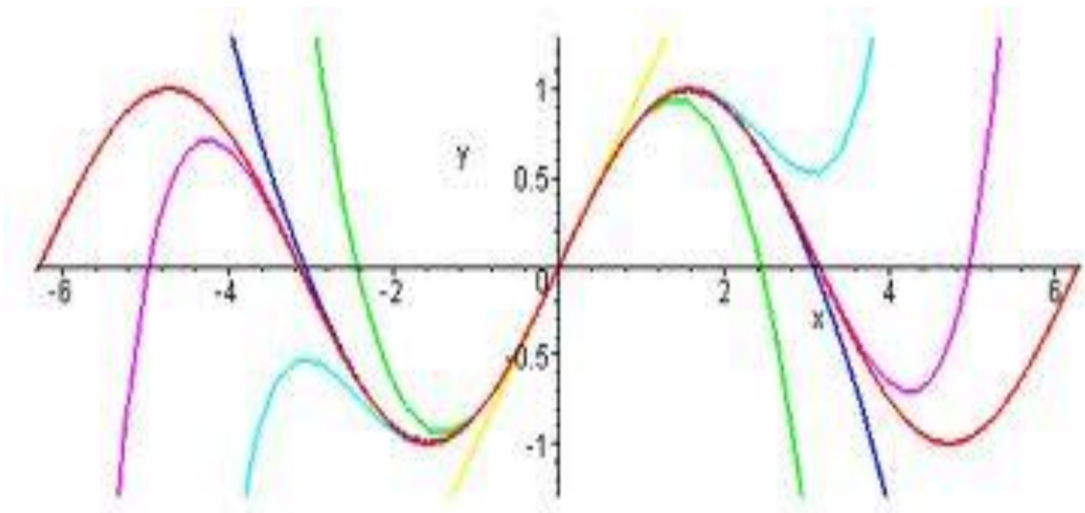
Calculus Part C



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} ....$$

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Advice for students:

1. This is an honors class. We will typically cover new material fairly quickly. Once we have gone over the key fundamentals with fairly rote problems, we will often apply the new material to more challenging non-routine problems. These can be difficult, and one of our main goals this semester is to help you develop important problem-solving skills. In the face of challenges, try to be persistent, patient, and creative. Be proactive about seeking help when you need it. We do not expect everyone to get all homework problems correct.

2. In sports, you have practices. In performing arts, you have rehearsals. In academics, you have homework. Homework in this class is designed to help you cement your understanding of the material by practicing straightforward problems and develop your problem-solving skills. Do your best to trying all the homework questions. If you have difficulty, come to the next class with specific questions that will help you advance your understanding.

3. Every problem set in this book comes with answers. They are at the end of each individual problem set. Checking your answers is essential. We recommend that you check answers after every few problems to make sure you are on the right track. We all make mistakes; please let us know if you think you have found an error in the answers.

4. HELP!!! Get help when you need it. Some places (in no particular order):
- Classmates
- Parents (?)
- Your teacher
- Internet (believe it or not, Google can help you find great explanations and practice problems)
- Khan Academy videos: goto www.khanacademy.org and scroll down to Calculus

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Unit 1 Handout #1: Integration Review

This should be pretty-much review (but not necessarily easy). Some hints:

-Basic anti-derivative rules:

$$\int u^n dx = \frac{u^{n+1}}{n+1} + C \quad for\ n \neq -1\ \ and\ \ \ln|u| + C\ \ for\ n = -1$$
$$\int e^u du = e^u + C$$
$$\int \cos u du = \sin u + C \quad and\ \int \sin u du = -\cos u + C$$

-You can split numerators so $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ but not denominators so $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$ (usually).

-When you have to integrate a fraction like $\frac{ax+b}{cx+d}$, you should consider dividing it.

-It usually helps to rewrite trigonometric expressions in terms of sines and cosines.

-To solve differential equations involving y's or x's and y's, separate the y terms onto one side and the x-terms onto the other.

-To integrate exponential functions where the base is not e, it may be best to rewrite them with a base of e. Remember that $e^{\ln a} = a$ so to integrate something like $\int 2^{5x} dx$, do the following:

$$2 = e^{\ln 2}\ \ so\ \int (e^{\ln 2})^{5x} dx = \int e^{5x\ln 2} dx = \frac{1}{5\ln 2}\int e^{5x\ln 2}(5\ln 2)dx = \frac{1}{5\ln 2}e^{5x\ln 2} = \frac{1}{5\ln 2}2^{5x} + C$$

Find the following indefinite integrals:

1. $\int (2x^3 + 3x - 1)dx$

2. $\int \frac{3x^2 - 1}{x} dx$

3. $\int xe^{2x^2}\,dx$

4. $\int x(x^2-5)^3\,dx$

5. $\int (x^2+4)^2\,dx$

6. $\int \frac{dx}{3x+5}$

7. $\int \cos(4x)dx$

8. $\int \frac{2x}{x^2+1}\,dx$

9. $\int \frac{4dx}{\sqrt[3]{6-x}}$

10. $\int 5x(2-x^2)^6\,dx$

11. $\int x(x^2+2)^4\,dx$

12. $\int \sqrt{2\sin(x)+7}\cos x\,dx$

13. $\int \frac{3dx}{\csc(5x)}$

14. $\int \tan x\,dx$ (rewrite with sines and cosines)

15. $\int \frac{x-1}{x^2-2x+5}\,dx$

16. $\int \frac{x^2-2x-5}{x-1}\,dx$

17. $\int \frac{6}{x^2+1}dx$ (inverse trig)

18. $\int x\sqrt{6-x}dx$ (try a u-substitution)

19. $\int x(x-4)^7 dx$ (don't FOIL- find a u)

20. $\int \sin x \cos x dx$ (several good ways…)

21. $\int \sin^3 x dx$ (substitute for $\sin^2 x$)

22. $\int \sin^2 x dx$ (cos2x = something with $\sin^2 x$ …)

23. $\int \left(e^{2x} + 2x^{e} + ex + e\right)dx$

24. $\int 6^{x}\,dx$ (can you write it as $e^{????}$)

25. $\int \cos x \csc x dx$

26. $\int \tan x \sec x dx$

27. $\int x\cos(\pi x^{2})dx$

28. $\int \frac{(\ln x)^{2}}{x}\,dx$

29. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

30. $\int \frac{e^{3\ln x}}{x}\,dx$ (a few ways…incl. simplifying 1st)

31. $\int \frac{x^2-4}{x+1}\,dx$

32. $\int \frac{1}{\sec x+\tan x}\,dx$

33. $\int 10^x\,dx$

34. $\int 3x\cdot 5^{x^2}\,dx$

Evaluate the following definite integrals without using your calculator:

35. $\int_{1}^{3}(x^2-1)dx$

36. $\int_{0}^{5}\frac{dx}{\sqrt{9-x}}$

37. $\int_{1/6}^{1/2}\cos(\pi x)dx$

39. What is the average value of the function $f(x)=3x^2-2x$ on the interval [1,4]?

39. The area under the graph of $f(x)=\frac{1}{x}$ (and above the *x*-axis) on the interval [2,*a*] is 1. What is *a*?

40. The area under the graph of $f(x) = xe^{-x^2}$ on the interval $[0,a]$ is 0.25. What is a?

41-44: Solve the following differential equations given the initial values.

41. $f''(x) = 2x - 7 \quad f'(1) = 4 \textit{ and } f(0) = 5$

42. $f'(x) = 3x^2 - \dfrac{1}{x}$ thru (1,2)

43. $\dfrac{dy}{dx} = 0.2y - 10$ thru (0,40)

44. $\dfrac{dy}{dx} = 2\sqrt{\dfrac{x}{y}}$ thru (4,9)

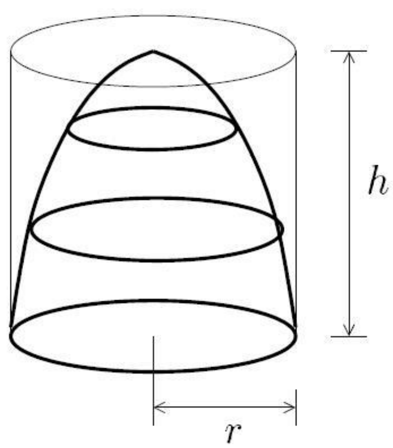
45-47: Find the areas and volumes below without using your calculators:

45. Find the area of the region in the first quadrant between the graphs of $f(x) = 4x$ and $g(x) = \sqrt[3]{x}$.

46. Find the volume of the solid that is formed by revolving the region bounded by $y = e^{-x}$, x=2, and the coordinate axes around the x-axis.

47. Find the volume of the solid that is formed by revolving the region bounded by $y = e^{-x}$, x=2, and the coordinate axes around the line y=-1.

48. A parabolic dish (a parabola revolved around its axis of symmetry) is inscribed in a cylinder. What percentage of the volume of the cylinder does the parabolic dish represent?



49. At noon there are 100 people in the park. Over the next 60 minutes, people enter at the rate of $E(t) = 10 + 2\sqrt{t}$ people per minute and leave at the rate of $L(t) = 3 + 0.5t$. Answer the following (please feel free to use fnInt—if you forget how, please ask).

a. The total number of people who entered in the first 30 minutes.

b. The number of people in the park at 1 pm.

c. The average rate that people enter between noon and 1 pm.

d. The average number of people in the park between noon and 1 pm.

Answers

1. $0.5x^4 + 1.5x^2 - x + C$ 2. $1.5x^2 - \ln|x| + C$ 3. $0.25e^{2x^2} + C$

4. $\frac{1}{8}(x^2 - 5)^4 + C$ 5. $\frac{x^5}{5} + 8\frac{x^3}{3} + 16x + C$ 6. $\frac{1}{3}\ln|3x+5| + C$

7. $0.25\sin(4x) + C$ 8. $\ln|x^2 + 1| + C$ 9. $-6(6-x)^{2/3} + C$ 10. $\frac{-2.5(2-x^2)^7}{7} + C$

11. $\frac{1}{10}(x^2 + 2)^5 + C$ 12. $\frac{1}{3}(2\sin(x) + 7)^{1.5} + C$ 13. $\frac{-3\cos(5x)}{5} + C$

14. $-\ln|\cos x| + C$ *or* $\ln|\sec x| + C$ 15. $0.5\ln|x^2 - 2x + 5| + C$ 16. $0.5x^2 - x - 6\ln|x-1| + C$

17. $6\tan^{-1}(x) + C$ 18. $-4(6-x)^{1.5} + 0.4(6-x)^{2.5} + C$ 19. $\frac{(x-4)^9}{9} + \frac{(x-4)^8}{2} + C$

20. $-0.25\cos(2x) + C$ *or* $\frac{\sin^2 x}{2} + C$ *or* $-\frac{\cos^2 x}{2} + C$ 21. $-\cos x + \frac{\cos^3 x}{3} + C$

22. $\frac{x}{2} - \frac{\sin(2x)}{4} + C$ 23. $\frac{e^{2x}}{2} + \frac{2}{e+1}x^{e+1} + \frac{ex^2}{2} + ex + C$ 24. $\int e^{(\ln 6)(x)}dx = \frac{e^{x\ln 6}}{\ln 6} = \frac{6^x}{\ln 6} + C$

25. $\ln|\sin x| + C$ 26. $\frac{1}{\cos x} + C$ *or* $\sec x + C$ 27. $\frac{\sin(\pi x^2)}{2\pi} + C$ 28. $\frac{(\ln x)^3}{3} + C$

29. $2e^{\sqrt{x}} + C$ 30. $\frac{x^3}{3} + C$ *or* $\frac{e^{3\ln x}}{3} + C$ 31. $\frac{x^2}{2} - x - 3\ln|x+1| + C$ 32. $\ln|1 + \sin x| + C$

33. $\frac{10^x}{\ln 10} + C$ 34. $\frac{3 \cdot 5^{x^2}}{2\ln 5} + C$ 35. 20/3 36. 2 37. $\frac{1}{2\pi}$ 38. 16

39. $\ln a - \ln 2 = 1$ so a=2e 40. $-0.5e^{-a^2} + 0.5 = 0.25$ so $-a^2 = \ln(0.5)$ *and* $a = \sqrt{\ln 2}$

41. $f(x) = \frac{x^3}{3} - \frac{7x^2}{2} + 10x + 5$ 42. $x^3 - \ln|x| + 1$

43. $y = -10e^{0.2x} + 50$ 44. $y^{1.5} = 2x^{1.5} + 11$ *so* $y = (2x^{1.5} + 11)^{2/3}$ 45. $\int_0^{1/8} (\sqrt[3]{x} - 4x)dx = 1/64$

46. $\int_0^2 \pi e^{-2x}dx = \frac{\pi}{2} - \frac{\pi}{2e^4}$ 47. $\int_0^2 \pi[(e^{-x} + 1)^2 - (1)^2]dx = \frac{5\pi}{2} - \frac{\pi}{2e^4} - \frac{2\pi}{e^2}$ 48. 50%

49a. $\int_0^{30} \left(10 + 2\sqrt{t}\right)dt = 519$ b. $100 + \int_0^{60} \left(10 + 2\sqrt{t}\right)dt - \int_0^{60} (3 + 0.5t)dt = 240$

c. $\frac{1}{60}\int_0^{60} \left(10 + 2\sqrt{t}\right)dt = 20.33$ per minute d. $\frac{1}{60}\int_0^{60} \left(100 + 10t + \frac{4}{3}t^{3/2} - 3t - 0.25t^2\right)dt \approx 257.87$

Unit 1 Handout #2: Integration by Parts

Integration by Parts is "undo-ing" the product rule:

Product Rule: $\dfrac{d(uv)}{dx} = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$ so $uv = \int u \cdot \dfrac{dv}{dx} + \int v \cdot \dfrac{du}{dx}$ and thus $\int u \cdot \dfrac{dv}{dx} = uv - \int v \cdot \dfrac{du}{dx}$

Example: Find $\int x \sin x dx$

Let $u = x$ *and* $dv = \sin x dx$. Then $du = dx$ *and* $v = -\cos x$.

So $\int x \sin x dx = -x \cos x - \int -\cos x dx$.

This is $-x \cos x + \sin x + C$

Check: taking the derivative of this yields $-x(-\sin x) + (-1)\cos x + \cos x = x \sin x$

These can all be solved by integration by parts, though other approaches may work for some:

1. $\int xe^{-2x} dx$

1.5. $\int x \cdot \cos x dx$

2. $\int x^2 \cos x dx$

3. $\int x^4 \ln x dx$

4. $\int \ln(x-3)dx$

5. $\int x\ln(x-3)dx$

6. $\int \tan^{-1}(x)dx$

7. $\int e^x \sin x dx$

8. $\int \sin^{-1} x dx$

9. $\int \cos^2 x dx$ hint: split it into cosx and cosx and later substitute for $\sin^2 x = 1 - \cos^2 x$ (or you can use the double-angle formula).

10. $\int \ln \sqrt{x} dx$ (rewrite first!)

11. $\int \ln(1 + x^2) dx$

12. $\int \frac{\ln x}{x^2} dx$

13. $\int \frac{x dx}{\sqrt{x - 6}}$

14. $\int \frac{x^3 dx}{\sqrt{x^2+1}}$ hint: x^2 is one of the two parts

15. $\int (\sin^{-1} x)^2 \, dx$ (you may need to iterate)

16. $\int (\ln x)^2 \, dx$

17. $\int x(\ln x)^2 \, dx$

18. If $\int_0^1 f'(x)g(x)dx = 10$ then find $\int_0^1 f(x)g'(x)dx$ given the following:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	3	2	-4	-2
1	4	1.5	-5	0

19. Let R be the region in the first quadrant bound by the coordinate axes and the graphs of $y = e^x$ and the $x = 2$.

a. Region R is revolved around *y*-axis. Find the volume of the resulting solid (shells recommended)

b. Region R is revolved around x=3. Find the volume of the resulting solid.

20. The region bounded by the graphs $y = 0$, $x = e$, *and* $y = \ln x$ is divided into two parts of equal area by a line. (Evaluate all integrals by hand – no fnInt – but calc intersect is OK to solve equations)

a. If the line is vertical, then find its equation.

b. If the line is horizontal, then find its equation.

21. A stadium is empty at noon. Thereafter, people enter the stadium at the rate of $f(t) = -0.01t^2 + t + 10$ people per minute on the interval [0,80], where t is measured in minutes. Two students are trying to compute average time of entry for those people.

a. Adrian says it is $\dfrac{\int_0^{80} (t)(-0.01t^2 + t + 10)dt}{\int_0^{80} (-0.01t^2 + t + 10)dt}$. Does her approach seem correct? Explain why or why not, being sure to specify what the integrals represent.

b. Bobby has a different approach. He says the number of people in the stadium at any point in time is $P(t) = \int_0^t f(x)dx = \dfrac{-0.01t^3}{3} + 0.5t^2 + 10t$. Thus the average arrival time is

80- $\dfrac{\int_0^{80} \left(\dfrac{-0.01t^3}{3} + 0.5t^2 + 10t \right) dt}{\int_0^{80} (-0.01t^2 + t + 10)dt}$. Does his approach seem correct? Explain why or why not, being sure to specify what the integrals and first 80 represent.

c. One way to relate the approaches of Adrian and Bobby is using the following equation, where $P(t)$ is the number of people in the stadium at any point in time.

$$\int t \cdot P'(t)dt = t \cdot P(t) - \int P(t)dt$$

Explain why this equation must be correct and how it relates the two students' approaches.

22. Let $N(t)$ represent the number of people in a stadium at time t minutes. What do each of the following represent? Be sure to include units! Assume $N'(t)$ is positive.

a. $N'(t)$

b. $\int_0^c N(t)dt$

c. $\int_0^c N'(t)\cdot(c-t)dt$

d. Why might the values of b and c not be the same?

e. Show mathematically that $\int_0^c N(t)dt = \int_0^c N'(t)\cdot(c-t)dt + cN(0)$ and explain why it must be the case by explaining what each side represents.

23. Challenging ones:

a. $\int \cos\sqrt{x}dx$

b. $\int \frac{(\ln x)^2}{x^2}dx$

ANSWERS 1. $-0.5xe^{-2x}-0.25e^{-2x}+C$ 1.5. $x\sin x+\cos x+C$ 2. $x^2\sin x+2x\cos x-2\sin x+C$

3. $\frac{x^5}{5}\ln x-\frac{x^5}{25}+C$ 4. $x\ln(x-3)-x-3\ln(x-3)+C$ 5. $\frac{x^2}{2}\ln(x-3)-\frac{1}{4}x^2-\frac{3}{2}x-\frac{9}{2}\ln(x-3)+C$

6. $x\tan^{-1}(x)-0.5\ln(x^2+1)+C$ 7. $\frac{e^x\sin x-e^x\cos x}{2}+C$ 8. $x\sin^{-1}x+\sqrt{1-x^2}+C$

9. $\frac{\cos x\sin x+x}{2}+C$ 10. $\frac{x\ln x-x}{2}+C$

11. $x\ln(1+x^2)-\int x\frac{2x}{1+x^2}dx=x\ln(1+x^2)-2\int\left(1-\frac{1}{1+x^2}\right)dx=x\ln(1+x^2)-2x+2\tan^{-1}x+C$

12. $\frac{-1}{x}\ln x-\frac{1}{x}+C$ 13. $\frac{2}{3}(x-6)^{3/2}+12(x-6)^{1/2}+C$ 14. $x^2\sqrt{x^2+1}-\frac{2}{3}(x^2+1)^{3/2}+C$

15. $x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$ 16. $x(\ln x)^2-2x\ln x+2x+C$

17. $\int x(\ln x)^2dx=\frac{x^2}{2}(\ln x)^2-\frac{x^2}{2}\ln x+\frac{x^2}{4}+C$

18. -18 since $\int_0^1 f'(x)g(x)dx=(fg)(1)-(fg)(0)-\int_0^1 f(x)g'(x)dx$

19a. $\int_0^2 2\pi xe^x dx=2\pi\left[xe^x-e^x\right]=2\pi(e^2+1)$ b. $\int_0^2 2\pi(3-x)e^x dx=2\pi\left[3e^x-xe^x+e^x\right]=2\pi(2e^2-4)$

20. total area of the region is 1

a. $\int_1^k \ln x dx=\frac{0.5}{2}$ so $k\ln k-k-(-1)=0.5$ *and* $k\ln k-k=-0.5$ *so* $k=2.155$

b. $\int_0^k (e-e^y)dy=0.5$ *so* $ek-e^k-(-1)=0.5$ *and* $ek-e^k=-0.5$ *and* $k=0.325$

21a. simple weighted-avg of arrival time, weighted by the number of people who arrived at that time.
b. the numerator is total people-minutes and the denom is people, so the fractional represents the average number of minutes spent in the stadium. Since no one left, 80 minus this is the average arrival time.
c. this formula is just integration by parts. Divide both sides by the number of people and make it a definite integral and you get the expressions that Adrian and Bobby used.

22a. rate of inflow of people in people per minute
b. the total people-minutes that were spent in the stadium from t=0 to t=c
c. The total people-minutes spent by people who entered the stadium between t=0 and t=c
d. both measure people minutes… integration by parts should make it work out!

23a. $\int 2\sqrt{x}\frac{\cos\sqrt{x}}{2\sqrt{x}}dx=2\sqrt{x}\sin\sqrt{x}-\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx=2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}+C$

23b. $\int\frac{(\ln x)^2}{x^2}dx=(\ln x)^2(-1/x)-\int\frac{2\ln x}{x}(-1/x)dx$ and $\int\frac{\ln x}{x^2}dx=(\ln x)(-1/x)-\int\frac{-1}{x^2}dx$

So $\int\ =\frac{-(\ln x)^2}{x}+2\left[\frac{-\ln x}{x}-\int\frac{-1}{x^2}dx\right]=\frac{-(\ln x)^2}{x}+\frac{-2\ln x}{x}-\frac{2}{x}+C$

Unit 1 Handout #3: Partial Fractions

If we have a rational expression, we can divide it to have a polynomial plus a rational expression that has a lower degree in the numerator than in the denominator. To handle a rational expression with a lower degree in the numerator than in the denominator, there are a few approaches we have used in Cal B:

Case #1: constant over a linear: $\int \frac{3}{2x-1}dx$ is just a natural log: $1.5\ln|2x-1|+C$

Case #2: constant over a quadratic $1+u^2$: $\int \frac{1}{1+x^2}dx$ is just inverse tangent: $\tan^{-1}(x)+C$

Case #3: linear over quadratic: $1+u^2$: $\int \frac{4x+7}{1+4x^2}dx$ can be split into a log and an inverse tangent:

$$\int \frac{4x}{1+4x^2}dx + \int \frac{7}{1+4x^2}dx = 0.5\ln|1+4x^2|+3.5\int \frac{2dx}{1+(2x)^2} = 0.5\ln|1+4x^2|+3.5\tan^{-1}(2x)+C$$

Case #4: denominator can be rewritten as a square plus a constant:

Example: $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{1+(x+2)^2} = \tan^{-1}(x+2)+C$

Uglier: $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{4+(x+2)^2} = \int \frac{(1/4)dx}{1+(\frac{x+2}{2})^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$

But what if we get $\int \frac{dx}{1-x^2}$? We can't write the denominator in the form $1+u^2$.

