{3}$ b. algebra: x is radius so $x\sqrt{3}$ is hgt: $\frac{1}{3}\pi x^2 x\sqrt{3}=\frac{500\pi\sqrt{3}}{3}$ and x≈7.94.

Scale: if Volume is ½ then lengths are $\sqrt[3]{1/2}$ which is about 0.794 so 10*0.794 is about 7.94

45a. 4/9 b. same base and same height c. 3/5 d. 6 e. 2/3 f. 9/25