This is a job for Partial Fractions!

<u>Example #1</u>

$\int \frac{dx}{x^2-1}$: write $\frac{1}{x^2-1}$ as $\frac{1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$ since (x+1) and (x-1) are factors of x^2-1.

Now multiply by x^2-1 to get $1 = A(x-1)+B(x+1)$ *or* $1 = Ax - A + Bx + B$

Since there are no x's on the left, we know $A+B=0$; since the constant on the left is 1 we know $B-A=1$. Solve this system to get $B=1/2$ *and* $A=-1/2$:

So $\int \frac{dx}{x^2-1} = \int\left(\frac{-0.5}{x+1}+\frac{0.5}{x-1}\right)dx = -0.5\ln|x+1|+0.5\ln|x-1|+C$

Note: using the laws of logs, this may also be written as $\ln\sqrt{\frac{x-1}{x+1}}+C$ *or* $0.5\ln\left|\frac{x-1}{x+1}\right|+C$

Example #2:

If one term is a quadratic that does not factor then the numerator on the fraction with it in the denominator may have a linear term and a constant.

$\int \frac{2x^2+5}{x^3+x}dx$ the degree of the numerator is below that of the denominator, so polynomial division will not help. We don't have CRS. So try partial fractions:

$\frac{2x^2+5}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ note that last fraction needs a linear x term because denominator is quadratic.

Solve to get $2x^2+5 = Ax^2+A+Bx^2+Cx$ and so

$$A+B=2$$
$$C=0 \qquad \rightarrow \text{ therefore A=5, B=-3, and C=0}$$
$$A=5$$

So $$\int \frac{2x^2+5}{x^3+x}dx = \int \frac{5}{x}dx + \int \frac{-3x}{x^2+1}dx = 5\ln|x| - 1.5\ln|x^2+1| + C$$

Example #3

If there is a repeated (double or triple) factor in the denominator then you need fractions that have that term once, twice…up to the number of times it occurs.

Example: $\int \frac{5x+1}{(x-1)^2}dx$ no CRS so try partial fractions:

$\frac{5x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$ note that we don't need Bx+C even though the denominator is quadratic because the term (x-1) is linear. [If we did Bx+C then we would get A=0, B=5, and C=1 which is right where we started!]

So $5x+1 = A(x-1)+B$ and A=5 and B=6

So $\int \frac{5x+1}{(x-1)^2}dx = \int \frac{5}{x-1}dx + \int \frac{6}{(x-1)^2}dx = 5\ln|x-1| - \frac{6}{x-1} + C$

Note: if the original problem was $\int \frac{5}{(x-1)^2}dx$ then we don't need partial fractions since we have CRS.

Solve the following—most will require partial fractions – but not all:

1. $\int \frac{3dx}{x^2 - x - 2}$

2. $\int \frac{2x + 4dx}{x^2 + x - 6}$

3. $\int \frac{5dx}{x^2 - 3x - 4}$

4. $\int \frac{3dx}{x^2 + 2x}$

5. $\int \frac{x+1}{x^2-4}dx$

6. $\int \frac{11x+17}{2x^2+7x-4}dx$

7. $\int \frac{3x^2+2}{x^3+x}dx$

8. $\int \frac{2x-10}{x^3+x}dx$

9. $\displaystyle\int_3^4 \frac{dx}{x^2-8x+12}$ (a definite integral!)

10. $\displaystyle\int \frac{dx}{x^2-a^2}$

11. $\displaystyle\int \frac{dx}{x^2+4x+4}$

12. $\displaystyle\int \frac{x^2-6}{x^2-9}\,dx$

13. $\displaystyle\int \frac{2x-3}{x^2-2x+1}dx$

14. $\displaystyle\int \frac{3x+1}{(x-2)^2}\,dx$

Solve the following; partial fractions may or may not be required:

15. $\int \frac{2x}{x^2+1}\,dx$

16. $\int \frac{2x-1}{x^2-1}\,dx$

17. $\int \frac{2x-1}{x^2+1}\,dx$

18. $\int \frac{dx}{(x+3)^2+1}$

19. $\int \frac{x^2-3x+5}{x^2+1}\,dx$

20. $\int \frac{x^3-6x+5}{x-2}\,dx$

21. $\int \frac{x^3 - 3x + 4}{x^2 - x - 2}\,dx$

22. $\int \frac{x^3 - 3x^2 + 7}{x^2 - 1}\,dx$

23. Challenging: Given that $\frac{dy}{dx} = 0.01y(100 - y)$ and y=50 when x=10 and $0 \le y \le 100$.

a. A quick look at the differential equation should reveal the horizontal asymptotes of the graph of *y*. What are they?

b. Find *y* when *x*=12. Hint: combine the logs and realize that what is inside the log is always negative!

c. Find *y* in terms of *x*.

24. Is there any rational function with a 2nd degree denominator that cannot be integrated by a combination of polynomial division, u-substitution, partial fractions, and inverse tangent?

Answers

1. $\ln|x-2|-\ln|x+1|+C$ 2. $0.4\ln|x+3|+1.6\ln|x-2|+C$

3. $\ln|x-4|-\ln|x+1|+C$ *or* $\ln\left|\frac{x-4}{x+1}\right|+C$ 4. $1.5\ln|x|-1.5\ln|x+2|+C$

5. $0.25\ln|x+2|+0.75\ln|x-2|+C$ 6. $2.5\ln|2x-1|+3\ln|x+4|+C$

7. $0.5\ln|x^2+1|+2\ln|x|+C$ 8. I get $\int\frac{-10}{x}dx+\int\frac{10x+2}{x^2+1}dx$ 2nd one is $\int\frac{10xdx}{x^2+1}+\int\frac{2dx}{x^2+1}$

so answer becomes $\int\frac{10xdx}{x^2+1}+\int\frac{2dx}{x^2+1}$ *so* $-10\ln|x|+5\ln|x^2+1|+2\tan^{-1}(x)+C$

9. $0.25\ln|x-6|-0.25\ln|x-2|$ *so* $-0.25\ln 3$ 10. $\frac{1}{2a}\ln|x-a|-\frac{1}{2a}\ln|x+a|+C$

11. this is $\int\frac{dx}{(x+2)^2}=-1(x+2)^{-1}+C$ because it satisfies the chain rule!

12. $\int\left(1+\frac{3}{x^2-9}\right)dx=x+0.5\ln|x-3|-0.5\ln|x+3|+C$

13. $2\ln|x-1|+\frac{1}{x-1}+C$ 14. $3\ln|x-2|-\frac{7}{x-2}+C$

15. $\ln|x^2+1|+C$ 16. $1.5\ln|x+1|+0.5\ln|x-1|+C$ 17. $\ln|x^2+1|-\tan^{-1}(x)+C$

18. $\tan^{-1}(x+3)+C$ 19. $x-1.5\ln|x^2+1|+4\tan^{-1}(x)+C$ 20. $\frac{x^3}{3}+x^2-2x+\ln|x-2|+C$

21. $0.5x^2+x+2\ln|x-2|-2\ln|x+1|+C$ 22. $0.5x^2-3x+2.5\ln|x-1|-1.5\ln|x+1|+C$

23a. the slope is zero when y=0 or y=100.

b. $\frac{dy}{y(100-y)}=0.01dx$ using partial fractions→ $\frac{0.01}{y}+\frac{0.01}{100-y}=0.01dx$ or $\frac{1}{y}+\frac{1}{100-y}=dx$

So $\ln|y|-\ln|100-y|=x+C$ and since y is restricted we can eliminate the abs values

So $\ln y-\ln(100-y)=x+C$ given (10,50) we get C=-10 so $\ln\left(\frac{y}{100-y}\right)=x-10$

When x=12 this is $\ln\left(\frac{y}{100-y}\right)=2$ so y=88.08

c. since $\ln\left(\frac{y}{100-y}\right)=x-10$ we get $\frac{y}{100-y}=e^{x-10}$ so y=$\frac{100e^{x-10}}{1+e^{x-10}}$

(note: you can multiply top&bottom by e^{10-x} to get $\frac{100}{1+e^{10-x}}$ 24. no

Unit 1 Handout #4: Some Integrals of Trigonometric Functions

Some things to remember:

1. $\sin^2 x = 1 - \cos^2 x$ *or* $\cos^2 x = 1 - \sin^2 x$ can be useful substitutions

2. $\tan^2 x + 1 = \sec^2 x$ → divide all terms in $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$ to get this

3. $\int \sec^2 x = \tan x + C$

4. $\int \sec x dx$ can be evaluated by multiplying by the fraction $\dfrac{\sec x + \tan x}{\sec x + \tan x}$, a similar thing works to integrate $\csc x$.

5. The most useful double-angle formula is $\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$ $(= \cos^2 x - \sin^2 x)$. This enables you to integrate $\sin^2 u$ *or* $\cos^2 u$ by rewriting as something involving $\cos(2u)$.

6. Products of powers of sines and cosines:

If both degrees are even, it stinks (use formula 5 above, maybe repeatedly)

If at least one degree is odd, keep one of that function and switch all others of them (an even number) to the other function, using the Pythagorean identity (#1 above). Then you have potentially several terms, each with CRS.

Example: $\int \sin^4 x \cos^5 x dx = \int \sin^4 x(1 - \sin^2 x)(1 - \sin^2 x)\cos x dx$ → now you get powers of sine, each with a cosine, giving you CRS

7. Products of powers of tangents and secants:

Likely use u-substitution with u being tanx or secx

Sometimes need to use Pythagorean identity #2 above to help

Examples: $\int \sec^4 x \tan x dx = \int (\sec x)^3 (\sec x \tan x dx) = 0.25\sec^4 x + C$

$$\int \sec x \tan^3 x dx = \int \sec x \tan x(\sec^2 x - 1)dx = \int (\sec^3 x \tan x - \sec x \tan x)dx$$

Which is $\sec^3 x / 3 - \sec x + C$

Find the following

1. $\int \sin^3 x dx$

2. $\int \sin^3 x \cos x dx$

3. $\int \cos^2 x \cdot \sin^3 x dx$

4. $\int \cos^5 x \cdot \sin x dx$

5. $\int \cos^5 x \cdot \sin^2 x dx$

6. $\int \cos^3 x \cdot \sqrt{\sin x} dx$

7. $\int \sec^2 x dx$

8. $\int \tan x \sec^2 x dx$

9. $\int \tan^2 x \sec^2 x dx$

10. $\int \tan(2x) dx$

11. $\int \sec^3 x \cdot \tan x dx$

12. $\int \sec^2 (2x-1) dx$

13. $\int \sin^2 x\,dx$

14. $\int \tan^2 x\,dx$

15. $\int \tan^4 x\,dx$

16. $\int \sin^5 x\,dx$

17. $\int \sec x\,dx$

18. $\int \tan^3 x\,dx$

19. $\int \csc x dx$ (use the "secant trick")

20. $\int \sin(2x)\cos(2x)dx$

21. $\int \sec^3 x dx$ (hint: you may need to use $1+\tan^2 x = \sec^2 x$ somewhere!)

22. $\int \sqrt{\tan x} \cdot \sec^4 x dx$ (hint: substitute 2 of the secants for a tangent)

23. $\int \frac{1+\sin x}{\cos x} dx$ (hint: "conjugate" of numerator)

24. $\int \frac{1}{1+\sin x} dx$

25. $\int \frac{1}{\cos x - 1} dx$

26. $\int \frac{\cos^3 x dx}{\sqrt{\sin x}}$

27. $\int \frac{\sec x dx}{\tan^2 x}$

28. What is the average value of $f(x) = \cot x$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. No calculator.

29. $\int \frac{\cos x dx}{2 - \sin x}$

30. $\int \frac{\cos^2 x}{2 - 2\sin x} dx$

31. $\int \sec x \csc x dx$

32. $\int \tan^5 x dx$

33. $\int (2+\tan x)^2\,dx$

34. $\int \frac{\sin x}{1+\sin x}\,dx$

35. $\int \sec^2 x \csc x\,dx$

36. (hard) $\int \cos^4 x\,dx$

Answers

1. substitute $\sin^2 x = 1-\cos^2 x$ $\rightarrow$ $-\cos x + \dfrac{\cos^3 x}{3} + C$ 2. CRS so $0.25\sin^4 x + C$

3. $-\dfrac{\cos^3 x}{3} + \dfrac{\cos^5 x}{5} + C$ 4. $-\dfrac{\cos^6 x}{6} C$ 5. $\dfrac{\sin^3 x}{3} - \dfrac{2\sin^5 x}{5} + \dfrac{\sin^7 x}{7} + C$

6. $\dfrac{2}{3}(\sin x)^{1.5} - \dfrac{2}{7}(\sin x)^{3.5} + C$ 7. $\tan x + C$ 8. CRS so $0.5\tan^2 x + C$ or $0.5\sec^2 x + C$

9. CRS so $(1/3)\tan^3 x + C$ 10. $-0.5\ln|\cos(2x)| + C$ 11. CRS so $(1/3)\sec^3 x + C$

12. $0.5\tan(2x-1) + C$ 13. $0.5x - 0.25\sin(2x) + C$

14. substitute for $\sec^2 x - 1$ so get $\tan x - x + C$

15. get $\int \tan^2 x(\sec^2 x - 1)dx$ which is $(1/3)\tan^3 x - \tan x + x + C$

16. $-\cos x + \dfrac{2\cos^3 x}{3} - \dfrac{\cos^5 x}{5} + C$

17. $\ln|\sec x + \tan x| + C$ 18. $0.5\tan^2 x + \ln|\cos x| + C$ 19. $-\ln|\csc x + \cot x| + C$

20. $0.25\sin^2(2x) + C$ *or* $-0.25\cos^2(2x) + C$

21. $\dfrac{\sec x \tan x}{2} + \dfrac{\ln|\sec x + \tan x|}{2} + C$ 22. $\dfrac{2}{3}(\tan x)^{1.5} + \dfrac{2}{7}(\tan x)^{3.5} + C$

23. $-\ln|1-\sin x| + C$ 24. $\tan x - \sec x + C$ 25. $\csc x + \cot x + C$

26. $2\sqrt{\sin x} - \dfrac{2}{5}(\sin x)^{5/2} + C$ 27. $-\csc x + C$

28. $\dfrac{1}{0.5\pi - 0.25\pi}\displaystyle\int_{0.25\pi}^{0.5\pi} \cot x$ which is $\dfrac{\ln(\sin(0.5\pi)) - \ln(\sin(0.25\pi))}{0.25\pi} = \dfrac{-4\ln(1/\sqrt{2})}{\pi} = \dfrac{2\ln 2}{\pi}$

29. $-\ln|2-\sin x| + C$ 30. $\displaystyle\int \frac{1+\sin x}{2}dx = \frac{x}{2} - \frac{\cos x}{2} + C$

31. $\int 2\csc(2x)dx = -\ln|\csc(2x) + \cot(2x)| + C$

32. $\displaystyle\int \left(\sec^4 x \tan x - 2\sec^2 x \tan x + \tan x\right)dx = \frac{\sec^4 x}{4} - \sec^2 x - \ln|\cos x| + C$

33. $\int \left(4 + 4\tan x + \sec^2 x - 1\right)dx = 3x - 4\ln|\cos x| + \tan x + C$

34. $\displaystyle\int \frac{\sin x(1-\sin x)}{1-\sin^2 x}dx = \int\left(\frac{\sin x}{\cos^2 x} - \tan^2 x\right)dx = \frac{1}{\cos x} + x - \tan x + C$

35. by parts… $\tan x \csc x - \int -\csc x \cot x \tan x dx = \tan x \csc x - \ln|\csc x + \cot x| + C$

36. ugly: $\cos 2x = 2\cos^2 x - 1$ *so* $\cos^2 x = \dfrac{1+\cos 2x}{2}$ *so* $\cos^4 x = \dfrac{1 + 2\cos 2x + \cos^2 2x}{4}$

And $\cos 4x = 2\cos^2 2x - 1$ *so* $\cos^2 2x = \dfrac{1+\cos 4x}{2}$ So $\displaystyle\int \cos^4 x dx = \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C$

`Unit 1 Handout #5: Integration by Trigonometric Substitution

Useful substitutions: these turn binomials into monomials:

if you have (*a* is a constant)	**then substitute**	**to get**
$a^2 - x^2$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Example: $\int \frac{dx}{\sqrt{1-x^2}}$

Make $x = \sin\theta$; so $dx = \cos\theta d\theta$ and $1 - x^2 = \cos^2\theta$

The integral is then $\int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta = \theta + C$

Now substitute back for *x*: since $x = \sin\theta$, $\theta = \sin^{-1} x$ so the integral is $\sin^{-1} x + C$, as some may have noticed before doing the trigonometric substitution.

Note: sometimes the substitution back for *x* is more complicated. For example, we may have set *x* equal to sine but then we need cotangent. In these cases, it is best to draw a right triangle where sine of theta is *x* over 1. Then solve for the missing side and get the necessary trig ratio or ratios.

Find the following. Most require trigonometric substitution, but some are solvable using other techniques:

1. $\int \frac{dx}{\sqrt{4-x^2}}$

2. $\int \sqrt{9-x^2}\,dx$

3. $\int \frac{x^3 dx}{\sqrt{9-x^2}}$

4. $\int \frac{dx}{\sqrt{9+x^2}}$

5. $\int \frac{x}{\sqrt{x^2+9}} dx$

6. $\int \frac{dx}{x^2+9}$

7. $\int \frac{\sqrt{16-x^2}}{x}\,dx$

8. $\int \frac{dx}{x^3+x}$

(do this both by partial fractions and by trig substitution)

9. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

10. $\int \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

11. $\int \frac{1}{x\sqrt{x^2+1}}\,dx$

12. $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

13. $\int \frac{\sqrt{x^2-4}}{x}\,dx$

14. $\int \frac{x^2\,dx}{\sqrt{25-x^2}}$

15. $\int \frac{xdx}{\sqrt{9-x^2}}$

16. $\int x^3\sqrt{x^2-1}dx$

A few review ones to keep early material fresh!

17. $\int \frac{x}{x^2-2x-3}dx$

18. $\int \frac{3dx}{x^2-5x}$

19. $\int xe^{2x}dx$

20. $\int x^3 \ln xdx$

21. A heavy box is located at the point (3,1). A person has a 5-unit rope tied to it. The person is initially at the point (0,5). The person walks up the *y*-axis, pulling the box. The box always moves in the direction the rope is pulling it.

a. Show that the slope of the box at any point (*x*,*y*) is given by the equation $\frac{dy}{dx} = \frac{-\sqrt{25-x^2}}{x}$.

b. Solve this differential equation to give the equation of the path that the box moves along. (It is called a tractrix).

Answers:

1. $x = 2\sin\theta \;\; dx = 2\cos\theta d\theta$ so $\int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} = \int d\theta = \theta + C$

Since $x = 2\sin\theta \;\; \theta = \sin^{-1}(0.5x)$ *so* $\theta = \sin^{-1}(0.5x) + C$

2. $x = 3\sin\theta \;\; dx = 3\cos\theta d\theta$ so $\int 9\cos^2\theta d\theta = 9\int 0.5(1+\cos(2\theta))d\theta$ so $9\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right] + C$

$\sin(2\theta) = 2\sin\theta\cos\theta$ and (using a triangle), if $\sin\theta = x/3$ *then* $\cos\theta = \sqrt{9-x^2}/3$ so

$4.5\sin^{-1}(x/3) + 4.5\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$

3. $x = 3\sin\theta$ *so* $27\int \sin^3\theta d\theta = 9\cos^3\theta - 27\cos\theta + C = 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 9\sqrt{9-x^2} + C$

4. $x = 3\tan\theta$ *so* $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$ 5. CRS: $\sqrt{x^2+9} + C$

6. $x = 3\tan\theta$ *so* $\int \frac{d\theta}{3} = \frac{\theta}{3} + C$ *or* $\frac{1}{3}\tan^{-1}(x/3) + C$ (or just manipulate it as an inv-tangent)

7. $x = 4\sin\theta$ so $4\int \frac{\cos^2\theta}{\sin\theta}d\theta = 4\int \frac{1-\sin^2\theta}{\sin\theta}d\theta = -4\ln|\csc\theta + \cot\theta| + 4\cos\theta + C$

$$-4\ln\left|\frac{4}{x} + \frac{\sqrt{16-x^2}}{x}\right| + \sqrt{16-x^2} + C$$

8. by trig substitution: $x = \tan\theta$ *so* $\int \cot\theta d\theta = \ln|\sin\theta| + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

By partial fractions becomes …. $\int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - 0.5\ln|x^2+1| + C$

9. $x = \tan\theta$ *so* $\int \frac{\cos\theta}{\sin^2\theta}d\theta = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{x^2+1}}{x} + C$

10. $x = \sin\theta$ *so* $\int 4\tan^2\theta d\theta = 4\int(\sec^2\theta - 1)d\theta = 4\tan\theta - 4\theta + C = \frac{4x}{\sqrt{1-x^2}} - 4\sin^{-1}x + C$

11. $x = \tan\theta$ *so* $\int \csc\theta d\theta = -\ln|\csc\theta + \cot\theta| + C = -\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C$

12. $x = \sin\theta$ *so* $\int \csc\theta d\theta = -\ln|\csc\theta + \cot\theta| + C = -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$

13. $x = 2\sec\theta$ *so* $2\int \tan^2\theta d\theta = 2\int(\sec^2\theta - 1)d\theta = 2\tan\theta - 2\theta = \sqrt{x^2-4} - 2\sec^{-1}(0.5x) + C$

14. $x = 5\sin\theta$ *so* $\int 25\sin^2\theta d\theta = \frac{25}{2}\int(1-\cos 2\theta)d\theta = \frac{25\theta}{2} - \frac{25\sin(2\theta)}{4} + C$ which, using the formula for sin(2x) becomes $12.5\sin^{-1}(x/5) - 0.5x\sqrt{25-x^2} + C$

15. CRS: $-(9-x^2)^{1/2} + C$

16. if $x = \sec\theta$ this is $\int \sec^4\theta \cdot \tan^2\theta d\theta = \int \sec^2\theta(\tan^4\theta + \tan^2\theta)d\theta = \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C$

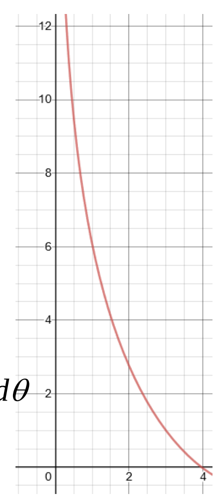
Which is $\frac{1}{5}(x^2-1)^{2.5} + \frac{1}{3}(x^2-1)^{1.5} + C$

17. $0.75\ln|x-3| + 0.25\ln|x+1| + C$

18. $-0.6\ln|x| + 0.6\ln|x-5| + C$

19. $\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

20. $\frac{x^4\ln x}{4} - \frac{x^4}{16} + C$



21a. by the Pythagorean theorem b. $\frac{dy}{dx} = \frac{\sqrt{25-x^2}}{x}$ so $x = 5\sin\theta$ then $y = \int \frac{5\cos\theta}{5\sin\theta}\cdot 5\cos\theta d\theta$

this is $5\int \frac{\cos^2\theta d\theta}{\sin\theta} = 5\int \frac{(1-\sin^2\theta)d\theta}{\sin\theta} = 5\int \csc\theta d\theta - 5\int \sin\theta d\theta = 5\ln|\csc\theta + \cot\theta| - 5\cos\theta + C$

$\sqrt{25-x^2} - 5\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right| + C$ and when x=3 y=1 so $y = -\sqrt{25-x^2} + 5\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right| + 5 - 5\ln 3$

Unit 1 Handout #6: Improper Integrals

Improper integrals may either have infinity or negative infinity as one or both bounds of integration or they may include a vertical asymptote within the bounds of integration. The result may be finite or may not be, in which case it is said to "diverge".

Example: Evaluate the following: a. $\int_1^\infty \frac{dx}{x}$ **and b.** $\int_1^\infty \frac{dx}{x^2}$

A formal solution to *a*: $\int_1^\infty \frac{dx}{x} = \lim_{k\to\infty} \int_1^k \frac{dx}{x} = \lim_{k\to\infty} \ln x \Big|_1^k = \lim_{k\to\infty}\left(\ln k - \ln 1\right) = \lim_{k\to\infty}(\ln k) = \infty$, so it diverges.

An informal solution to *b*: $\int_1^\infty \frac{dx}{x^2} = \frac{-1}{x}\Big|_1^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 0 + 1 = 1$

1. $\int_0^\infty xe^{-x^2}\,dx$

2. $\int_2^6 \frac{dx}{(6-x)^{1.5}}$

3. $\int_{-\infty}^5 \frac{dx}{(6-x)^{1.5}}$

4. $\int_1^\infty \frac{1}{x+3}\,dx$

5. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

6. $\displaystyle\int_6^9 \frac{dx}{x^2-6x}$

7. $\displaystyle\int_0^\infty \frac{3}{(2x+1)^2}\,dx$

8. $\displaystyle\int_1^2 \frac{x^2}{x^3-1}\,dx$

9. $\displaystyle\int_4^\infty \frac{x^2}{x^3-1}\,dx$

10. $\displaystyle\int_{-1}^8 \frac{2}{\sqrt[3]{x}}\,dx$

11. $\displaystyle\int_{-1}^1 \frac{4}{x^3}\,dx$

12. $\int_0^1 \ln x dx$

13. $\int_1^4 \frac{x}{x^2-9} dx$

14. $\int_3^4 \frac{x}{\sqrt{x-3}} dx$ hint: try u=x-3

15. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

16. $\int_{2/\pi}^{\infty} \frac{\sin(1/x)}{x^2} dx$ hint: CRS

17. $\int_e^{\infty} \frac{1}{x \ln x} dx$ (CRS)

18. $\int_{-\infty}^{0} e^{x} \cos x dx$

19 $\int_{1}^{\sqrt{5}} \frac{1}{\sqrt{x^2-1}} dx$ (trig substitution)

20. $\int_{\sqrt{5}}^{\infty} \frac{1}{\sqrt{x^2-1}} dx$

21. If $\int_{0}^{\infty} e^{kx} dx = 3$ what is k?

22. $\int_3^\infty x^p\,dx$ converges for what values of p?

23. $\int_0^3 x^p\,dx$ converges for what values of p?

Answers

1. $-0.5e^{x^2}$ so ½

2. $\frac{2}{\sqrt{6-x}}$ so diverges

3. 2

4. $\ln(x+3)$ so diverges

5. $-\sqrt{1-x^2}$ so 1

6. $-\frac{1}{6}\ln|x|+\frac{1}{6}\ln|x-6|$ diverges

7. $\frac{-1.5}{2x+1}$ *so* 1.5

8. $\frac{1}{3}\ln|x^3-1|$ so diverges

9. $\frac{1}{3}\ln|x^3-1|$ so diverges

10a. $3x^{2/3}$ on [-1,0] is -3 and on [0,8] is 12 so 9

11. $\frac{-2}{x^2}$ on [-1,0] diverges so diverges

12. $x\ln x - x$ so ??? what is 0ln(0)??? Integral is same magnitude as $\int_{-\infty}^{0} e^x\,dx$ which is 1 so -1

13. $0.5\ln|x^2-9|$ diverges at x=3 so diverges

14. becomes $\frac{2}{3}(x-3)^{1.5}+6\sqrt{x-3}$ so 20/3

15. $\tan^{-1}x$ so π

16. $\cos(1/x)$ so 1

17. $\ln(\ln x)$ diverges

18. Recursive parts gives $0.5e^x(\cos x+\sin x)$

this is ½ since $0.5e^x(\sin x+\cos x)$ is squeezed between e^x *and* $-e^x$ and both go to zero as x goes to -∞

19. $\ln|\sec\theta+\tan\theta| = \ln|x+\sqrt{x^2-1}|$ so $\ln(2+\sqrt{5})$

20. diverges

21. $\frac{e^{kx}}{k}$ and k<0 so k=-1/3

Unit 1 Handout #7: L'Hopital's Rule

L'Hopitals Rule is a great way to evaluate limits where they are 0/0 or $\pm\infty/\pm\infty$, and can be applied to some other interesting cases like 0^0. Here's how it works:

If $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}\ or\ \dfrac{\pm\infty}{\pm\infty}$ then the limit is equal to $\dfrac{f'(c)}{g'(c)}$.

Some examples:

#1. Find $\lim\limits_{x\to 1}\dfrac{x^3-2x^2+1}{x-1}$. If you substitute, you get 0/0. In cal A you'd have to do polynomial division and then evaluate the quotient at *x*=1. With l'Hopitals this limit is $\dfrac{3x^2-4x}{1}$ at *x*=1, which is -1.

#2. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$. You could remember that this is 1 or use l'Hopitals. Then you get $\dfrac{\cos x}{1}=1$.

#3. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^2}$ is ∞/∞ so use l'Hopitals and get $\dfrac{e^x}{2x}$, which is still ∞/∞. So use it again and get $\dfrac{e^x}{2}$, which is ∞. (no surprise—we know that exponential growth trumps polynomial growth).

#4. $\lim\limits_{x\to 0^+} x\ln x$ is $0\cdot(-\infty)$ which is unclear. So write it as a ratio to use l'Hopitals. You get $\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x}$ which is -∞/∞ and using l'Hopitals you get $\dfrac{1/x}{-1/x^2}=-x$, which as *x*→0 is 0.

#5. $\lim\limits_{x\to 0}\dfrac{\sin x}{1+\cos x}$. You can't use l'Hopitals since it is 0/2, which is just 0.

#6. tricky one: $\lim\limits_{x\to 0} x^x$ is undefined. Zero to the anything is 0. Anything to the 0th power is 1. Which wins? Try this: $y=x^x$ so $\ln y = x\ln x=\dfrac{\ln x}{1/x}$ as *x*→0 $\dfrac{\ln x}{1/x}\to 0$ so $\ln y\to 0$ and if $\ln y = 0\ \ then\ \ y=e^0=1$ so the limit is 1.

#7. Same ln trick again. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$: is 1 to the infinity. You can try l'Hopitals here:

$\ln y = x\ln(1+1/x) = \ln(1+1/x)/(1/x)$ and using l'Hoptials this is $\dfrac{\frac{1}{1+1/x}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \dfrac{x}{x+1}$ and as $x\to\infty$

this approaches 1. So $\ln y \to 1$ and thus $y \to e$. Hopefully some of you recognized the original limit as one definition of *e*.

#8. You can sometimes use it for limits that are $\infty-\infty$ by using the "e-trick". In f&g above we took the ln of *y*; we can also take e to the y (this turns subtraction into division). Example: find

$\lim_{x\to\infty}(x-\ln(2e^x+1))$ → $y = x-\ln(2e^x+1)$ *so* $e^y = e^{x-\ln(2e^x+1)} = \dfrac{e^x}{e^{\ln(2x^x+1)}} = \dfrac{e^x}{2e^x+1}$. As $x\to\infty$ this

clearly approaches ½ so $e^y \to 0.5$ *and* $y \to \ln 0.5$, which is the limit.

Evaluate the following limits:

1. $\lim_{x\to 4}\dfrac{\sqrt{x+5}-3}{x-4}$

2. $\lim_{x\to 0}\dfrac{\tan(2x)}{x}$

3. $\lim_{x\to 0}\dfrac{\tan(x)}{x^2}$

4. $\lim_{x\to 0^+}\sqrt{x}\ln x$

5. $\lim_{x\to(\pi/2)^-}\sec x-\tan x$

6. $\lim_{x\to\infty}\dfrac{e^x}{x^{10}}$

7. $\lim_{x \to 0^+} \sin x \ln x$

8. $\lim_{x \to 0} \frac{\sin(5x)}{2x}$

9. $\lim_{x \to 0^+} \frac{1}{x} + \ln x$

10. $\lim_{x \to \infty} \sqrt[x]{x}$

11. $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

12. $\lim_{x \to \infty} \ln(2x) - \ln(x+1)$

13. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

14. $\lim_{x \to 0} \left(\frac{1}{x^2}\right)^x$

15. $\lim_{x \to 0^+} (\tan x)^x$

16. $\lim\limits_{x\to 0^+} (\sin x)^{\tan x}$

17. $\lim\limits_{x\to 0^+} \sqrt[x]{x}$

18. $\lim\limits_{x\to 0} \left(e^x + x\right)^{1/x}$

19. $\lim\limits_{x\to \infty} x - \ln(3e^x + 5)$

20. $\lim\limits_{x\to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$

21. $\lim\limits_{x\to \infty} \left(1 + \dfrac{2}{x}\right)^x$

22. $\lim\limits_{x\to \infty} \dfrac{x^{-1}}{\sin(1/x)}$

23. $\lim\limits_{x\to 0} \dfrac{x - \tan^{-1} x}{x^3}$

24. $\lim\limits_{x\to \infty} \left(\ln x\right)^{1/x}$

25. $\lim_{x\to\pi/2}\left(\left(0.5\pi - x\right)\tan x\right)$

26. Evaluate $\lim_{x\to 0}\left(\frac{1}{x} - \csc x\right)$. Then explain why your result means that $\frac{1}{x}$ is a good approximation for $\csc x$ near x=0. Plug in some numbers to see how close it is. Also, does this at all relate to $\lim_{x\to 0}\frac{\sin x}{x}$?

27. The function $f(x)$ is continuous and twice differentiable over all real numbers. The table below shows some values of $f(x)$ and its derivatives:

x	-1	0	3	Lim as x → ∞
$f(x)$	0	2	0	∞
$f'(x)$	3	1	0	3
$f''(x)$	0	-2	3	0

Evaluate the following:

a. $\lim_{x\to 0}\frac{f(x)}{x^2}$

b. $\lim_{x\to -1}\frac{f(x)}{x+1}$

c. $\lim_{x\to 3}\frac{f(x)}{e^x - e^3}$

d. $\lim_{x\to 0}\frac{f'(x)-1}{3x}$

e. $\lim_{x\to 0}\frac{\left[f(x)-2\right]^2}{x^2}$

f. $\lim_{x\to\infty}\frac{f(x)}{x^2}$

g. $\lim_{x\to\infty}\frac{f(x)}{\ln x}$

28. Given that $\frac{dy}{dx} = 2y(x+3)$ and y is 4 when x=2, find

a. $\lim_{x\to 2} \frac{y-4}{x-2}$

b. $\lim_{x\to 2} \frac{y^2 - y - 12}{x^2 - 3x + 2}$

29. Evaluate the following: $\lim_{x\to 3} \frac{x}{x-3} \int_3^x \frac{\sin t}{t}\, dt$?

30. (from AP Cal BC exam 2013)

Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is $\lim_{x\to 1} \frac{\int_1^x g(t)\, dt}{g(x) - 6}$?

Answers

1. 1/6 2. 2 3. $+\infty$ 4. 0; write as $\frac{\ln x}{1/\sqrt{x}}$ 5. 0 6. $+\infty$

7. write as $\frac{\ln x}{\csc x}$ so it becomes $\frac{-\sin x \tan x}{x}$ with l'hopitals → do it again and get 0

8. 2.5 9. $e^y = \frac{e^{1/x}}{1/x}$ which becomes ∞ 10. use logs and get 1 11. 1 12. ln2 13. 0 14. 1 15. 1

16. 1 17. 0 18. e^2 19. ln(1/3) or –ln3 20. ½ 21. e^2 22. 1 23. 1/3 24. 1 25. 1

26. 0 so $\csc x \approx \frac{1}{x}$ near x=0…and it is pretty close! If $\lim_{x\to 0} \frac{\sin x}{x} = 1$ then sinx ≈x near 0, so cscx ≈ 1/x

27 a. either + or - ∞ b. 3 c. 0 d. -2/3 e. 1 f. 0 g. ∞ 28a. 40 b. 280 29. sin3 30. 2

Unit 1 Handout #8: Review Problems

1. Evaluate the following limits:

a. $\lim\limits_{x\to 0} x\ln(\sqrt{x})$

b. $\lim\limits_{x\to \pi/2} \dfrac{\cos x}{x-\pi/2}$

c. $\lim\limits_{x\to\infty} \dfrac{e^{\sqrt{x}}}{x^2}$

d. $\lim\limits_{x\to 0^+} (x)^x$

e. $\lim\limits_{x\to 0^+} x\ln x$

f. $\lim\limits_{x\to\infty} (2x)^{3/x}$

g. $\lim\limits_{x\to\infty} \left(e^x + x\right)^{2/x}$

h. $\lim\limits_{x\to 1^+} \left(\tan\dfrac{\pi x}{4}\right)^{3/(x-1)}$

i. $\lim\limits_{x\to 0} \dfrac{1}{e^x - 1} - \dfrac{1}{x}$

2. Find the following:

a. $\int \frac{1-\ln x}{x} dx$

b. $\int \frac{2x}{e^x} dx$

c. $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

d. $\int \frac{x-2}{x^2-4x} dx$

e. $\int \frac{x^2-4x}{x-2} dx$

f. $\int \frac{x^2}{\sqrt{2+x}} dx$

g. $\int x^2 e^{2x} dx$

h. $\int \sec^3 x \cdot \tan x dx$

i. $\int x^2 \sin x dx$

j. $\int \frac{2x}{x^2 - 2x - 15} dx$

k. $\int \sin^3 x \cos^2 x dx$

l. $\int_{-2}^{2} \frac{1}{x^2} dx$

m. $\int_0^4 \frac{1}{\sqrt{x}} dx$

n. $\int_0^\infty e^{-x} dx$

o. $\int e^x \cos x dx$

p. $\int \frac{\cos^3 x}{\sqrt{\sin x}} dx$

q. $\int \sin^3 x \cos^4 x dx$

r. $\int \frac{\sec x dx}{\tan^2 x}$

s. $\int_0^1 -\ln x dx$

t. $\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

u. $\int x\sqrt{x^2-9}dx$

v. $\int \frac{\sqrt{x^2-9}}{x}dx$

w. $\int \frac{x^2 dx}{\sqrt{4-x^2}}$

x. $\int \sqrt{x^2+9}dx$

3. If $\frac{dy}{dx} = \frac{2}{x^2-4}$ and y=7 when x=1 then find y in terms of x and find $\lim\limits_{x \to 2^-} y$

4. What is the area under the graph of $f(x) = \dfrac{6}{x^2 + 1}$ over its entire domain?

5. The R be the region bound by the *x*-axis and $y = \cos x$ on the interval [0,π/2].

a. What is the volume of the solid generated by revolving R around the *x*-axis?

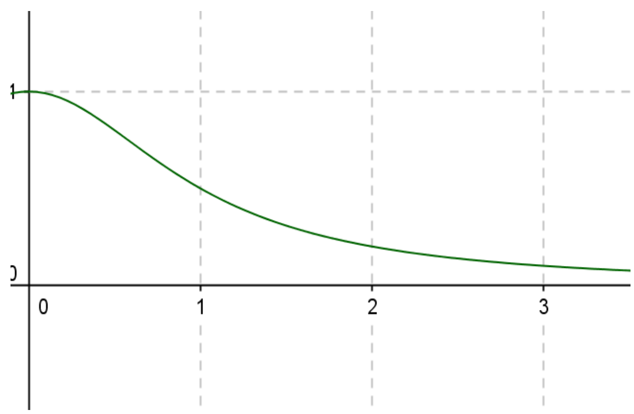
b. What is the volume of the solid generated by revolving R around the line *x*=2.

6. The region R is defined as the region in the first quadrant bounded by the *y*-axis, the graph of $y = e^x$ and the line $y = e^2$. No fnInt on this question.

a. Find the volume of the solid resulting from revolving R around the line *y*=-2.

b. Find the volume of the solid resulting from revolving R around the *y*-axis using both shells and discs

7. The graph below shows $f(x) = \dfrac{1}{1+x^2}$ for positive numbers.



a. find the area under the curve in the first quadrant

b. find the volume of the solid that results when the graph is revolved around the *y*-axis.

c. find the volume of the solid that results when the graph is revolved around the *x*-axis.

Answers

1a. this is 0.5xlnx which → 0 b. -1 c. this is e^u/u^4 so ∞ d. 1 e. 0 f. 1 g. e^2 h. $e^{1.5\pi}$ i. -0.5

2a. $\frac{-(1-\ln x)^2}{2}+C$ (CRS) b. $-2xe^{-x}-2e^{-x}+C$ c. diverges d. $0.5\ln|x|+0.5\ln|x-4|+C$

e. $0.5x^2-2x-4\ln|x-2|+C$ f. use u=2+x and get $\frac{2}{5}(2+x)^{5/2}-\frac{8}{3}(2+x)^{3/2}+8(2+x)^{1/2}+C$

g. $e^{2x}\left(0.5x^2-0.5x+0.25\right)+C$ h. $\frac{\sec^3 x}{3}+C$ i. $-x^2\cos x+2x\sin x+2\cos x+C$

j. $1.25\ln|x-5|+0.75\ln|x+3|+C$ k. $-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$ l. diverges m. 4 n. 1

o. $\frac{e^x\sin x+e^x\cos x}{2}+C$ p. $2(\sin x)^{0.5}-\frac{2}{5}(\sin x)^{2.5}+C$ q. $-\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+C$

r. $-\csc x+C$ s. 1 t. $\frac{x}{\sqrt{x^2+1}}+C$ u. $\frac{\left(x^2-9\right)^{1.5}}{3}+C$ v. $\sqrt{x^2-9}-3\sec^{-1}(x/3)+C$

w. x=2sinθ so $\int 4\sin^2\theta d\theta$ so $2\sin^{-1}(x/2)-\frac{x\sqrt{4-x^2}}{2}+C$ x. $0.5x\sqrt{x^2+9}+4.5\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|+C$

3. $y=0.5\ln|x-2|-0.5\ln|x+2|+7+0.5\ln 3$ and $\lim_{x\to 2^-} y=-\infty$ 4. 6π

5a. $\int_0^{0.5\pi}\pi\cos^2 x\,dx=\pi\left[\frac{x}{2}+\frac{\sin(2x)}{4}\right]=\frac{\pi^2}{4}$ b. $\int_0^{0.5\pi}2\pi(2-x)\cos x\,dx = 4\pi\sin x-2\pi\left(x\sin x+\cos x\right)= 6\pi-\pi^2$

6a. $\int_0^2\pi[R^2-r^2]dx=\int_0^2\pi[(2+e^2)^2-(2+e^x)^2]dx=\pi\int_0^2(4e^2+e^4-4e^x-e^{2x})dx = \pi(4e^2+1.5e^4+4.5)$

b. shells: $\int_0^2 2\pi x(e^2-e^x)dx=\pi e^2x^2-2\pi\int xe^x dx$ do the 2nd integral by parts: $\int xe^x dx=xe^x-e^x$

so $\pi e^2x^2-2\pi xe^x+2\pi e^x=4\pi e^2-4\pi e^2+2\pi e^2-(0+0+2\pi)=2\pi(e^2-1)$

b. discs. $\int_1^{e^2}\pi x^2 dy=\int_1^{e^2}\pi(\ln y)^2 dy$ use parts where u=(lny)^2 and v'=dy so u'=2lny/y and v=y

so $\int(\ln y)^2 dy=y(\ln y)^2-2\int\ln y\,dy$ this last integral (by parts) is $y\ln y-y$

so we get $\pi\int_1^{e^2}(\ln y)^2 dy=\pi\left[y(\ln y)^2-2y\ln y+2y\right]$ or $\pi[(4e^2-4e^2+2e^2)-(0-0+2)]=2\pi(e^2-1)$

7a. $\int_0^\infty\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_0^\infty=\frac{\pi}{2}$ b. shells: $\int_0^\infty\frac{2\pi x}{1+x^2}dx=\pi\ln(1+x^2)=\infty$

c. $\int_0^\infty\frac{\pi}{(1+x^2)^2}dx$ using trig substitution get $\pi\int\cos^2\theta d\theta=\pi\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right] = \pi\left[\tan^{-1}x+\frac{x}{1+x^2}\right] = \frac{\pi^2}{4}$

Additional Unit 1 Practice Problems

1. Find $\int \sin(\ln x)dx$

2. Find $\int \frac{dx}{x^2-1}$

   a. Using partial fractions
   b. Using trig substitution
   c. Use it to find $\int_2^{\infty} \frac{dx}{x^2-1}$

3. Evaluate the following limits:

a. $\lim_{x\to 0^+} \frac{\sin x - x}{\tan x - x}$
b. $\lim_{x\to 0^+} \left(e^x - 1\right)^x$
c. $\lim_{x\to 2} \frac{x^a - 2^a}{x^b - 2^b}$ (a & b are constants)

4. Answer the following questions about $f(x) = \frac{\ln x}{x}$.

   a. What are $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to\infty} f(x)$?
   b. Find the zeros of $f(x)$ and determine when $f(x) > 0$.
   c. Find the coordinates of all relative max and min of $f(x)$.
   d. Use your answers to parts *a*, *b*, and *c* to sketch a graph of $f(x)$.
   e. Evaluate $\int_1^{e^3} f(x)dx$
   f. Evaluate $\int_1^{\infty} f(x)dx$
   g. The R be the region between the *x*-axis and the graph of $f(x)$ where *x*>1. Find the volume of the solid created when R is revolved around the *x*-axis. [this one part covers a lot!]
   h. The R be the region between the *x*-axis and the graph of $f(x)$ where *x*>1. Find the volume of the solid created when R is revolved around the *y*-axis.

Solutions

1. $\int \sin(\ln x)dx$ by parts

$u = \sin(\ln x) \quad du = \dfrac{\cos(\ln x)}{x}dx$ so $\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$

$dv = dx \qquad v = x$

$\int \cos(\ln x)dx$ by parts:

$u = \cos(\ln x) \quad du = \dfrac{-\sin(\ln x)}{x}dx$ so $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$

$dv = dx \qquad v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - \left[x\cos(\ln x) + \int \sin(\ln x)dx\right]$$

So $2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x)$ and $\int \sin(\ln x)dx = \dfrac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$

2a. $\displaystyle\int \frac{dx}{x^2-1} = \int\left(\frac{-0.5}{x+1} + \frac{0.5}{x-1}\right)dx = 0.5\ln|x-1| - 0.5\ln|x+1| + C$

b. $x = \sec\theta$ *and* $dx = \sec\theta\tan\theta d\theta$ so $\displaystyle\int \frac{dx}{x^2-1} = \int \frac{\sec\theta\tan\theta d\theta}{\tan^2\theta} = \int \csc\theta d\theta$

which is $\displaystyle -\ln\left|\csc\theta + \cot\theta\right| = -\ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right| + C = -\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right| + C = -\ln\left|\sqrt{\frac{x+1}{x-1}}\right| + C$

which, using laws of logs, is the same as in 2a (but much less efficient!) 2c. $0.5\ln 3$

3a. -1/2 b. $\displaystyle\lim_{x\to 0^+}\left(e^x - 1\right)^x$ let $y = (e^x-1)^x$ then $\ln y = x\ln(e^x-1) = \dfrac{\ln(e^x-1)}{1/x}$ at x=0 this is -∞/∞

use L'hopitals: $\displaystyle\lim_{x\to 0^+}\ln y = \frac{\frac{e^x}{e^x-1}}{-1/x^2} = \frac{-x^2e^x}{e^x-1} = \frac{0}{0}$ → $\displaystyle\lim_{x\to 0^+}\ln y = \frac{-x^2e^x - 2xe^x}{e^x} = 0$ so lny →0 & y → 1

3c. $\dfrac{a}{b}x^{a-b}$

4a. $\displaystyle\lim_{x\to 0^+} f(x) = \frac{-\infty}{0} = -\infty \qquad \lim_{x\to\infty} f(x) = \frac{\infty}{\infty} = \frac{1/x}{1} = 0$ by l'hopitals

b. zero is x=1 and $f(x) > 0$ whenever $x > 1$

c. $f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = 0$ when $\ln x = 1$ *so* $x = e$ and the point is (e,1/e)

since $f'(x) > 0$ to the left of x=1 and <0 to the right of x=1, this is a maximum

e. $\displaystyle\int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2$ since it has CRS (u=lnx) so $\displaystyle\int_1^{e^3} f(x)dx = \frac{1}{2}(\ln e^3)^2 - \frac{1}{2}(\ln 1)^2 = 4.5$

f. $\int_1^\infty f(x)dx$ diverges since it is $\frac{1}{2}(\ln\infty)^2 - \frac{1}{2}(\ln 1)^2$ and $\lim_{x\to\infty} \ln x = \infty$ (ie, DNE)

g. $\int_1^\infty \pi\left(\frac{\ln x}{x}\right)^2 dx$ $\int \frac{(\ln x)^2}{x^2}dx$ by parts:

$u = (\ln x)^2$ $du = \frac{2\ln x}{x}dx$ so $\int \frac{(\ln x)^2}{x^2}dx = \frac{-(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2}dx$

$dv = dx/x^2$ $v = -1/x$

And $\int \frac{\ln x}{x^2}dx$ by parts $u = \ln x$ $du = \frac{1}{x}dx$ so $\int \frac{\ln x}{x^2}dx = \frac{-\ln x}{x} + \int \frac{dx}{x^2} = \frac{-\ln x}{x} - \frac{1}{x}$

$dv = dx/x^2$ $v = -1/x$

$\int \frac{(\ln x)^2}{x^2}dx = \frac{-(\ln x)^2}{x} - 2\left(\frac{\ln x + 1}{x}\right)$ and volume = $\pi\left(\frac{-(\ln\infty)^2 - 2\ln\infty - 2}{\infty}\right) - \pi\left(\frac{-(\ln 1)^2 - 2\ln 1 - 2}{1}\right)$

The first expression needs l'Hopitals: $\lim_{x\to\infty} \frac{-(\ln x)^2 - 2\ln x - 2}{x} = \frac{-2\ln x/x - 2/x}{1} = \frac{-2\ln x - 2}{x}$

Use l'hopitals again and get 0 so volume = 2π …..Whew!

4h. Definitely use shells since I don't want to find the inverse of $f(x) = \frac{\ln x}{x}$

$$\int_1^\infty 2\pi r h dx = \int_1^\infty 2\pi x \cdot \frac{\ln x}{x}dx = 2\pi \int_1^\infty \ln x dx$$

To find $\int \ln x dx$ use parts and get $x\ln x - x$ or know it must diverge since the area under the log function on the interval [1,∞) is infinite (the function is positive and increasing)

So it diverges

Unit 2 Handout #1: Introduction to Sequences

1. Write out the general term of each sequence below. Then determine if it converges and, if so, what it converges to:

a. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots.$ b. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots..$ c. $1,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4\cdot 5,\ 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7, \ldots$

2. Determine if each sequence below converges and, if so, what it converges to. Some require l'Hopitals.

a. $s_n = \cos(\pi n)$ b. $s_n = \sin(\pi n)$ c. $s_n = \frac{\sin(n)}{n}$

d. $s_n = 2 - \frac{10}{n^2}$ e. $s_n = \frac{3n+1}{2n-5}$ f. $s_n = \frac{100n + 5000}{0.1n^2 - 10000n}$

g. $s_n = \sqrt[n]{n}$ h. $s_n = \left(1 + \frac{2}{n}\right)^n$

3. For each sequence below, determine if it converges and, if so, what it converges to. Some may require l'Hopitals.

a. $\left\{ \dfrac{n^2}{3n(n+1)} \right\}_{n=1}^{\infty}$ b. $\left\{ \dfrac{(-1)^n}{2n+1} \right\}_{n=1}^{\infty}$ c. $\left\{ \dfrac{(-1)^n \cdot n}{2n+1} \right\}_{n=1}^{\infty}$

d. $\left\{ n \sin\left(\dfrac{\pi}{n} \right) \right\}_{n=1}^{\infty}$ e. $\left\{ \dfrac{e^n}{\pi^n} \right\}_{n=1}^{\infty}$ f. $\left\{ \dfrac{n^e}{e^n} \right\}_{n=1}^{\infty}$

4. For the following recursively defined sequences, find the first three terms and then determine if the sequence converges and, if so, what it converges to.

a. $a_1 = 20 \quad a_n = 0.8a_{n-1} + 25$ b. $a_1 = 50 \quad a_n = a_{n-1} + 0.01(a_{n-1})(200 - a_{n-1})$

5. For each sequence below, define the sequence recursively and then determine if it converges, and if so, what it converges to. The quadratic formula or completing the square may be required.

a. $a_1 = \frac{1}{2} \quad a_2 = \frac{1}{1+\frac{1}{2}} \quad a_3 = \frac{1}{1+\frac{1}{1+\frac{1}{2}}}$

b. $a_1 = \sqrt{7} \quad a_2 = \sqrt{7+\sqrt{7}} \quad a_3 = \sqrt{7+\sqrt{7+\sqrt{7}}}$

6. For each geometric series below, find the sum. If it only converges for certain *x* values, then determine which ones.

a. $3 + \frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \ldots..$

b. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots.$

c. $x - \frac{x^3}{3} + \frac{x^5}{9} - \frac{x^7}{27} +$

d. $2 + \frac{(x-1)}{3} + \frac{(x-1)^2}{18} + \frac{(x-1)^3}{108} + \ldots$

Answers

1a. $s_n = \frac{1}{n^2}$ converges to 0 b. $a_n = (-1)^{n+1} \cdot \frac{1}{2^{n-1}}$ converges to 0 c. $t_n = (2n-1)!$ diverges

2a. diverges (alternatives between -1 and +1) b. converges to 0 c. converges to zero

d. converges to 2 e. converges to 1.5 f. converges to 0 g. converges to 1 (l'Hopitals)

h. converges to e^2 (using l'Hopitals)

3a. converges to 1/3 b. converges to 0 c. diverges (gets close to flipping between 0.5 and-0.5)

d. converges to π (L'hopitals) e. converges to 0 f. converges to 0

4a. converges to 125 b. converges to 200

5a. converges to $\frac{-1+\sqrt{5}}{2}$ b. converges to $\frac{1+\sqrt{29}}{2}$

6a. 27/8 b. $\frac{1}{1+0.5x}$ for |x/2|<1 or |x|<2 c. $\frac{x}{1+x^2/3}$ *for* $|x|<\sqrt{3}$ d. $\frac{2}{1-(x-1)/6}$ for -5<x<7

Unit 2 Handout #2: Sequence and Series Convergence

1. For each below, determine whether (a) the sequence $\{a_n\}$ in convergent and (b) whether the series $\sum_{n=1}^{\infty} a_n$ is convergent. If you know the number it converges to, then give it.

a. $a_n = \dfrac{n}{n+1}$

b. $a_n = \dfrac{3}{2^n}$

c. $a_n = \dfrac{n^2}{12n+1}$

d. $a_n = (-1)^{n+1} \cdot \dfrac{4^n}{3^{2n+1}}$

e. $a_n = 5 + 0.2^n$

2. For each series, find the sum if it converges.

a. $\sum_{n=1}^{\infty} 5(-0.2)^n$

b. $\sum_{n=1}^{\infty} \cos(n\pi)$

c. $\sum_{n=1}^{\infty} \left(\cos(\pi/6)\right)^n$

d. $\sum_{n=1}^{\infty} \dfrac{7}{10^n}$

e. $1 - \dfrac{4}{3} + \dfrac{16}{9} - \dfrac{64}{27} \ldots.$

f. $\sum_{n=1}^{\infty} e^{-n}$

g. $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}$

h. $\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

i. $\sum_{n=1}^{\infty} \frac{3^n - 2 \cdot 2^{n+1}}{5^n}$ NEW ONE

j. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6}$

k. $\sum_{n=3}^{\infty} \frac{1}{n^2 - 1}$

l. $\ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \ln \frac{4}{5} + \ldots.$

3. Show that the series $\sum_{k=0}^{\infty} (-1)^k (2x)^k = \frac{1}{1+2x}$ so long as -0.5 < x < 0.5

4. Find the sum of the series $\sum_{k=0}^{\infty}(x-3)^k$ (in terms of *x*) and determine for what values of *x* the formula holds.

5. Find the values of *x* that makes each geometric series converge and find the function of *x* that the series represents when it does converge:

a. $1+2x+4x^2+8x^3+.....$

b. $1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+....$

c. $\sum_{n=0}^{\infty}(-1)^n(x+1)^n$

d. $\sum_{n=0}^{\infty}3(0.5x-1)^n$

e. $\sum_{n=0}^{\infty}(-0.5)^n(x-3)^n$

6. For what values of x will each geometric series converge?

a. $\sum_{n=1}^{\infty}(\sin x)^n$ b. $\sum_{n=0}^{\infty}(\tan x)^n$

7. Does the series $\sum_{n=0}^{\infty}\frac{e^{\pi n}}{\pi^{ne}}$ converge or diverge? Explain how you know. (calculator OK if you need to, but try without it first!) You may want to consider extrema of the function $f(x)=\frac{e^x}{x^e}$.

8. Write a power series (in Σ notation) expressing each function and give its interval of convergence (ie, for what values of x will it converge).

a. $\frac{1}{1+3x}$ b. $\frac{x}{1-2x}$ c. $\frac{3}{1-x^3}$

d. $\dfrac{x}{2-x}$ (careful!)

e. $\dfrac{1}{x}$ (hint: re-write is as $\dfrac{1}{1-u}$)

9. The series $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\ldots..$ has an interesting property. (note: since it is **not** a geometric series we cannot easily add it up). Take the derivative of the series term-by-term and compare it to the original series. What function might it be? Graph the first several terms to confirm.

10. Given the sequence $\left\{\sin\left(0.5\pi-\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$:

a. Does the sequence converge? If so, where to?

b. Does the related series $\sum_{n=1}^{\infty}\sin\left(0.5\pi-\dfrac{1}{n}\right)$ converge or diverge? Why or why not?

11. Does the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.$ converge or diverge? Based on your answer and knowing how each series below relates to the harmonic, can we know for certain if they converge or diverge? (Hint: think squeeze)

a. $\sum_{n=1}^{\infty}\frac{1}{n+1}$ b. $\sum_{n=1}^{\infty}\frac{1}{n-0.5}$ c. $\sum_{n=1}^{\infty}\frac{0.1}{n}$

12. True or False

a. If $\lim_{n\to\infty} a_n$ is defined then $\sum a_n$ converges.

b. If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges.

c. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

d. If $\sum a_n$ diverges then $\lim_{n\to\infty} a_n \neq 0$.

e. If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Answers

1a. sequence converges to 1; series diverges b. sequence converges to 0; series converges to 3
c. both sequence and series diverge

d. sequence converges to zero; series converges to 4/39 (write it as $\frac{-1}{3}\cdot\left(\frac{-4}{9}\right)^n$

e. Sequence converges to 5; series diverges

2a. -5/6 b. diverges c. $\frac{\sqrt{3}}{2-\sqrt{3}}$ d. 7/9 e. diverges since |r|>1 f. $\frac{1}{e-1}$ g. 6

h. it telescopes and the sum is ½ i. 1.5 – 8/3 = -7/6 j. telescopes to 1/3

k. telescopes and sum is 5/12 l. condenses to $\ln\frac{1}{n}$ as n→ infinity which is -∞ so diverges

3. geometric series with first term of 1 and r of 2x as long as |2x|<1

4. $\frac{1}{4-x}$ so long as |x-3|<1 or 2<x<4

5a. $f(x)=\frac{1}{1-2x}$ $for\ -0.5<x<0.5$ b. $f(x)=\frac{1}{1+0.5x}$ $for\ -2<x<2$

c. $f(x)=\frac{1}{2+x}$ $for\ -2<x<0$ d. $f(x)=\frac{3}{2-0.5x}$ $for\ 0<x<4$

e. $f(x)=\frac{1}{0.5x-0.5}$ $for\ 1<x<5$ this looks wrong

6a. for all except x=odd number of π's divided by 2

b. from –π/4 to π/4 and then repeated each π so from $\pi n-\frac{\pi}{4}\ to\ \pi n+\frac{\pi}{4}$ for any integer n

7. diverges since $e^{\pi}>\pi^{e}$

8a. $\sum_{n=0}^{\infty}(-3x)^n$ $for\ -1/3<x<1/3$ b. $\sum_{n=1}^{\infty}x(2x)^{n-1}$ $for\ -1/2<x<1/2$

c. $\sum_{n=0}^{\infty}3(x)^{3n}$ $for\ -1<x<1$ d. write as $\frac{.5x}{1-.5x}$ so $\sum_{n=1}^{\infty}(0.5x)^n$ $for\ -2<x<2$

e. write as $\frac{1}{1-(1-x)}$ so $\sum_{n=0}^{\infty}(1-x)^n$ $for\ 0<x<2$

9. its derivative is the same as itself, so maybe e^x *or generally* ae^x

10a. converges to 1 b. diverges since you keep adding 1's

11. the harmonic diverges a. diverges since always smaller than harmonic (term by term)
b. diverges since you get a few larger terms at first then a bunch of terms that are smaller than corresponding harmonic terms c. 1/10th of the harmonic, so it must also diverge

12a. false (sequence may converge to 1 or converge slowly to zero)

b. false: may converge slowly (harmonic) c. true d. false e. true

Unit 2 Handout #3: Some Convergence Tests

Will a series converge? Possible answers:

1. **Yes**: such as the geometric series $\sum_{n=1}^{\infty}(0.5)^n$, which converges to 1.

Note: sometimes we know it will converge but don't know what the sum will converge to….

2. **No**: the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, where the terms converge to zero but too slowly, so the sum doesn't.

3. **For some values of *x* only**: ie $\sum_{n=1}^{\infty}x^n$ is geometric and sums to $\frac{x}{1-x}$ so the sum will converge as long as $|x|<1$. (Note, these ones that converge for only some values of x are defined in terms of x's and n's)

How can we tell if a series converges? There are a number of tests. We'll work on these over the next week or so.

1. **Geometric series:** will converge if $|r|<1$ only; the sum can be determined

2. **"Telescoping series"** will often converge to a known sum—think partial fractions!

3. **"N^th^ term test":** if the limit of the individual terms is not zero, then the series cannot converge. But if the nth term is zero, the sum ***may or may not*** converge (remember the harmonic!).

4. "**P-test**" : for a series of the form $\sum\frac{1}{x^p}$; if $p>1$ it will converge else it will not converge

5. **"Comparison test"**: if each term of a series is (positive and) below corresponding (all positive terms) of a known convergent series, then it must converge. If it is term-by-term larger than a known divergent series, it must diverge. This is somewhat like the Squeeze Theorem for limits.

6. **Integral test**: if the integral of the series expression converges, then the series will (though probably not to the same number). If the integral diverges, then so will the series.

7. **Limit comparison test**: convenient test—coming soon

8. **Ratio test.** It is especially useful when the series contains factorials—coming soon

1. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$.

2. The "nth term test" will show that some of these series cannot converge, since the individual terms of the sequence do not converge to zero. Determine which of these series cannot converge for this reason. Note: you do not need to determine if the series actually does converge, because as you (hopefully) know, the terms of the sequence converging to zero is necessary for the series to converge, but not sufficient. l'Hopitals may be quite useful here.

a. $\sum_{n=1}^{\infty} 1 + \frac{1}{n}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

c. $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$

d. $\sum_{n=1}^{\infty} \sin(\pi n)$

e. $\sum_{n=1}^{\infty} \sqrt[n]{n+1}$

f. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\ln n}$

g. $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

h. $\sum_{n=1}^{\infty} \frac{1}{n} \tan^{-1}(n)$

i. $\sum_{n=1}^{\infty} (ne^{1/n} - n)$

3. Use the comparison test to determine which of these converge and which diverge. Compare them to know convergent or divergent series (like a *p* series or a geometric series).

a. $\sum_{n=1}^{\infty} \frac{0.5^n}{n}$

b. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

c. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

d. $\sum_{n=1}^{\infty} \frac{1.5}{n}$

e. $\sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{n^2+3}$

f. $\sum_{n=3}^{\infty} \frac{1}{n-2}$

g. $\sum_{n=1}^{\infty} \frac{1+\ln n}{n}$

h. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}-2}$

i. $\sum_{n=1}^{\infty} \frac{2^n}{3^n+7}$

j. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

k. $\sum_{n=1}^{\infty} \frac{3}{n \cdot 2^n}$

l. $\sum_{n=1}^{\infty} \frac{n+1}{n \cdot 0.99^n}$

m. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

n. $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^2}$

o. $\sum_{n=1}^{\infty} \frac{3+\cos n}{2^n}$

p. $\sum_{n=1}^{\infty} \frac{11}{3^n + 2}$

4. Use the integral test to determine whether each series below converges or diverges.

a. $\sum_{n=1}^{\infty} ne^{-n^2}$ b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ (try $u=\sqrt{n}$) e. $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ f. $\sum_{n=1}^{\infty} \frac{1}{(n+4)^2}$

g. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Answers

1. $\int_1^{\infty} \frac{1}{x^p}dx = \frac{x^{-p+1}}{-p+1} = \frac{\infty^{-p+1}}{-p+1} - \frac{1}{-p+1}$ for p≠1 which → ∞ if p<1; If p=1 the integral is lnx which diverges

2a. cannot converge b. may converge (but does not b/c of p-test) c. may converge
d. may converge (does) e. cannot converge f. cannot converge g. cannot converge
h. may converge i. cannot converge (terms approach 1 →factor the x out)

3a. < convergence geometric so converges b. < or = to 1/n^2 which converges, so converges
c. < than 1/n^2 which converges so converges d. > 1/n which diverges, so diverges
e. < 1/n^1.5 which converges, so converges f. larger than 1/n so diverges
g. larger than 1/n so diverges h. > 1/n^0.5 so diverges i. < (2/3)^2 so converges
j. > than 1/n so diverges k. < ½^n so converges l. > 1/0.99^n so diverges
m. < 1/n^1.5 so converges n. smaller than 1/n^3 so converges
o. < 5/2^n so converges p. < 11/(3^n) so converges

4a. converges b. ln(lnx) so diverges c. converges d. diverges e. diverges f. conv g. diverges

Unit 2 Handout #4: More Convergence Tests (Limit Comparison Test and Ratio Test)

1. Use the limit comparison test to determine if each series below converges or not. You will usually want to use a p-series $\left(\sum \frac{1}{n^p}\right)$ but in part *d*, a geometric series may be better.

a. $\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$

b. $\sum_{n=1}^{\infty} \frac{n-2}{n^2 + 3n + 14}$

c. $\sum_{n=1}^{\infty} \frac{\sqrt{n+3}}{n^2 - 2n + 11}$

d. $\sum_{n=1}^{\infty} \frac{3}{2^n - 3}$

e. $\sum_{n=1}^{\infty} \frac{7}{\sqrt[3]{n^4 + 6n^2 - 1}}$

f. $\sum_{n=1}^{\infty} \frac{7}{\sqrt[4]{n^3 + 6n^2 - 1}}$

g. $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n^2 - 11n + 7}}$

h. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ hint: compare to $\sum \frac{1}{n}$

2. Use the ratio test to determine if each series below converges or not. Some may be inconclusive. In all cases, give the value that the ratio of successive approaches.

a. $\sum_{n=1}^{\infty} \frac{6^n}{(n-1)!}$

b. $\sum_{n=1}^{\infty} \frac{6^n}{n^6}$

c. $\sum_{n=1}^{\infty} \frac{n!}{n^6}$

d. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

e. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (l'Hopital's?)

f. $\sum_{n=1}^{\infty} \frac{3^n}{5^n+7}$

g. $\sum_{n=1}^{\infty} \frac{3^n}{10n^2+3}$

h. $\sum_{n=1}^{\infty} \frac{(n+3)!}{n! \cdot 3^n}$

Answers

1a. converge b. diverge c. converge d. converge (compare to 1/(2^n) or 3/(2^n)) e. converge f. diverge g. diverge h. diverge (limit =1 and harmonic diverges)

2a. converge (0) b. diverge (6) c. diverge (∞) d. converge (1/3) e. converge (1/e) f. converge (3/5) g. diverge (3) h. converge (1/3)

Unit 2 Handout #5: Convergence Practice Questions

1. Each of the following series converges. To what does each converge?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-3}}{3^{n+1}}$

b. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2 - n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1^n + 2^n + 3^n}{6^n}$

2. Determine whether each series converges. Try to use an efficient test.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 3\sqrt{n}}{n^2 + 7}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2 + 4}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2 + 4}$

d. $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$

e. $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right)$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(2n)|}{n\sqrt{n}}$

g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5+2^{-n}}$

h. $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$

i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

j. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(n+1)\cdot 3^{2n}}$

k . $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+4}$

l. $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^2+4}}$

m. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{2^n}$

n. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n\cdot 2^n}$

o. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\cdot 2^n}$

p. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

q. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

r. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

s. $\sum_{k=1}^{\infty} \frac{k^{0.5k}}{k!}$

t. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

u. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

3. Let x be some unspecified **positive** constant in each of the series. Determine which positive values of x (if any) make the series converge and which make it diverge. The ratio test tends to work well. ***You may need a different test to see what is going on at the endpoints (ie, when the ratio turns out to be 1 and the ratio test is inconclusive.)***

Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$ The ratio test: $\lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{x^1}{n^2 3^n}} = \frac{x}{3} \cdot \frac{(n+1)^2}{n^2}$ as n→∞ the 2nd fraction is 1 so this converges for positive numbers less than 3 (ie, when 0< x/3 < 1). At the endpoint, when x=3, this series is $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-test. Therefore this converges for positive numbers less than or equal to 3.

a. $\sum_{n=1}^{\infty} \frac{5x^n}{3^n}$ b. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ c. $\sum_{n=1}^{\infty} \frac{x^n}{(n+1)2^n}$

d. $\sum_{n=1}^{\infty} \frac{x^n}{(n^2+1)2^n}$ e. $\sum_{n=1}^{\infty} 2(x-3)^n$ f. $\sum_{n=1}^{\infty} n!x^n$

Answers

1a. Geometric and sums to 5/12 b. sums to 2 c. sum of 3 geometrics is 1.7

2a. converge (I used the limit comparison test) b. diverge (LCT) c. diverge by nth-term test

d. converge by integral test (or ratio test) e. diverges—seems to telescope to ln2 but last term is infinit

f. converges by comparison test → always less than or equal to 1/n^1.5 which converges

g. diverges by nth term test h. diverges by LCT → compare to 1/n and get ratio of 1

i. converges by ratio test (ratio approaches 1/3)

j. converges by ratio test or comparison test (always below 5/9 to the n, which is geometric)

k. converges by LCT l. diverges by nth term test m. converges by ratio test

n. converges by comparison test as it eventually is lower than ½^n which converges

o. converges by ratio test p. diverges by ratio test

q. diverges by comparison test (eventually each term is greater than terms in 1/n)

r. diverges by comparison test as each term eventually is greater than 1/sqrt(n), which diverges

s. converges using ratio test limit of ratio is $\lim_{k\to\infty} \frac{1}{\sqrt{k+1}} \cdot \left(\frac{k+1}{k}\right)^{0.5k}$ which is sqrt(e)/sqrt(k+1) or zero

t. ratio test shows it to diverge u. converges by comparison test.. always smaller than $\frac{e^n}{e^{2n}} = \frac{1}{e^n}$ which is convergent geometric →or use limit comparison test comparing it to the series 1/(e^n)

3a. ratio test indicates will converge when x<3; at x=3 it diverges by the nth-term test (because it is geometric it will actually converge from -3<x<3.)

b. ratio test becomes x/(n+1) which, as n→ infinity is zero so it converges for all values of x

c. x<2 by ratio test and at x=2 exactly this is 1/(n+1) which diverges

d. x≤2; by ratio test we get x<2 and when x=2 exactly this is 1/(n^2+1) which converges

e. 2 < x < 4 using either geometric or ratio test f. only when x=0

Unit 2 Handout #6: Alternating and Monotone Series

1. Show that the terms in each series either always decrease or eventually always decrease. You can do it by differences (subtraction), ratios (division), or derivatives.

a. $\left\{\dfrac{1}{2n-3}\right\}_{n=2}^{\infty}$ b. $\left\{\dfrac{n}{2n-1}\right\}_{n=1}^{\infty}$ c. $\left\{\dfrac{n^2}{e^n}\right\}_{n=1}^{\infty}$

2. Determine if each alternating series converges. You do not need to formally prove the terms decrease in magnitude; a brief rationale is okay.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{\sqrt{n}}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n}{n+1}$ c. $\displaystyle\sum_{n=3}^{\infty}(-1)^{n+1}\cdot\frac{\ln n}{n}$

d. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n}{e^n}$ e. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt[3]{n}}$ f. $\displaystyle\sum_{n=3}^{\infty}(-1)^{n+1}\cdot\frac{1}{\ln n}$

g. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+4}$ h. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+3}$ i. $\displaystyle\sum_{n=1}^{\infty}\frac{n(-1)^n}{3n+2}$

j. $\sum_{n=1}^{\infty} \frac{n^3}{(-3)^n}$ k. $\sum_{n=1}^{\infty} \frac{\cos(0.5n\pi)}{n!}$ l. $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$

m. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ n. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

Answers

1a. derivative is $\frac{-2}{(2n-3)^2}$ which is always negative

b. derivative is $\frac{-1}{(2n-1)^2}$ which is always negative; or differences are $\frac{n+1}{2n+1} - \frac{n}{2n-1} = \frac{-1}{4n^2-1} < 0$

c. derivative is $\frac{e^n(2n) - n^2e^n}{e^{2n}} = \frac{n(2-n)}{e^n}$ which is negative for n>2 so eventually always decreases

2a. converges because terms get smaller and approach zero

b. diverges because terms do not approach zero

c. converges: terms get smaller since derivative of ln(n)/n = (1-ln(n))/n^2 is always negative for n>e

d. converges: terms get smaller (can use derivative)

e. converges: terms get smaller and alternate in sign

f. converges: terms get smaller and alternate in sign

g. converges h. converges i. diverges—terms don't approach 0

j. converges—terms decrease when n>3/ln(3)

k. simplifies to $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ which converges l. diverges: terms grow in magnitude (ratio)

m. converges—terms decrease once n>2 n. diverges: terms do not approach zero

Unit 2 Handout #7: Absolute and Conditional Convergence

1. For each series below, determine whether it diverges, converges conditionally, or converges absolutely. Note: to show conditional convergence you need to show that it converges but diverges absolutely.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{10^n}{n^{10}}$

b. $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1+n}{n^2}$

d. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{2n}$

e. $\sum_{n=1}^{\infty}\frac{\sin(\pi n)}{2n}$

f. $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{3n-4}$

g. $\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$

h. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$

i. $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^3}$

j. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(3n)}{n!}$

k. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2+1}$

l. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$

m. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 3^n}$

n. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}$

o. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$

p. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-10)^n}$

q. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots . (2n-1)}$

2. Give a few examples of series whose absolute values diverge but whose actual terms converge, if possible.

3. Give a few examples of series whose absolute values diverge according to the ratio test but whose actual terms converge, if possible.

4. Determine whether each series below converges.

a. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

b. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

c. $\sum_{n=2}^{\infty} \frac{\ln n}{\ln(n^2)}$

d. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (be creative)

e. $\sum_{n=2}^{\infty} \frac{5}{n\sqrt{n^2-1}}$

f. $\sum_{n=1}^{\infty} n^{-1.3}$

g. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

h. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

i. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n\sqrt{n}}$

j. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n+5^n}$

k. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

l. $\displaystyle\sum_{n=1}^{\infty} 2^{1/n}$

Answers

1a. using absolute ratio test ratio > 1 so it **diverges**

b. it satisfies alternating series test since terms approach zero and get smaller in magnitude; absolute value of it is larger than harmonic by comparison test so diverges so it is **conditionally convergent.** I'd avoid ratio tests since in involves a logarithm

c. it always declines in magnitude (negative first derivative) and magnitude approaches zero so it converges by alternating series test; absolutely we compare it to harmonic using limit comparison test and it diverges so it is also **conditionally convergent**

d. the numerator makes it alternative +/- so it is alternating harmonic which is conditionally convergent (passes alternating series test but absolute value diverges in limit comp test to harmonic): **conditionally convergent**

e. **absolutely convergent** since each term is zero

f. terms approach 2/3 in magnitude so **divergent**

g. using absolute ratio test we get ratio below 1 so it **converges absolutely**

h. absolute terms converge by limit comparison test so **converges absolutely**

i. using absolute ratio test we get ratio>1 so it **diverges**

j. series does not alternative but has positive and negatives… if we take the absolute value of terms, it is always less than or equal to 1/n! which converges, so it **converges absolutely**

k. using absolute ratio test we get ratio>1 so it **diverges**

l. passes alternating series test for convergence but absolute values diverge (using limit comparison test relative to harmonic) so **conditionally convergent**

m. by absolute ratio test ratio < 1 so it **converges absolutely**

n. **converges absolutely** by absolute ratio test

o. **converges absolutely** since absolute value of terms is always below $1.1/n^2$

p. **diverges** by absolute ratio test

q. **converges absolutely** by absolute ratio test

2. Alternating harmonic and ones like that.

3. Impossible—if the ratio test on the absolute values shows divergence then the magnitude of the terms is eventually increasing. Therefore even a series with some positive and some negative cannot converge since the later terms are the largest and will likely change the sign of the sum (so it can't converge).

4a. converges by LCT vs 1/n^1.5

b. converges: geometric with |r|<1

c. diverges by nth-term test since terms approach (are always equal to) ½

d. using integral test we get -1/(lnn) which converges

e. converges using limit comparison test relative to 1/n^2

f. converges- p series with exponent > 1

g. converges by the ratio test

h. converges by the ratio test

i. converges by comparison test to 1/n^1.5

j. comparison test vs (3/5)^n – always smaller than convergent geom. Series so converges

k. converges by integral test

l. diverges by nth term test

Unit 2 Handout #8: The Absolute Ratio Test

In each question below, determine the *x*-values for which the series converges. Use the absolute ratio test and then check the endpoints with other convergence tests.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n+3}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n} x^n}{n^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot 2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)(2x+3)^n}{n^2 \cdot 2^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(2x-1)^n$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$

10. $\sum_{n=1}^{\infty}\frac{(x-3)^n 2^n}{n+3}$

11. $\sum_{n=1}^{\infty} n!(2x-1)^n$

12. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{5n}$

13. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{5n\sqrt{n}}$

Answers

1. $-1\le x<1$ 2. all x 3. $-5<x<5$ 4. $-\frac{1}{9}\le x\le\frac{1}{9}$ 5. $-2<x\le 2$ 6. $-2.5\le x\le -0.5$

7. $-1.5<x<2.5$ 8. $0<x\le 2$ 9. all x since ratio is always 0 1. $2.5\le x<3.5$ 11.1/2 only

12. $4<x<6$ by ratio test and at 4 it will and at 6 it won't so $4\le x<6$ 13. $4\le x\le 6$

Unit 2 Handout #9: Practice

1. Find the sums of these telescoping and geometric-ish series.

a. $\sum_{n=1}^{\infty} \frac{4}{n^2 + 2n}$ b. $\sum_{n=1}^{\infty} \frac{5 + (-2)^n}{3^n}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

2. Convergence of sequences. To what value does each converge, or does it diverge?

a. $\left\{ \left(\frac{n-2}{n} \right)^n \right\}_{n=1}^{\infty}$ b. $\left\{ \frac{\sin n}{n^2} \right\}_{n=1}^{\infty}$ c. $\left\{ \frac{2n-3}{3n+2} \right\}_{n=1}^{\infty}$

3**. Determine whether each series below converges. Describe how you know (ie, which test.. if a comparison test then comparing it to what series… if test involves a ratio then give the value of the ratio).

a. $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$ b. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2}$ c. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

**Some of these problems are from "Calculus: Single Variable" by Stewart

d. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

e. $\sum_{n=1}^{\infty} (-e)^n$

f. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

g. $\sum_{n=1}^{\infty} \left(\frac{n^2+3}{2n^2+5} \right)^n$

h. $\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2+5}$

i. $\sum_{n=1}^{\infty} \frac{n+5}{5^n}$

j. $\sum_{n=2}^{\infty} (-1)^n 2^{1/n}$

k. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

l. $\sum_{n=1}^{\infty} \frac{3^n}{(2n+1)!}$

4**. Determine whether each sequence converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos n}{n\sqrt{n}}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n}$

5**. Find the x's for which each series converges. Use the absolute ratio test and then check the endpoints using other convergence tests we have done.

a. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 5^n}$ b. $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n+3}$ c. $\sum_{n=1}^{\infty}\frac{(-1)^n(x-2)^n}{2^n}$

d. $\sum_{n=1}^{\infty}2^n(x)^{2n}$ e. $\sum_{n=1}^{\infty}nx^n$

Answers

1a.3 since it is (2/n)-2/(n+2) b. 2.1 c. 1 2a. 1/e^2 b. 0 c. 2/3

3a. converges by comparison test or LCT comparing to $(1/3^n)$

b. converges by alternating series test→ terms decrease in magnitude towards zero

c. converge by ratio test—limit of ratio of successive terms is 1/e, which is <1

d. converge by comparison test… this quickly becomes less that sqrt(n)/n^2, which converges

e. diverges by nth term test f. converges by ratio test… limit approaches zero

g. converges by limit comparison test, as this approaches $(1/2)^n$ h. diverges by nth term test

i. converges by ratio test… ratio of successive terms approaches 1/5

j. div since terms → 1 k. conv by ratio test—ratio →0 l. converges by ratio test—ratio → 0

4a. converges conditionally b. converges absolutely c. converges absolutely by ratio test

5a. $-2\le x<8$ b. $-4\le x<-2$ c. $0<x<4$ d. $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ e. $-1<x<1$

Unit 3 Handout #1: Power Series Geometrically

1. Write the following series in Σ notation. (You will soon see that these are MacLaurin series.)

a. $1+x+x^2+x^3+x^4+\ldots.$

b. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots.$ (this is e^x)

c. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}\ldots$

(this is sinx)—hint—expect some (2x) or (2x+1)'s

d. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}\ldots$ (this is cosx)

2. Write polynomial expansions of each function below using the infinite geometric series definition. Write them using Σ notation. Give the *x* values for which the series converges. In some case there may be multiple correct answers.

a. $f(x)=\frac{1}{1-x^2}$

b. $f(x)=\frac{1}{1+2x}$

c. $f(x)=\frac{5}{2-x^3}$

d. $f(x)=\frac{3}{x^2}$ (hint: write it as something over 1-_)

3. Writing a polynomial expansion of $f(x)=\dfrac{1}{2-x}$, Ben decides to divide all terms by 2 and write it as $f(x)=\dfrac{0.5}{1-0.5x}$, yielding $\dfrac{1}{2}+\dfrac{x}{4}+\dfrac{x^2}{8}+\dfrac{x^3}{16}+\dfrac{x^4}{32}+.....$ Teddy, his twin brother, decides to write it as $f(x)=\dfrac{1}{1-(x-1)}$, yielding $1+(x-1)+(x-1)^2+(x-1)^3+.....$ They look at each other's answers, realize they are very different functions, and start arguing about who is correct. Help them resolve their disagreement. A graph may help.

4. In question #3, for what x values does each twin's series converge to $f(x)$?

5. Answer the following questions about $f(x) = \dfrac{1}{1+x}$.

a. Write a power series for $f(x)$ in Σ notation.

b. Use that power series to write power series for $g(x) = \dfrac{x}{1+x}$ *and* $h(x) = \dfrac{x^2}{1+x}$.

c. You want to approximate the integral $\int_0^1 \dfrac{x^2}{1+x}\,dx$. Use the first five or six terms of the series you wrote for $h(x)$ to approximate the integral (average the two sums). Then find the exact value of the integral by using anti-derivatives (hint: divide it)

6. You want to approximate the integral $\int_0^1 \frac{\sqrt{x}}{1+x}dx$. Use the first five or six terms of a series to approximate the integral (average the two sums). Then find the exact value of the integral by using anti-derivatives (I recommend u-substitution and don't forget to substitute for the dx term!)

7. Write a power series for $f(x) = \tan^{-1} x$ by integrating a series for $\frac{1}{1+x^2}$. Use Σ notation. Do you need to worry about the C when you integrate? Why or why not?

8. Starting with your answer to question 7 above, write power series for each of the following. Adjust the series term-by-term.

a. $x^2 \tan^{-1} x$

b. $\dfrac{\tan^{-1} x}{x}$

c. $\displaystyle\int \frac{\tan^{-1} x}{x}\, dx$

9. Given the power series you have for $f(x) = \tan^{-1} x$ above, can you think of it as a power series for $\tan^{-1} u$ and substitute any expression for *u* in it? In other words: you want a power series for $\tan^{-1}(x^2)$. Try to substitute *x*-squared for *u*. Is this valid? How can you use your calculator to check?

10. Here's another (?) way to get a power series for $\tan^{-1}(x^2)$: take its derivative, write its derivative as a power series, and then integrate that. Try it and see what you get.

11. Write a power series for $f(x) = \ln x$ by integrating a series for its derivative (note—it will be in terms of powers of (1-*x*) not of *x*). Also note that $\ln 1 = 0$ so find an appropriate C.

12. Write a power series for $f(x) = \ln(x^2 + 1)$ in two ways:

a. By substituting into your answer to 11.

b. By taking its derivative, writing it as a power series, and integrating it.

13. Since $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, a power function for it should be the derivative of the power function for $\frac{1}{1-x}$. Find it, and write it using Σ notation.

14. Use a power series to approximate $\int_0^1 \frac{x}{1+x^6}\,dx$. Use the average of the first 5 and 6 terms.

15. Find a power series for $\frac{x}{x^2-3x+2}$ by using partial fractions. Give the interval of convergence.

16. Given the differential equation $\frac{dy}{dx} = \frac{1}{1-x^3}$ and an initial condition of (0,2), use a series approximation to estimate the *y* value when *x*=0.5. Hint: write a series for *dy/dx* and then integrate it—make sure you solve for C. What happens if you try to solve for *y* when *x*=2?

17. Convergence of power series involving *x*. The three possibilities for where they converge are: (1) for all *x*; (2) for one value of *x* only; and (3) for *x* within a given distance R of a given point (the "*c*"). The ratio test will help us find these, though in scenario (3) it is inconclusive at the endpoints of the interval and they need to be separately tested using another convergence test.

Find the radius of convergence and the interval of convergence for each power series below. Be sure to check the endpoints of the interval when appropriate.

a. $\sum_{n=1}^{\infty} \frac{x^n}{2n^2}$ b. $\sum_{n=1}^{\infty} \frac{nx^n}{10^n}$ c. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$

d. $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ e. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n}$ f. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$

g. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$ h. $\sum_{n=1}^{\infty} \frac{n^2 (x-7)^n}{2n^2+5}$ i. $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+3)^n$

18. Errors in alternating series: Given the infinite sum (convergent) $\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12}$..... we want to approximate it with the first four terms only. Show that the actual sum must be above the sum of the first four terms. Then show that the sum must be smaller than the sum of the first five terms. Give an interval that the actual sum must lie in.

19. If we have the sum of a convergent alternating series, the largest error possible by approximating the sum using the sum of the first n terms is the (n+1)st term. Use that fact to answer the following questions:

a. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, what is the maximum absolute error by using the first 4 terms only?

b. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, how many terms are necessary to have the error be < 0.01?

c. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$, what is the maximum absolute error by using the first 4 terms only?

d. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$, how many terms are necessary to have the error be < 0.00001?

e. Given the series $\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2^n \cdot n!}$, approximate the sum to four decimal places.

Answers

1a. $\sum_{n=0}^{\infty} x^n$ b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ c. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$ d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

2a. $1 + x^2 + x^4 + ... = \sum_{n=0}^{\infty} x^{2n}$ converges where -1 < x < 1

b. $1 - 2x + 4x^2 - 8x^3 + ... = \sum_{n=0}^{\infty} (-2x)^n$ converges where -0.5<x<0.5

c. write as $f(x) = \frac{2.5}{1 - 0.5x^3}$ so $2.5 + 2.5\frac{x^3}{2} + 2.5\frac{x^6}{4} + ... = \sum_{n=0}^{\infty} 2.5 \cdot \frac{x^{3n}}{2^n}$ converges for $-\sqrt[3]{2} < x < \sqrt[3]{2}$

or $f(x) = \frac{5}{1 - (x^3 - 1)}$ so $5 + 5(x^3 - 1) + 5(x^3 - 1)^2 + ...\sum_{n=0}^{\infty} 5(x^3 - 1)^n$ converges for $0 < x < \sqrt[3]{2}$

d. $f(x) = \frac{3}{1 - (1 - x^2)} = 3 + 3(1 - x^2) + 3(1 - x^2)^2 + ... = \sum_{n=0}^{\infty} 3(1 - x^2)^n$ converges for 0 < x < $\sqrt{2}$

3. both seems OK

4. Ben's converges for -2<x<2 and Teddy's for 0<x<2

5a. $f(x) = 1 - x + x^2 - x^3 + ... = \sum_{n=0}^{\infty} (-1)^n x^n$

b. $g(x) = x(1 - x + x^2 - x^3 + ...) = \sum_{n=0}^{\infty} (-1)^n x^{n+1}$ & $h(x) = x^2(1 - x + x^2 - x^3 + ...) = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

c. $\int_0^1 (x^2 - x^3 + x^4 - x^5 + x^6 - x^7)dx = \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8}$

five terms yields 109/420 or 0.259; six terms yields 113/840 or 0.135 (big difference); avg is 0.197

exact: divide it or split it and get $\int_0^1 \frac{x^2 - 1}{x + 1} + \frac{1}{x + 1} = 0.5x^2 - x + \ln(x + 1) = \ln 2 - 0.5 \approx 0.193$.

Note: as a geometric series, our power series converges where -1<x<1. From the integral, when x=1 we get most terms of an alternating harmonic, so it will converge at x=1 but not above 1.

6. the series $x^{1/2} - x^{3/2} + x^{5/2} - x^{7/2} + x^{9/2} - x^{11/2}$;

The integral is $\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{9}x^{9/2} + \frac{2}{11}x^{11/2} - \frac{2}{13}x^{13/2}$

First 5 terms is 0.512 and6 terms is 0.358 so average is 0.435

Integrating: $u = \sqrt{x}$ $du = \frac{dx}{2\sqrt{x}}$ $so \int \frac{u(2udu)}{u^2 + 1} = \int \frac{2u^2 du}{u^2 + 1} = \int \frac{2u^2 + 2 - 2}{u^2 + 1} du = 2u - 2\tan^{-1} u$

So $2\sqrt{x} - 2\tan^{-1}(\sqrt{x})$ from 0 to 1 is 2-0.5π=0.429. the approximation is pretty close!

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$ since $\tan^{-1} 0 = 0$ we're OK with not having a C explicitly

8a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n-1}$ b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{2n-1}$ c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)^2}$

9. if $\tan^{-1} u = u - \frac{u^3}{3} + \frac{u^5}{5} + \ldots \text{ } then \text{ } \tan^{-1}(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-2}}{2n-1}$

Can graph the first several terms and the function $\tan^{-1}(x^2)$ and compare

10. $\tan^{-1}(x^2) = \int \frac{d}{dx}(\tan^{-1}(x^2))dx = \int \frac{2x}{1+x^4} dx = \int (2x) + (2x)(-x^4) + (2x)(-x^4)^2 + \ldots$

Which is $x^2 - \frac{2x^6}{6} + \frac{2x^{10}}{10} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-2}}{2n-1}$ as in question #9. Math is so cool!

11. $\ln x = \int \frac{1}{x} dx = \int \frac{1}{1-(1-x)} dx = \int 1 + (1-x) + (1-x)^2 + (1-x)^3 = x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots$

But since ln1=0 we need a -1; we can combine the -1 with the first x to get -(1-x) and thus

The series is thus $\sum_{n=1}^{\infty} \frac{-1(1-x)^n}{n}$

12a. If $\ln u = \sum_{n=1}^{\infty} \frac{-1(1-u)^n}{n}$ then $\ln(x^2+1) = \sum_{n=1}^{\infty} \frac{-1(-x^2)^n}{n} \text{ } or \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n}$

b. $\ln(x^2+1) = \int \frac{2x}{x^2+1} dx = \int \frac{2x}{1-(-x^2)} dx = \int 2x + 2x(-x^2) + 2x(-x^2)^2 + \ldots = x^2 - \frac{2x^4}{4} + \frac{2x^6}{6} - \ldots$

no C is necessary since when x=0 ln(x^2+1)=0; this is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n}$ as in 12a

13. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ its derivative is $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots. = \sum_{n=1}^{\infty} nx^{n-1}$

14. $\int x - x^7 + x^{13} - x^{19} + x^{25} - x^{31} = \frac{x^2}{2} - \frac{x^8}{8} + \frac{x^{14}}{14} - \frac{x^{20}}{20} + \frac{x^{26}}{26} - \frac{x^{32}}{32}$ avg of 5th and 6th terms is 0.419

15. x = A/x-2 + B/x-1 → x=A(x-1)+B(x-2) when x=1 B=-1 A=2

$$\frac{x}{x^2-3x+2} = \frac{2}{x-2} - \frac{1}{x-1} = -\frac{1}{1-0.5x} + \frac{1}{1-x} = -[1 + 0.5x + (0.5x)^2 + \ldots] 1 + x + x^2 + x^3 + \ldots$$

$= \frac{x}{2}+\frac{3x^2}{4}+\frac{7x^3}{8}+\ldots = \sum_{n=1}^{\infty}\frac{(2^n-1)x^n}{2^n}$ as long as -1<x<1 (it checks graphically- needs lots of terms to stay close as x gets closer to 1)

16. $y' = 1+x^3+x^6+x^9+\ldots$ so $y = C+x+x^4/4+x^7/7+x^{10}/10+\ldots = 2+\sum_{n=1}^{\infty}\frac{x^{3n-2}}{3n-2}$. When x=0.5, using four terms plus the constant we get 2.51684. A closer answer, using fnInt is 2.51685. Looking at it as a geometric series, it converges from -1<x<1; from the power-series, the ratio test shows it converging on the interval [-1,1). So it is not helpful for x=2 (it would give an infinitely large answer!). See below (after #19) for a way to integrate it without using series to get the exact answer.

17a. radius =1 converge for -1≤x≤1 b. radius = 10 converges for -10<x<10

c. radius = 3; converges for -1≤x<5 d. converges for all x

e. radius=3; converges for -4<x<2 f. radius=1; converges for -3≤x<-1

g. radius=2; converges for -3<x<1 h. radius=1; converges 6<x<8

i. radius=4; converges for -7<x<1

18. the first four terms sum to 13/120…. The sum of the 5th and 6th is positive, as are the sums of the 7th and 8th, 9th and 10th, and so on. Thus the terms we exclude (the fifth on) have a positive sum. The first five terms sum to 23/120. The sum of the 6th and 7th is negative, as is the sum of the 8th and 9th, 10th and 11th. So the terms after the fifth have a negative sum and thus we know the actual sum is in the rather wide interval (13/120, 23/120). Pretty wide range of possibilities!

19a. the 5th term which is 1/25 b. 10terms c. 1/120 d. 9th term is first below it so 8th terms

e. 0.6065 .. $1-\frac{1}{2}+\frac{1}{8}-\frac{1}{48}+\frac{1}{384}-\frac{1}{3840}+\frac{1}{46080}$.... stop after the 6th term….

FYI: integrating #16

Start with partial fractions, remembering that $a^3 - x^3 = (a-x)(a^2 + ax + x^2)$

So $\int \frac{1}{1-x^3}\,dx = \int\left(\frac{1/3}{1-x} + \frac{1/3x + 2/3}{1+x+x^2}\right)dx$

Now split the last term into something with like 2ax+a for CRS as a ln of something:

$$\int\left(\frac{1/3}{1-x} + \frac{x/3+2/3}{1+x+x^2}\right)dx = \frac{-1}{3}\ln|1-x| + \int \frac{x/3+1/6}{1+x+x^2}dx + \int \frac{1/2}{1+x+x^2}dx$$

Integrate the middle term and complete the square on the last term's denominator:

$$= \frac{-1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \int \frac{1/2}{(x+1/2)^2+3/4}dx$$

The last term will be an inverse tangent… once we get chain rule satisfaction. First we need the ¾ to be a 1, so multiply all terms by 4/3:

$$= \frac{-1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \int \frac{2/3}{4/3(x+1/2)^2+1}dx$$

Now make the first term in the denominator "something squared":

$$= \frac{-1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \int \frac{2/3}{\left[(2/\sqrt{3})(x+1/2)\right]^2+1}dx$$

Now get chain rule satisfaction by adjusting the numerator:

$$= \frac{-1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\int \frac{2/\sqrt{3}}{\left[(2/\sqrt{3})(x+1/2)\right]^2+1}dx$$

And now use inverse tangent:

$$= \frac{-1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(2/\sqrt{3})(x+1/2)\right) + C$$

Find C and then plug in 0.5; yielding an answer of 2.5168491839…

Power Series Handout #2: Taylor and Maclaurin Series

1. Find the Maclaurin series for $f(x)=\cos x$ and show that it converges for all x (use the absolute ratio test). Write the series in sigma notation.

2. Use your answer to question #1 above to do the following:

a. Find the Maclaurin series for $f(x)=\cos(x^2)$.

b. Use the first four terms of that series to approximate the value of $\int_0^1 \cos(x^2)dx$.

c. What is the largest possible error for your approximation in part b? Also, you know the exact answer must be between what numbers (give the narrowest range possible).

d. Use fnInt to find the exact answer.

3. Find the interval of convergence for each power series below. Be sure to check the endpoints of the interval when appropriate.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n+4}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{2^n}$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{n(x-2)^n}{n^2+1}$

d. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x+1)^n}{\sqrt{n}}$

e. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)(x-4)^n}{10^n}$

f. $\displaystyle\sum_{n=1}^{\infty}\frac{n!x^n}{100^n}$

g. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2x-1)^n}{n\cdot 6^n}$

h. $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n+1}}{(-4)^n}$

i. $\displaystyle\sum_{n=1}^{\infty}\frac{2^nx^{2n}}{(2n)!}$

4. Use the formulas for Taylor and Maclaurin series to write series for each of the following functions. Write the first four terms out individually; then try using Σ notation if you recognize a pattern.

a. $f(x) = e^x$ at x=0

b. $f(x) = \dfrac{1}{\sqrt{x}}$ at x=1

c. $f(x) = \ln x$ at x=3

d. $f(x) = \int_{0.5}^{x} \sin(\pi t)\,dt$ at x=0.5

5. Use your answer to question 4a above and substitution to find Maclaurin series for $f(x) = e^{3x}$, $f(x) = e^{-x^2}$, *and* $f(x) = 7xe^{x}$.

6. How does the Maclaurin series for e^x give one common definition of *e*, which is $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

Answers

1. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-.....=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}$ convergence: ratio is $\frac{x^2}{(2n+2)(2n+1)}$ which approaches zero as n approaches infinity for all values of x.

2a. $1-\frac{x^4}{2!}+\frac{x^8}{4!}-.....=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{4n}}{(2n)!}$

b. $\int\cos(x^2)dx=x-\frac{x^5}{5\cdot 2!}+\frac{x^9}{9\cdot 4!}-\frac{x^{13}}{13\cdot 6!}....$ first 4 terms gives 0.904523 as fraction 25399/28080

c. error is less than the next term so less than 1/(17*8!) = 1/685440. So answer is between 25399/28080 and this plus 1/685440 -→ between 0.90452279 and 0.90452425

d. fnInt gives 0.90452424

3a. -1≤x<1 b. -2<x<2 c. 1≤x<3 d. -2<x≤0 e. -6<x<14 f. x=0 only g. -2.5<x≤3.5

h. -2<x<2 i. everywhere

4a. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

b. $1-\frac{1}{2}(x-1)+\frac{3}{4}\cdot\frac{(x-1)^2}{2!}-\frac{15}{8}\cdot\frac{(x-1)^3}{3!}+...=\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot..(2n-1)}{2^n\cdot n!}(x-1)^n$

c. $\ln 3+\frac{1}{3}(x-3)-\frac{1}{9}\cdot\frac{(x-3)^2}{2!}+\frac{2}{27}\cdot\frac{(x-1)^3}{3!}-\frac{6}{81}\cdot\frac{(x-4)^3}{4!}+...=\ln 3+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n-1)!}{3^n\cdot n!}(x-3)^n$

d. $(x-0.5)-\frac{\pi^2}{3!}(x-0.5)^3+\frac{\pi^4}{5!}(x-0.5)^5+...=\sum_{n=1}^{\infty}(-1)^{n+1}\pi^{2n-2}\frac{(x-0.5)^{2n-1}}{(2n-1)!}$

5. You can get the Maclaurin for $f(x)=e^{3x}$ by just substituting “3x” everywhere “x” appears in the Maclaurin series for $f(x)=e^x$ (no “chain rule” adjustment like multiplying by 3).

So the series for e^{3x} will be $1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+....=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$

For e^{-x^2} we get $1+(-x^2)+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+....=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}$ or $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

For $7xe^x$ multiply the series for $f(x)=e^x$ by 7x to get $\sum_{n=0}^{\infty}\frac{(7x)x^n}{n!}$

6. Take the series for e^x and plug x=1 in, yielding the series for e

Power Series Handout #3: More Taylor and Maclaurin Series

1. Write the Taylor series for each function below at the specified point. Try to use Σ notation if you can.

a. $f(x) = \sqrt[3]{x}$ at x=1

b. $f(x) = \int_0^x \frac{1}{(t+1)^2}\,dt$ at x=0

c. $f(x) = e^x$ at x=2

2. Using what you (hopefully) know about the Maclaurin series for sine, cosine and e^x, write Maclaurin series for each function below. You should NOT have to do it by taking nth derivatives of the functions! Hint: there are some that might be easiest done by rewriting them.

a. e^{-x^2}

b. $\cos(x^2)$

c. $\frac{\sin x}{x}$

d. $\frac{1-e^{-x}}{x}$

e. $\frac{\sin^2 x}{x}$

f. $\sin x \cos x$

g. $e^x \cos x$

(5th degree polynomial)

3. Using your work in part 2c above, show that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

4. In accelerated trig (or maybe regular trig), you should have learned that you can write complex numbers in cis form, where $cis(\theta)=\cos\theta+i\sin\theta$. Use the Maclaurin series results for sine, cosine, and e^x to show that $e^{i\theta}=\cos\theta+i\sin\theta$ Then show (as Euler did) that $e^{i\pi}+1=0$. [This famous equation relates five of the most important numbers in math.]

5. Find the radius of convergence and the interval of convergence for each of the following power series. Be sure to check the endpoints!

a. $1-\frac{3x}{\sqrt{2}}+\frac{9x^2}{\sqrt{3}}-\frac{27x^3}{\sqrt{4}}+\frac{81x^4}{\sqrt{5}}+\ldots.$

b. $\sum_{n=1}^{\infty}(\tan x)^n$ on interval [-0.5π,0.5π]

c. $\sum_{n=0}^{\infty} \frac{nx^n}{10^n}$

d. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

e. $\sum_{n=0}^{\infty} \frac{(x-1)^n \sqrt{n}}{3^n}$

f. $1+\frac{1}{2}\cdot\frac{x}{2}+\frac{1}{3}\cdot\frac{x^2}{4}+\frac{1}{4}\cdot\frac{x^3}{8}+\ldots$

g. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

h. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$

6. Does the following converge? [one idea: write it in terms of an *x* and see which *x*'s it converges for.]

$$1+\frac{1.01}{2}+\frac{1.01^2}{3}+\frac{1.01^3}{4}+\frac{1.01^4}{5}+\ldots$$

7. Answer the following questions:

a. Write the Maclaurin series for $f(x) = \cos(x^2)$. Use substitution!

b. Use your answer to part *a* to find the Maclaurin series for $2x\cos(x^2)$.

c. Write the Maclaurin series for $f(x) = \sin(x^2)$.

d. Use the series you wrote to show that the derivative of $\sin(x^2) = 2x\cos(x^2)$.

8. Given that $f(1) = 2,\ f'(1) = -1,\ f''(1) = 0,\ and\ f'''(1) = 1$ do the following:

a. Write the third degree Taylor series for $f(x)$ at x=1.

b. Use it to approximate $f(1.2)$.

c. Use this to approximate the average value of $f(x)$ on the interval [1,1.25]

9. The function $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ is called the hyperbolic cosine function and comes up often in engineering. (I believe it describes the shape of a cable hung from its ends—like on a suspension bridge). Write the Maclaurin series for it by manipulating the series for e^x.

10. Each alternating series below converges. Find the upper bound of the absolute error for the *n*th partial sum (this means the sum of the first *n* terms).

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ first 5 terms

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ first 5 terms

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ first 4 terms

11. Each alternating series below converges. To have the absolute value of error within the given bounds, how many terms are necessary?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $|\text{error}|<0.0001$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $|\text{error}|<0.001$

12. Answer the questions about the series $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n\cdot 3^n}$

a. What is its radius of convergence and interval of convergence?

b. Use the first three terms to approximate $f(-1)$.

c. What is the upper bound on the absolute error after 3 terms?

d. Still approximating $f(-1)$, how many terms do you need to have the absolute error below 0.001?

13. You want to use the Maclaurin series for e^x to approximate $1/e$.

a. What is the interval of convergence for the series?

b. If you use the first four terms, what is your approximation?

c. What is the upper bound on your error from this approximation?

d. You want to approximate $1/e$ to the nearest one-hundredth, so you need to have an error of no more than 0.005. How many terms do you need?

e. How do your answers to parts b, c, and d change if you want to use the series to approximate e^2?

14. You want to approximate the integral $\int_0^1 \sin(x^2)dx$.

a. Write the first four terms in the Maclaurin series for $\sin(x^2)$. (Use substitution)

b. Integrate it term by term.

c. Evaluate it on this interval.

d. What is the largest possible error bound on your answer?

15. You want to approximate the integral $\int_0^1 e^{-x^2}dx$.

a. Write the first four terms in the Maclaurin series for e^{-x^2}

b. Integrate it term by term.

c. Evaluate it on this interval (approximating with the first four terms.)

d. What is the largest possible error bound on your answer?

16. You want to approximate the integral $\int_0^1 \frac{\sin x}{x}\,dx$. Approximate it with the first four non-zero terms and then give the largest possible error.

ANSWERS

1a. $1+\frac{1}{3}(x-1)-\frac{2}{9}\cdot\frac{(x-1)^2}{2!}+\frac{10}{27}\cdot\frac{(x-1)^3}{3!}-\frac{80}{81}\cdot\frac{(x-1)^4}{4!}+\ldots=1+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\cdot5\cdot8\cdot\ldots(3n-4)}{3^n}\cdot\frac{(x-1)^n}{n!}$

b. $x-2\frac{x^2}{2!}+6\frac{x^3}{3!}-24\cdot\frac{x^4}{4!}+\ldots.=\sum_{n=1}^{\infty}(-1)^{n+1}x^n$

c. $e^2+e^2(x-2)+\frac{e^2(x-2)^2}{2!}+\frac{e^3(x-2)^3}{3!}+..\sum_{n=0}^{\infty}\frac{e^2(x-2)^n}{n!}$

2a. $1+(-x^2)+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}=\sum_{n=0}^{\infty}\frac{(-x)^{2n}}{n!}$

b. $1-\frac{(x^2)^2}{2!}+\frac{(x^2)^4}{4!}-\frac{(x^2)^6}{6!}+..=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$

c. $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^5}{7!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}$

d. $1-\frac{x}{2!}+\frac{x^2}{3!}-\frac{x^3}{4!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(n+1)!}$

e. rewrite as $(1-\cos(2x))/(2x)$ and then it becomes $\frac{2x}{2!}-\frac{(2x)^3}{4!}+\frac{(2x)^5}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+2)!}$

f. rewrite as $\sin(2x)/2$ and get $\frac{2x}{2}-\frac{(2x)^3}{2\cdot3!}+\frac{(2x)^5}{2\cdot5!}-\frac{(2x)^7}{2\cdot7!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{2\cdot(2n+1)!}$

g. $1+x-\frac{x^3}{6}-\frac{x^4}{6}-\frac{x^5}{30}$???? (it looked pretty close on my calculator!)

3. as x → zero all of the x terms drop out and you get 1 (cool!)

4. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\ldots$ $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\ldots$

$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+..$ so $e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+..$

Which is $e^{i\theta}=1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+..$ given the powers of *i*

And this is $\cos\theta+i\sin\theta$; so $e^{i\pi}=\cos(\pi)+i\sin(\pi)=-1$

5a. this is $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(3x)^n}{\sqrt{n+1}}$ and by ratio test its radius of convergence is 1/3 and interval is $\left(\frac{-1}{3},\frac{1}{3}\right]$

b. this is geometric in tan so converges when $|\tan x|<1$ *so* $-0.25\pi<x<0.25\pi$

radius of convergence is 0.25π and interval is (0.25π,0.25π)

c. radius is 10 and interval is (-10,10) d. the ratio is x^2/(n+1) so it converges for all x

e. radius is 3 and interval is (-2,4) f. this is $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)\cdot2^n}$ so radius is 2 and interval is [-2,2)

g. radius = 2 interval is (-3,1) h. radius is 1 and interval is [-3,-1)

6. write it as $1+\frac{x}{2}+\frac{x^2}{3}+....+\frac{x^n}{n+1}$ which converges for -1≤x<1 so does **not** converge at x=1.01 (or note that it is term by term at least as large as the harmonic, so it must diverge)

7a. $\cos(x^2)=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\ldots$ b. $2x\cos(x^2)=2x-\frac{2x(x^4)}{2!}+\frac{2x(x^8)}{4!}-\frac{2x(x^{12})}{6!}+\ldots$

c. $\sin(x^2)=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\frac{x^{18}}{9!}+\ldots$ d. $\frac{d(\sin(x^2))}{dx}=2x-\frac{2x^5}{2!}+\frac{2x^9}{4!}-\frac{2x^{13}}{6!}+\ldots$

8a. $2-(x-1)+\frac{1}{6}(x-1)^3$ b. $f(1.2)\approx 1.80133$ c. $f(x)\approx 3-x+\frac{1}{6}(x-1)^3$ *so* $\frac{1}{0.25}\int_1^{1.25} f(x)dx\approx 1.8757$

9. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+..$ $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-..$ so $\cosh(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\frac{x^{2n}}{(2n)!}$

10a. 1st 5 terms → error is bound by 6th term which is 1/36 b. 1/720 c. 1/10

11a. 100 terms since 100th is 0.0001 and we want it to be smaller than that…

b. 1000 terms since the 1000th is 0.001 and we want smaller error…

(the fact that these error bounds were less than and not less-than-or-equals adds a term in these cases)

12a. radius is 3 and converges [-3,3) b. $f(1)=\frac{x}{3}+\frac{x^2}{18}+\frac{x^3}{81}$ so -47/162

c. the fourth term, which is 1/324 d. 5th term is 1/1215 so 4 terms is fine

13a. all numbers

a. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+..$ so it is $1-1+1/2-1/6=1/3$ b. the 5th term, which is 1/24

d. 1/n! < 1/200 so n=6 and need the term with the 5! -so you need six terms

e. approx is 1+2+2+8/6=19/3 and you cannot use the error approx from alternating series

14a. $\sin(x^2)=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots$ b. $\int\sin(x^2)=\frac{x^3}{3}-\frac{x^7}{7\cdot 3!}+\frac{x^{11}}{11\cdot 5!}-\frac{x^{15}}{15\cdot 7!}+\ldots$

c. this is 1/3 - 1/42 + 1/1320 - 1/75600 or 0.3103

d. next term is 1/(19*9!) or about 1/7000000

15a. $e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+..$ b. $\int e^{-x^2}dx=x-\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+..$

c. 1 -1/3 + 1/10 – 1/42 = 156/210 d. next term, which is 1/(9*4!) or 1/216

16. $1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}-\frac{1}{7\cdot 7!}$ which is approximately 0.94608; next term is 1/(9*9!)= 1/ 3265920

Unit 3 Sheet #4: Miscellaneous Series Questions

1. A game works like this. You flip a fair coin. You keep flipping until you get a tails, in which case the game is over. You get $1 for each head that occurs before your first tail.

a. What is the probability that you receive $0? $1? $2?

b. Show that the expected value of the payoff is $\frac{1}{2}\cdot 0+\frac{1}{4}\cdot 1+\frac{1}{8}\cdot 2+....\frac{n-1}{2^n}$

c. What is the expected value of the payoffs equal to numerically?

2. Given that $\frac{dy}{dx} = xy + y$ and that y=2 when x=0, do the following:

a. Approximate y when x=1 using linearization.

b. Compute $\frac{d^2y}{dx^2}$ at (0,2). Hint: implicit differentiation. What does this tell you about the likely direction of the error to your answer to part *a* above?

c. Use Euler's method with 2 steps of equal size to approximate y when x=1.

d. Find the 2^{nd} degree MacLaurin series for y. Use it to approximate y when x=1.

e. Solve the differential equation analytically. Use this to find the exact value of y when x=1.

3. Find a function whose Maclaurin series is $1x+2x^2+3x^3+....+nx^n+....$

4. One question in the Quiz Bowl several years ago was to evaluate the following: $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx$.

a. Why will this integral have the same value as $2\int_0^{\infty}\frac{\sin^2 x}{x^2}dx$?

b. This integral cannot be solved easily using anti-derivatives. Write a power series for the term inside the integral. Hint: rather than writing an infinite polynomial for sin *x* and squaring it, substitute for $\sin^2 x$ as you would if you were just trying to do $\int \sin^2 x dx$.

c. Now integrate this power series, yielding a power series for the answer. Since the upper limit of integration is ∞, your answer should look something like $\lim_{x\to\infty}\sum_{n=1}^{\infty}....$

d. Former chemistry teacher Zhe correctly guessed that the value of the integral is π. Use a spreadsheet to show that this is approximately correct. (When I used Excel, I could not use a value of *x* greater than 20 or so).

5. Given a function of the form $f(x)=(x+1)^p$ where *p* is some constant do the following:

a. Write the Maclaurin series for $f(x)$ in terms of *p*.

b. The coefficients may look familiar. Can you write them as permutations or combinations?

c. If *p* is a positive integer, is the series infinite or finite? It should look familiar: from where?

d. Binomial expansion works for non-negative integers only; does this have the same limitation? (note: this is Newton's Binomial Theorem).

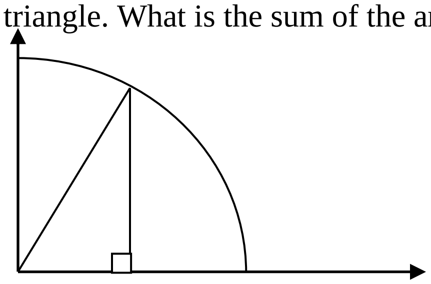
e. what happens when *p*= -1? Write out the expansion and see if there was an easier way to get it.

6. (continuation). One way to approximate π:

a. Use your result to question 3 above to write an expansion for $\sqrt{1-x}$. Write the first four terms and simplify them. Graph it on your calculator and compare it to the original function.

b. Now write the first four terms of the expansion for $\sqrt{1-x^2}$.

c. The diagram below shows a unit circle centered at the origin. The area in the first quadrant under the circle on the interval [0,0.5] can be expressed as $\int_0^{0.5} \sqrt{1-x^2}\,dx$ and also as the sum of a sector and a right triangle. What is the sum of the area of the sector and right triangle?



d. Now use the first four terms of the expansion from part *f* above to approximate the integral. Assuming this is exact, set the two expressions for area equal to each other and solve for π.

e. Add one more term to the series to improve the approximation.

f. Will this over- or under-estimate the value of π? Why?

7. (continuation): Do the same thing with a different circle:

a. Draw the circle $(x-0.5)^2 + y^2 = 0.25$. Show that the equation of the upper semi-circle can be given by $y = \sqrt{x-x^2} = \sqrt{x} \cdot \sqrt{1-x}$.

b. Write the first five terms a series for this using your answers to question 4b and 4e above. (Multiply the series for $\sqrt{1-x}$ by $\sqrt{x}$).

c. Integrate this from *x*=0 to *x*=0.25 to get the area of the part of the semi-circle. Give a decimal approximation.

d. Give an expression including π for that same area: sectors and a triangle!

e. Assuming the approximation in part *c* is exact, what must π be equal to?

8. Show that $\frac{1}{0!} + \frac{1\cdot 3}{1!} + \frac{1\cdot 3\cdot 5}{2!} + \frac{1\cdot 3\cdot 5\cdot 7}{3!} = \sum_{n=0}^{\infty} \frac{(2n+1)!}{n!\cdot 2^n \cdot n!}$

9. Use the results from the binomial expansion of $f(x)=(1+x)^p$ and some manipulation to find the first four terms in the Maclaurin series for $f(x)=\dfrac{1}{\sqrt{1-x^2}}$. Then use that result to find the beginning of Maclaurin series for $f(x)=\sin^{-1}(x)$. Use the first four terms of this to approximate π.

10. Find the exact value of the following sums. Most may be recognizable as the power series of a common function.

a. $5-\dfrac{5(0.2)^2}{2!}+\dfrac{5(0.2)^4}{4!}-\dfrac{5(0.2)^6}{6!}+\dfrac{5(0.2)^8}{8!}-\cdots$

b. $(0.2)^2-\dfrac{(0.2)^4}{3!}+\dfrac{(0.2)^6}{5!}-\dfrac{(0.2)^8}{7!}+\cdots$

c. $5-\dfrac{5(0.2)^2}{1!}+\dfrac{5(0.2)^4}{2!}-\dfrac{5(0.2)^6}{3!}+\dfrac{5(0.2)^8}{4!}-\cdots$

d. $\dfrac{5(0.2)^2}{2}+\dfrac{5(0.2)^4}{2}+\dfrac{5(0.2)^6}{2}+\dfrac{5(0.2)^8}{2}-\cdots$

e. $5+5+\dfrac{5}{2!}+\dfrac{5}{3!}+\dfrac{5}{4!}+\cdots$

f. $1+2+\dfrac{4}{2!}+\dfrac{8}{3!}+\dfrac{16}{4!}+\cdots$

g. $1-5+\dfrac{25}{2!}-\dfrac{125}{3!}+\dfrac{625}{4!}+\cdots$

11. For a long time, mathematicians tried to find the sum of the series $1+\frac{1}{4}+\frac{1}{9}+\ldots\frac{1}{n^2}$. This problem will walk you through Euler's solution.

a. How can we know that it must converge?

b. What are the zeros of the function $f(x)=\frac{\sin x}{x}$?

c. Given the zeros, write $f(x)=\frac{\sin x}{x}$ as an infinite polynomial in factored form. (as, for instance, a polynomial with zeros of 1, 2, and 7 can be written as $f(x)=a(x-1)(x-2)(x-7)$)

d. Use your answer to part *c* above to show that $f(x)=\frac{\sin x}{x}$ can also be written as $a(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\ldots$ which is also $a\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots$

e. Given what you know about $\lim_{x\to 0}\frac{\sin x}{x}$, what must the constant *a* be equal to?

f. When this messy expression $\frac{\sin x}{x}=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots$gets multiplied out, what will the coefficient of the x^2 term be? Hint: think about which products can generate x^2 terms.

g. Write the Maclaurin series for the function $f(x)=\frac{\sin x}{x}$.

h. Given two ways of writing $\frac{\sin x}{x}$ as infinite polynomials (from parts *f* and *g* above), the term-by-term coefficients must be equal. So set the coefficients of the x^2 terms equal to each other and solve for $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

12. Answer the following questions about series. Look for familiar patterns!

a. Solve for *x*: $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=1$

b. Find $x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\ldots$.

c. Find $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$.

d. If $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-\pi)^n}{n^2}$ find $f'''(\pi)$.

e. If $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{n!}$ find $f^{(7)}(5)$.

Some answers

1a. P(0)=0.5; P(1)=0.25 (need HT) P(2)=0.125 (need HHT) c. \$1 .. here's why:

$$\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\ldots=\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)+\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\right)+\left(\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots\right)$$

Each expression in parentheses is geometric, so this is $\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{8}\right)+\ldots=\frac{1/2}{1-1/2}=1$

2a. $\frac{dy}{dx}=2$ so 4 b. $\frac{d^2y}{dx^2}=x\frac{dy}{dx}+y+\frac{dy}{dx}$ and at (0,2) $\frac{dy}{dx}=2$ so 4. Too low; slope is becoming more positive to the right of x=0.

c. $f(0.5)\approx 3$ and $f(1)\approx 5.25$ d. $y\approx 2+2x+2x^2$ so y≈6

e. $\ln y=0.5x^2+x+C$ *so* $y=Ce^{0.5x^2+x}$ (0,2) → C=2 so $y=2e^{0.5x^2+x}$ and when x=1 $y=2e^{1.5}\approx 8.96$.

3. two ways to proceed from the teachers in room 206:

Howie's solution: $1x+2x^2+3x^3+....+nx^n=\quad x(1+2x+3x^2+4x^3+..)$.

Let this second term be called y: so the sum is xy where $y=1+2x+3x^2+4x^3+....$

Integrate y to get $\int y=\int 1+2x+3x^2+4x^3+....$ so $\int y=x+x^2+x^3+x^4+\ldots$

This is a geometric series so $\int y=\frac{x}{1-x}$ and so $y=\frac{(1-x)(1)-(x)(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}$

And thus the original sum, which was xy, is $\frac{x}{(1-x)^2}$

Mark's solution: $1x+2x^2+3x^3+4x^4....+nx^n$ is the sum of the following series:

Series #1 $x+x^2+x^3+x^4....+x^n$

Series #2 $x^2+x^3+x^4....+x^n$

Series #3 $x^3+x^4....+x^n$

…..

Series #n x^n

Each is geometric with ratio *x*: sum of series #1 is $\frac{x}{1-x}$; sum of #2 is $\frac{x^2}{1-x}$; #3 is $\frac{x^3}{1-x}$

Now sum the infinite series of these sums: these ***sums*** are geometric with ratio x so the total sum is

$$\frac{\frac{x}{1-x}}{1-x}=\frac{x}{(1-x)^2}$$

5a. $f(x) \approx 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + ...$ b. $f(x) \approx 1 + {}_pC_1 x + {}_pC_2 \cdot x^2 + {}_pC_3 \cdot x^3 + ...$

c. finite: we eventually get zeros as coefficients; it is binomial expansion / Pascal's Triangle

d. No, it always works! e. geometric.. expand 1/(1+x)

6a. $1 + \frac{1}{2}(-x) + \frac{1}{2}\cdot\left(\frac{-1}{2}\right)\left(\frac{1}{2!}\right)(-x)^2 + \left(\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\right)\frac{1}{3!}\cdot(-x)^3 = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16}$ b. $1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}$

c. $\frac{\pi}{12} + \frac{\sqrt{3}}{8}$ d. $\int_0^{0.5}\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}\right)dx = 0.5 - \frac{0.5^3}{6} - \frac{0.5^5}{40} - \frac{0.5^7}{112}$ which is about 0.478316

so $\frac{\pi}{12} + \frac{\sqrt{3}}{8}$ =0.478316 and thus π is about 3.1417.

e. with one more term: $\int 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{15x^8}{384}dx = 0.5 - \frac{0.5^3}{6} - \frac{0.5^5}{40} - \frac{0.5^7}{112} - \frac{15\cdot 0.5^9}{9\cdot 384} = 3.1416\ldots$

f. always over-estimate since all excluded terms are negative

7b. $\sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128}\right)$ c. 0.076774 d. 1/4 circle – triangle – sector $= \frac{\pi}{16} - \frac{\sqrt{3}}{32} - \frac{\pi}{48} = \frac{\pi}{24} - \frac{\sqrt{3}}{32}$

e. $0.076774 = \frac{\pi}{24} - \frac{\sqrt{3}}{32}$ so π=3.14161…. when it rounds to 3.14159. These 5 terms approximate π to about 10^{-5}; 10 terms gives accuracy of about 10^{-9} and 15 terms to within 10^{-12}. Pretty cool!

9. $(1+u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 \qquad + \ldots \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}{2^n\cdot n!}u^n$

So $(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots + \ldots \frac{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$

So $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{3}{8}\cdot\frac{x^5}{5} + \frac{5}{16}\cdot\frac{x^7}{7} + + \ldots \frac{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}{2^n\cdot n!}\cdot\frac{x^{2n+1}}{2n+1}$

$\frac{\pi}{2} = \sin^{-1}(1)$ so $\pi = 2\sin^{-1}1 = 2\left(1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{3}{8}\cdot\frac{1}{5} + \frac{5}{16}\cdot\frac{1}{7} + ...\right)$ which is 4321/1680.. not so good

10a. $5\cos(0.2)$ b. $0.2\sin(0.2)$ c. $5e^{-0.04}$ d. geometric: $\frac{0.1}{1-0.04} = \frac{5}{48}$ e. $5e$ f. e^2 g. e^{-5}

11a. p-series test b. π, 2π, 3π, -π, -2π all integer multiples of π except for 0 (not in the domain)

c. $\frac{\sin x}{x} = a(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$ e. a=1

f. $-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \ldots.$ since x^2 terms arise when all terms in a product except one are equal to 1

g. $\frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots.$ h. $-\frac{1}{3!} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ so $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

12a. π/2, 5π/2, 9π/2, …. (sinx=1) b. $x\sin x$ c. $\ln|1+k|$ d. -2/3 e. -8

Unit 3 Handout #5: AP Free-Response Questions on Series

2006 Form B #6: The function $f(x)$ is defined as $f(x)=\dfrac{1}{1+x^3}$. The Maclaurin series for *f* is given by $1-x^3+x^6-x^9+\ldots\ldots+(-1)^n x^{3n}$ which converges to $f(x)$ *for* $-1<x<1$.

a. Find the first three non-zero terms and the general term for the Maclaurin series for $f'(x)$.

b. Use your results from part (*a*) to find the sum of the infinite series $-\dfrac{3}{2^2}+\dfrac{6}{2^5}-\dfrac{9}{2^8}+\ldots.(-1)^n\dfrac{3n}{2^{3n-1}}$.

c. Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)dt$.

d. Use the first three nonzero terms of the infinite series found in part (*c*) to approximate $\int_0^{1/2} f(t)dt$.

What are the properties of the terms of the series representing $\int_0^{1/2} f(t)dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of this integral?

2009 #6: The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + ..$ The continuous function *f* is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1.$ The function *f* has derivatives of all orders at $x = 1$.

a. Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

b. Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for *f* about $x = 1$.

c. Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

d. Use the Taylor series for *f* about $x = 1$ to determine whether the graph of *f* has any points of inflection

2009 Form B #6: The function f is defined by the power series

$f(x) = 1 + (x+1) + (x+1)^2 + \ldots..(x+1)^n + \ldots. = \sum_{n=0}^{\infty}(x+1)^n$ for all real numbers x for which the series converges.

a. Find the interval of convergence of the power series for f. Justify your answer.

b. The power series above is the Taylor series for f about x = -1. Find the sum of the series for f.

c. Let g be the function defined $g(x) = \int_{-1}^{x} f(t)dt$. Find the value of $g(-0.5)$ if it exists, or explain why $g(-0.5)$ cannot be determined.

d. Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first three nonzero terms and the general term of the Taylor series for h about x = 0, and find the value of $h(0.5)$.

2005 #6: Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at x=2 is 0. When n is even and $n \geq 2$, the nth derivative of f at x=2 is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

a. Write the 6th degree Taylor polynomial for f about x=2.

b. In the Taylor series for f about x=2, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

c. Find the interval of convergence of the Taylor series for f about x=2. Show the work that leads to your answer.

2013 #6. The function *f* has derivatives of all orders at *x*=0. Let $P_n(x)$ denote the nth degree Taylor polynomial for *f* about *x*=0.

a. It is known that $f(0) = -4$ and that $P_1(1/2) = -3.$ Show that $f'(0) = -2.$

b. It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

c. The function *h* has first derivative given by $h' = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for *h* about *x*=0.

Answers

2006 form b #6: a. $f'(x) = -3x^2 + 6x^5 - 9x^8 + \ldots (-1)^n \cdot 3n \cdot x^{3n-1}$

b. that is just $f'(0.5)$ and since $f'(x) = \dfrac{-3x^2}{(1+x^3)^2}$, it is just -16/27

c. $x - \dfrac{x^4}{4} + \dfrac{x^7}{7} - \dfrac{x^{10}}{10} + \ldots \dfrac{(-1)^n x^{3n+1}}{3n+1}$ d. $\dfrac{1}{2} - \dfrac{1}{64} + \dfrac{1}{896}$ since the series is alternating and monotonically descending in magnitude, the error will be less than the absolute value of the next term, which is $\dfrac{1}{10240}$, which is less than $\dfrac{1}{10{,}000}$

2009 #6: a. $1 + (x-1)^2 + \dfrac{(x-1)^4}{2!} + \dfrac{(x-1)^6}{3!} + \ldots \dfrac{(x-1)^{2n}}{n!}$ b. $1 + \dfrac{(x-1)^2}{2!} + \dfrac{(x-1)^4}{3!} + \dfrac{(x-1)^6}{4!} + \ldots \dfrac{(x-1)^{2n}}{(n+1)!}$

c. (-∞,∞) d. take the 2nd derivative of (b): $1 + \dfrac{(x-1)^2}{2!} + \dfrac{(x-1)^4}{3!} + \dfrac{(x-1)^6}{4!} + \ldots \dfrac{(x-1)^{2n}}{(n+1)!}$ and find that all terms are positive, so the second derivative is positive everywhere. Thus the graph of f is always concave up and has no inflection points.

2009 Form B #6: a. using the ratio test…|x+1|<1 so (-2,0) and it diverges at endpoints so -2<x<0 (you can also look at it as a geometric series and set |r|<1).

b. geometric series so $\dfrac{a}{1-r} = \dfrac{1}{1-(x+1)} = \dfrac{-1}{x}$

c. $\displaystyle\int_{-1}^{x} \frac{-1}{t}\,dt = -\ln|x| - -\ln|-1| = -\ln|x| \;\; so \;\; -\ln 0.5 \;\; or \;\; \ln 2$

d. $h(x) = 1 + x^2 + x^4 + x^6 + \ldots + x^{2n}$ so $h = \dfrac{1}{1-x^2}$ *and* $h(0.5) = 4/3$

2005 #6: a. $7 + \dfrac{1}{18}(x-2)^2 + \dfrac{1}{3^4 \cdot 4}(x-2)^4 + \dfrac{1}{3^6 \cdot 6}(x-2)^6$ b. $\dfrac{1}{2n \cdot 3^{2n}}$

c. ratio test: $\displaystyle\lim_{n\to\infty}\left|\frac{\dfrac{(x-2)^{2n+2}}{2(n+1)\cdot 3^{2n+2}}}{\dfrac{(x-2)^{2n}}{2n\cdot 3^{2n}}}\right| = \lim_{n\to\infty}\left|\frac{n(x-2)^2}{(n+1)\cdot 9}\right|$ which is < 1 when $(x-2)^2 < 9$ so $(-1,5)$.

Now check endpoints: if x=-1 this is $\dfrac{(-3)^{2n}}{2n \cdot 3^{2n}}$ which is harmonic and diverges.

if x=5 this is $\dfrac{(3)^{2n}}{2n \cdot 3^{2n}}$ which is also harmonic and diverges. So the answer is $-1 < x < 5$.

2013 #6: a. $P_1(x) = f(0) + f'(0)(x-0)$ so -3=-4+ $f'(0) * (1/2)$ so $f'(0) = 2$

b. $P_3(x) = f(0) + f'(0)(x) + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3$ so it is $-4 + 2x - \dfrac{1}{3}x^2 + \dfrac{1}{18}x^3$

c. $\displaystyle h(x) = h(0) + \int_0^x f(2x)\,dx = 7 + \int_0^x \left(-4 + 4x - \frac{4}{3}x^2 + \frac{4}{9}x^3 - \ldots\right)dx = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

Unit 4 Handout #1: Arc Length

1. To estimate the length of the parabola $y = x^2$ on the interval [0,4] do the following:

a. Assume it is the hypotenuse of a single right triangle. Convert your estimate to a decimal.

b. Break it into two pieces, that on the interval [0,2] and that on the interval [2,4]. Now use two right triangles to approximate its length.

c. Do the same with four right triangles.

d. What can you do to get an extremely accurate approximation for the exact length?

2. In finding area "under a curve" (ie, between a curve and the x-axis), we add up very thin rectangles. If there are a finite number, we get an expression like $\sum bh = \sum y \cdot \Delta x$ or $\sum f(x)\Delta x$. If we want to express the area exactly, we need an infinite number of rectangles, so we get $\int ydx \ or \int f(x)dx$. When we find a volume of a solid of revolution using discs, we add up discs—circles with tiny height (or width), getting an expression like $\int \pi r^2 h \ or \int \pi y^2 dx$.

a. Explain why, when we want to get the exact length of a curve in the *xy*-plane, the expression $\int \sqrt{(dx)^2 + (dy)^2}$ makes sense.

b. Do some algebraic manipulation to show that arc length can be written as $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

3. Use the formula from 2*b* above to find the length of the given curves on the specified intervals. For parts *a* and *d* do **not** use fnInt. (*d* will require trig substitution or just thinking)

a. The length of $f(x) = 2x^{3/2}$ on the interval [0,4]

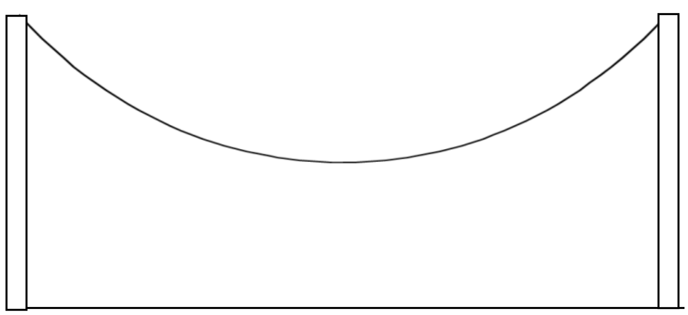
b. $f(x) = x^2$ on [0,4]

c. $f(x)=\sin x$ on $[0,2\pi]$

d. $f(x)=\sqrt{1-x^2}$ on [-1,1].

4. True or false: if the length of $f(x)$ on the interval [a,b] is k, then the length of $2f(x)$ on the interval [a,b] must be 2k.

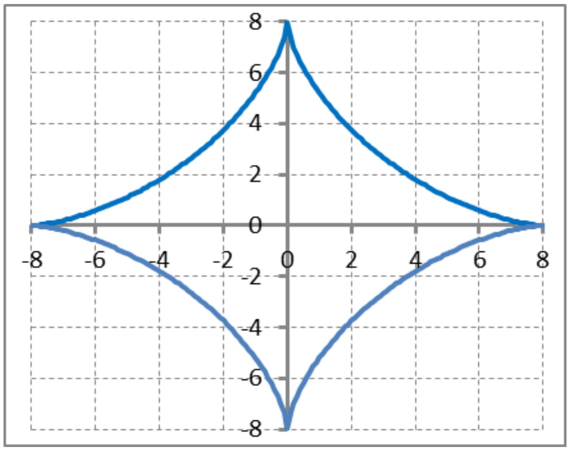
5. In the picture below, there are two towers 200 feet apart with a wire hanging from them (its shape is called a catenary). The origin is on the ground midway between the two towers. The equation of the cable is $y=75\left(e^{x/150}+e^{-x/150}\right)$.



a. How high are the towers?

b. How long is the wire? Try it without using your calculator if you want a challenge; otherwise you can use your calculator to evaluate the integral.

6. The graph below is called an astroid. Its equation is $x^{2/3}+y^{2/3}=4$. What is the perimeter of the astroid? This is doable without a calculator if you work your way through it carefully.



Answers

1a. sqrt(272)=16.492 b. 16.63 c. 16.747 d. more hypotenuses! 2a. adding up hypotenuses

3a. $\int_0^4 \sqrt{1+9x}dx=\frac{1}{9}\cdot\frac{2}{3}(1+9x)^{1.5}=\frac{2}{27}(37\sqrt{37}-1)\approx 16.6$ b. $\int_0^4 \sqrt{1+4x^2}dx\approx 16.819$ c. 7.640

d. π since it is $\int_{-1}^{1}\sqrt{1+\frac{x^2}{1-x^2}}dx=\int_{-1}^{1}\sqrt{\frac{1}{1-x^2}}dx=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x$

5a. $f(100)=75(e^{2/3}+e^{-2/3})\approx 184.59$ ft

b. $\int_{-100}^{100}\sqrt{1+\left(0.5e^{x/150}-0.5e^{-x/150}\right)^2}dx=\int_{-100}^{100}\sqrt{1+0.25e^{x/75}-0.5+0.25e^{-x/75}}dx=\int_{-100}^{100}(0.5e^{x/150}+0.5e^{-x/150})dx$

which is $75e^{x/150}-75e^{-x/150}=75e^{2/3}-75e^{-2/3}-75e^{-2/3}+75e^{2/3}=150(e^{2/3}-e^{-2/3})\approx 215.15$

6. $4\cdot\int_0^8\sqrt{1+\left(4-x^{2/3}\right)\left(x^{-2/3}\right)}dx=4\cdot\int_0^8\sqrt{1+4x^{-2/3}-1}dx=4\cdot\int_0^8\sqrt{4x^{-2/3}}dx=4\cdot\int_0^8 2x^{-1/3}dx=12x^{2/3}=48$

Unit 4 Handout #2: Parametric Equations

1. These three pairs of parametric equations each describe all or part of the parabola $y = x^2$. Indicate what part they each describe. (think domain!)

a. $x = \sqrt{t-1}$, $y = t - 1$

b. $x = e^t$, $y = e^{2t}$

c. $x = \sin t$, $y = 1 - \cos^2 t$

2. Eliminate the parameter to write the equation of the curve described by each pair of parametric equations. You do not need to solve for *y*.

a. $x = 1 - t^2$, $y = 2t$

b. $x = \cos^{-1}(t)$, $y = 2t^2 - 1$

c. $x = \sec t$, $y = \tan t$

d. $x = 2t + 3$, $y = 4t^2 - 9$

e. $x = 2\sin t$, $y = 3\cos t$

3. Find the equation of the tangent line to each curve at the specified value of the parameter. Remember, you do not need to eliminate the parameter to do this.

a. $\begin{cases} x = e^t \\ y = e^{-t} \end{cases}$ $t = \ln 2$

b. $\begin{cases} x = \sqrt{t} \\ y = 0.25(t^2 - 4) \end{cases}$ $t = 4$

c. $\begin{cases} x = \ln t \\ y = \sin^{-1} t \end{cases}$ $t = 0.5$

4. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

a. $\begin{cases} x = \cos t \\ y = \cos(2t) \end{cases}$

b. $\begin{cases} x = \ln t \\ y = t \ln t \end{cases}$

5. Find the arc length of the curve described in each part below: (finInt is OK)

a. $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \quad 0 \le t \le 2\pi$

b. $\begin{cases} x = t^2 \\ y = \dfrac{t^3}{3} - t \end{cases} \quad 0 \le t \le 4$

6. During the time period $t \ge 0$, the position of a particle is given by $x(t) = 1 - t^2$ *and* $y(t) = \dfrac{2t^3}{3}$.

a. Find $\dfrac{dy}{dx}$ in terms of *t*.

b. Write the equation of the tangent line to the path of the particle at *t*=3.

c. How far did the particle travel between *t*=0 and *t*=3 no calculator on this one?

7. Use your calculator to sketch the graph of $x(t) = \cos^3 t$ *and* $y(t) = \sin^3 t$ for t=0 to 2π.

a. Find the length of the curve. Try it without fnInt.

b. Write the equation for the tangent line to the curve when $t = \dfrac{\pi}{3}$.

c. For what values of t is the slope of the curve undefined, and what are the xy coordinates of the corresponding points?

8. A curve C is defined by the parametric equations $x(t) = 2t + 1$ *and* $y(t) = t^2 - 3t + 5$.

a. Find $\dfrac{dy}{dx}$ in terms of t.

b. When (and where) is the tangent line to C horizontal?

c. For what values of t is the x-coordinate increasing and the y-coordinate decreasing? Find analytically and confirm graphically.

9. (From 2013 AP Calc BC Exam)

The position of a particle moving in the xy-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. At which points (x, y) is the particle at rest?

Answers

1a. only makes sense when t≥1 so x≥0 and y≥0 b. x>0 and y>0

c. x must be between $-1 \le x \le 1$ and $0 \le y \le 1$

2a. $y^2 = 4 - 4x$ b. $y = 2\cos^2 x - 1$ c. $x^2 - y^2 = 1$ d. $y = x^2 - 6x$ e. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3a. y=-0.25x+1 b. y-3=8(x-2) c. $y - \pi/6 = \frac{\sqrt{3}}{3}(x - \ln 0.5)$

4a. $y' = \frac{2\sin(2t)}{\sin t}$ $y'' = \frac{2\sin(2t)\cos t - 4\sin t\cos(2t)}{\sin^3 t}$ OR simplify y’ to be 4cost so $y'' = 4$

b. $y' = t\ln t + t$ $y'' = 2t + t\ln t$

5a. $\int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt = 8$ b. $\int_0^4 \sqrt{4t^2 + (t^2 - 1)^2}\,dt = 76/3$

6a. $-t$ b. $y - 18 = -3(x + 8)$ c. $\int_0^3 \sqrt{4t^2 + 4t^4}\,dt = 20.415$

7a. $4\int_0^{\pi/2} \sqrt{(-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2}\,dt = 4\int_0^{\pi/2} \sqrt{9\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)}dt$

Which is $4\int_0^{\pi/2} 3\sin t\cos t dt = \frac{12\sin^2 t}{2} = 6$

*note.. if you try to do it as $\int_0^{2\pi} 3\sin t\cos t dt$ you get 0 because it should be $\int_0^{2\pi} |3\sin t\cos t| dt$

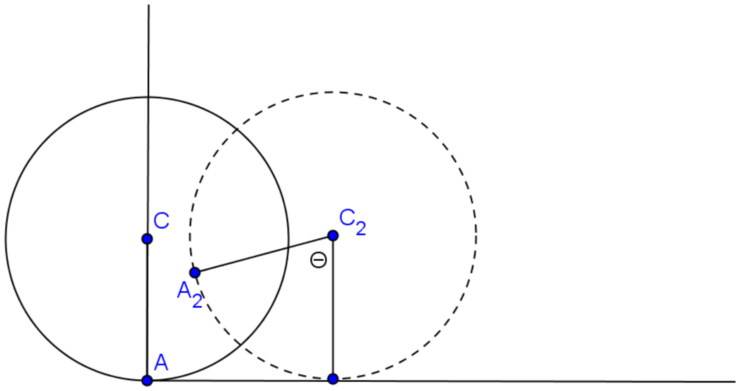
b. $\frac{dy}{dx} = \frac{3\sin^2 t\cos t}{3\cos^2 t(-\sin t)} = -\tan t$ so slope is $-\sqrt{3}$ and tangent line is $y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right)$

c. slope is undefined when t= π/2and 3π/2 and points are on the axes (0,1) and (0,-1) → slope appears to be 0/0 at t=0 and π but looking at graph and reducing slope expression (since slope is a limit of secant lines as distance horizontal approaches zero) it seems clear that the slopes are 0 at these points

8a. $\frac{2t-3}{2}$ b. when t=1.5 and the point is (4,2.75) c. t<1.5 9. (-4,12)

Unit 4 Handout #3: Cycloid Exploration

Point A is on circle C with radius 10 cm. Initially, A is located at the origin and C is at the point (0,10). Then circle then to rolls to the right, making one revolution every 2π seconds. The dashed circle in the diagram below shows the location of points A and C after a little time has elapsed.



1. Write parametric equations showing the coordinates of point A as a function of time as follows:

   a. Write parametric equations for the coordinates of C as a function of time.

   b. How does Θ – the angle measured clockwise from due south to point A –relate to time?

   c. Write parametric equations for location of A ***relative to C*** as a function of time.

   d. Combine parts *a* and *c* above to show that point A's coordinates can be given by
   $x(t) = 10t - 10\sin t \quad y(t) = 10 - 10\cos t$

2. What is the total distance that A travels in the 2π seconds it takes the circle to make one revolution?

Note: if you want to integrate this by hand, use the equation $\cos(0.5x) = \sqrt{\dfrac{1-\cos x}{2}}$ → this comes from $\cos(2u) = 1 - 2\cos^2 u$ with $u = 0.5x$.

3. What is the equation of the tangent line to point A when t=0.5π seconds?

4. What is the rate of change in the slope of the tangent line to A when t=0.5π seconds?

5. At *t*= 2π/3, what is the speed of point A and what angle does its path make with a horizontal line?

6. Write the function describing the speed of A at time *t*. When is the speed at its highest and lowest values and where on the wheel is point A at those times?

7. What is the area of one loop of the cycloid?

8. How far from the origin is point A when t=4π/3 and how quickly is point A's distance from the origin changing at this time?

9. What angle is the path of A making with a horizontal line at t=π/6 and how quickly is this angle changing? Answer in degrees for both.

Answers

1a. y=10 x=10t b/c when t=2π circle has made one revolution and A has traveled 20π to the right… so 20π units in 2π seconds means 10 units per second.

b. 2π radians every 2π seconds so 1 radian per second c. $x = -10\sin t \quad y = -10\cos t$

2. $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{200 - 200\cos t}\, dt = 80$ → 4 diameters... cool! (π diameters spinning around the center, and the other 0.85 or so diameters due to the movement of the center)

3. $x(0.5\pi) = 5\pi - 10 \quad y(0.5\pi) = 10$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10\sin t}{10 - 10\cos t} = \frac{10}{10} = 1$ so $y = x + 20 - 5\pi$

4. $\frac{d(dy/dx)/dt}{dx/dt} = \frac{(10-10\cos t)(10\cos t) - (10\sin t)(10\sin t)}{(10-10\cos t)^2 \cdot (10-10\cos t)}$

$\frac{(10-10\cos t)(10\cos t) - (10\sin t)(10\sin t)}{(10-10\cos t)^2 \cdot (10-10\cos t)} = \frac{100\cos t - 100}{(10-10\cos t)^3} = \frac{-10}{(10-10\cos t)^2} = -\frac{1}{10}$

note that this is always negative except when its undefined at t=2πn

5. $\frac{dx}{dt} = 10 - 10\cos t \quad \frac{dy}{dt} = 10\sin t$

So $\frac{dx}{dt} = 10 - 10\cos(2\pi/3) = 15$ and $\frac{dy}{dt} = 10\sin(2\pi/3) = 5\sqrt{3}$

A’s speed at this time is the magnitude of the velocity vector: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 10\sqrt{3}$ cm/second

Its angle is $\tan\phi = \frac{dy}{dx} = \frac{1}{\sqrt{3}}$ so $\phi = \frac{\pi}{6}$

6. speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ which is cool → the derivative of arc length function….

It simplifies to $\sqrt{200 - 200\cos t}$

lowest is at t=0+2πn; speed is 0 then… when point A is touching the ground it is stationary

highest is t=π + 2πn; speed is then 20 cm/second → its velocity is parallel to C’s velocity…

7. $\int_0^{2\pi} y\,dx = \int_0^{2\pi} (10-10\cos t)(10-10\cos t)dt =$

$\int (100+100\cos^2 t - 200\cos t)dt = \int (100-200\cos t + 50\cos(2t)+50)dt$

$150t - 200\sin t + 25\sin(2t) = \left(300\pi - 200\sin(2\pi) + 25\sin(4\pi)\right) - \left(0 - 200\sin(0) + 25\sin(0)\right) = 300\pi$

Cool—three times the area of the circle…

8. Distance from the origin is $D = \sqrt{x^2+y^2}$

$D^2 = x^2+y^2 = (10t-10\sin t)^2 + (10-10\cos t)^2 = 100t^2 - 200t\sin t + 100\sin^2 t + 100 - 200\cos t + 100\cos^2 t$

So $D^2 = 100t^2 - 200t\sin t - 200\cos t + 200$

At t=4π/3, $D^2 = 1600\pi^2/9 - 200\cdot\frac{4\pi}{3}\cdot\left(-\frac{\sqrt{3}}{2}\right) + 300$ so D≈52.73 cm

$2D\frac{dD}{dt} = 200t - 200\sin t - 200t\cos t + 200\sin t = 200t(1-\cos t)$

At t=4π/3 we get $\frac{dD}{dt} = \frac{100t(1-\cos t)}{D} \approx 11.92$ cm/second

9. $\frac{dx}{dt} = 10 - 10\cos\left(\pi/6\right) = 10 - 5\sqrt{3}$ and $\frac{dy}{dt} = 10\sin(\pi/6) = 5$

Its angle is $\tan\phi = \frac{dy}{dx} = \frac{5}{10-5\sqrt{3}}$ so $\phi = 75^\circ$

In general $\tan\phi = \frac{dy}{dx} = \frac{10\sin t}{10-10\cos t}$

To find $\frac{d\phi}{dt}$: $\sec^2\phi\cdot\frac{d\phi}{dt} = \frac{(10-10\cos t)(10\cos t)-(10\sin t)(10\sin t)}{(10-10\cos t)^2} = \frac{100\cos t - 100}{(10-10\cos t)^2} = \frac{-10}{10-10\cos t}$

So $\frac{d\phi}{dt} = \frac{-10\cos^2\phi}{10-10\cos t} \approx -0.5$ radians per second or -28.65 degrees per second

Unit 4 Handout #4: More Parametrics and Vectors

1. Given the following: $\begin{cases} x_t = 0.2t\cos t \\ y_t = 0.1t^2 \end{cases} \quad t \geq 0$

a. Write the equation of the tangent line to the graph when $t=\pi/2$

b. What is the length of the path traveled in the first 20 units of time?

c. What is the average speed in the first 20 units of time?

d. What is the velocity at time 20?

e. What is the acceleration vector at time t?

f. What is $\dfrac{d^2y}{dx^2}$ when $t=\pi$?

2. A position vector is given by $\left\langle \ln\left(1+t^2\right), \sin t \right\rangle$. It may also be written as $\ln(1+t^2)\cdot i \oplus \sin t \cdot j$

a. What are the velocity and acceleration vectors?

b. What is the velocity at time t=3? The acceleration vector?

c. What is the slope of the path of the object at t=2?

d. What positive acute angle is the object moving (relative to the x-axis) at time t=2?

e. What is the total distance travelled in the first 2 seconds?

3. A moving particle has position $\langle x(t), y(t) \rangle$ at time t. The position at time 2 is $\langle -3, 5 \rangle$ and the velocity vector at time *t*>0 is given by $\left\langle -2t,\ 3t^2 \right\rangle$.

a. Find the acceleration vector at time *t*=2.

b. What is the position of the particle at time *t*=3?

c. At what time *t*>0 does the tangent line to the path of the particle have slope -3?

d. What is the speed of the particle at time *t*=3 and what (positive) angle is it moving (relative to the positive *x*-axis)?

e. What is the distance traveled from *t*=0 to *t*=3? Try it without your calculator.

f. What is the *x* position when the *y* position is 35?

4. Given the following: $\begin{cases} x_t = (1+0.1t)\cos t \\ y_t = (1+0.1t)\sin t \end{cases}$ $t \geq 0$

a. Without using your calculator, what do you think the graph will look like (in the *xy*-plane)?

b. Graph it to confirm.

c. What is the slope of the tangent line to the graph when $t=\pi/2$.

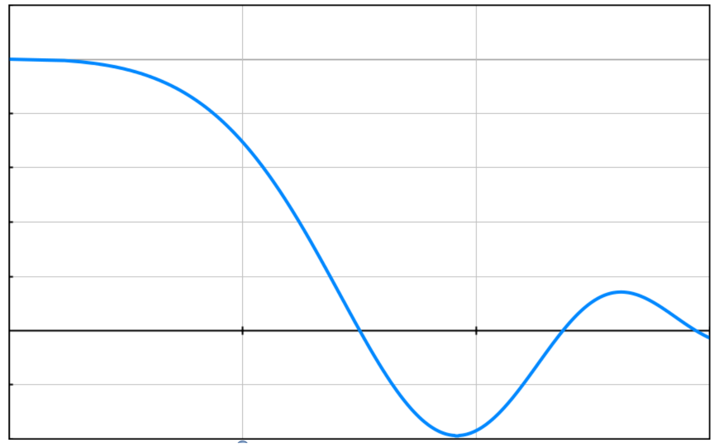
d. What is the length of the path traveled in the first 20 units of time?

e. What is the average speed in the first 20 units of time?

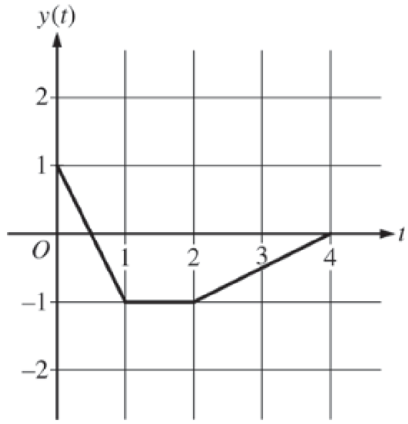
f. What is the velocity at time 20?

5. The graph below shows the parametric equations $x(t)=\sqrt{t}+\sqrt[3]{t}$ and $y(t)=5e^{-t}\cos(\pi t)$ for $0\le t\le 3$.

a. Find the equation of the line tangent to the curve when t=1.
b. What are the coordinates of the absolute minimum shown on the graph?
c. Find the x-coordinate of the first point where the graph first crosses the x-axis.
d. Let R be the region bound by the graph and the coordinate axes until the first time the graph crosses the x-axis. What is the perimeter of R?
e. What is the area of R?



6. (AP Calc BC exam #2 from 2016). Calculator OK.



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

   (a) Find the position of the particle at $t = 3$.

   (b) Find the slope of the line tangent to the path of the particle at $t = 3$.

   (c) Find the speed of the particle at $t = 3$.

   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

Answers

1a. point is $(0,\pi^2/40)$ slope is dy/dt / dx/dt = $\frac{0.2t}{0.2\cos t - 0.2t\sin t} = \frac{t}{\cos t - t\sin t}$ = -1 → so $y-\pi^2/40=-x$

b. $\int_0^{20} \sqrt{(0.2\cos t - 0.2t\sin t)^2 + 0.04t^2}\, dt = 48.503$ c. 48.503/20=2.425

d. $\sqrt{(0.2\cos t - 0.2t\sin t)^2 + 0.04t^2}$ when t=20 is 5.361 e. $(-0.4\sin t - 0.2t\cos t,\ 0{,}2)$

f. $\frac{dy}{dx} = \frac{t}{\cos t - t\sin t}$ so $\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{(\cos t - t\sin t) - t(-2\sin t - t\cos t)}{(\cos t - t\sin t)^2} \cdot \frac{1}{0.2\cos t - 0.2t\sin t}$

at t=π this is $\frac{-1-\pi^2}{1} \cdot \frac{1}{-0.2} = 5 + 5\pi^2$

2a. velocity is $\left\langle \frac{2t}{1+t^2}, \cos t \right\rangle$ and acceleration is $\left\langle \frac{2-2t^2}{\left(1+t^2\right)^2}, -\sin t \right\rangle$

b. velocity vector is $\left\langle \frac{6}{10}, \cos 3 \right\rangle$ so velocity is $\sqrt{(0.6)^2 + \cos^2 3}$ =1.158; accel vector is $\left\langle \frac{-16}{100}, -\sin 3 \right\rangle$

c. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1+t^2)\cos t}{2t} = -0.520$ d. draw a triangle…. $\tan^{-1}(0.520)$ =0.48 radians or about 27.5°

e. $\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{\left(\frac{2t}{1+t^2}\right)^2 + (\cos t)^2}\, dt$ =2.096

3a. $\langle -2, 6t \rangle$ so $\langle -2, 12 \rangle$ b. x is $-3 + \int_2^3 (-2t dt) = -8$ (or define x(t) as $-t^2+C$ and plug in initial to get $-t^2+1$

y is $5 + \int_2^3 3t^2 dt = 24$ so <-8,24> c. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -3 = \frac{3t^2}{-2t}$ so t=2

d. speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{4t^2 + 9t^4}$ =27.66 and angle is $\tan^{-1}(27/6) = 1.352$ *radians*

e. $\int_0^3 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^3 \sqrt{4t^2 + 9t^4}\, dt = 28.728$ f. y= $t^3 - 3$ → t=$\sqrt[3]{38}$ x=$1-t^2$ *or* -10.303

4a. like a circle with an increasing radius, so a spiral c. $\frac{dy}{dt} = 0.1$ $\frac{dx}{dt} = -1 - 0.05\pi$ *so* $\frac{dy}{dx} = -\frac{2}{20+\pi}$

d. $\int_0^{20} \sqrt{((1+0.1t)(-\sin t) + 0.1\cos t)^2 + ((1+0.1t)(\cos t) + 0.1\sin t)^2}\, dt = 40.05$ e. 2.003 f. 3.002

5a. $y + \frac{5}{e} = \frac{6}{e}(x-2)$ b. $\frac{dy}{dx} = 0$ so $\tan(\pi t) = \frac{-1}{\pi}$ and $t \approx 0.902$ so about (1.916,-1.933)

c. y=0 so t=0.5 and $x = \sqrt{0.5} + \sqrt[3]{0.5} \approx 1.501$ d. $5 + 1.5 + \int_0^{0.5} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \approx 12.009$

e. $\int_{t=0}^{t=0.5} y dx = \int_{t=0}^{t=0.5} 5e^{-t}\cos(\pi t)\cdot\left(0.5t^{-1/2} + (1/3)t^{-2/3}\right) dt \approx 5.563$

6a. $x(3) = x(0) + \int_0^3 x'(t) dt = 14.377$ and y(3)=-0.5 from graph so (14.377, -0.5)

b. $\frac{y'(3)}{x'(3)} = \frac{0.5}{9+\sin 27} \approx 0.0502$ c. $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{9.956^2 + 0.5^2} = 9.969$

d. split integral into 2 pieces because dy/dt is -2 on [0,1] and 0 on [1,2]

$\int_0^1 \sqrt{(-2)^2 + \left(\frac{dx}{dt}\right)^2}\, dt + \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + (0)^2}\, dt$ which is about 4.350 (don't cancel the squared and the square root on the second integral b/c we want it to always be positive. (it is in this case, but won't always be)

Unit 4 Handout #5: Polar Coordinates Review

1. The polar point $(4,5\pi/3)$ has what rectangular coordinates?

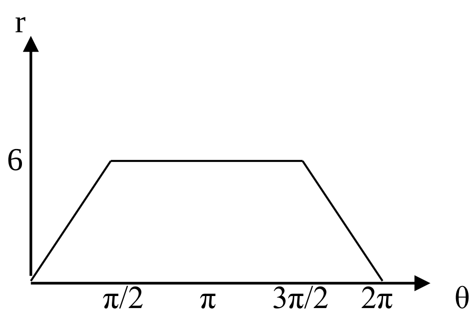
2. The rectangular point (3,-4) has what polar coordinates? Give a few possibilities (and we are in radians, of course!). Calc OK.

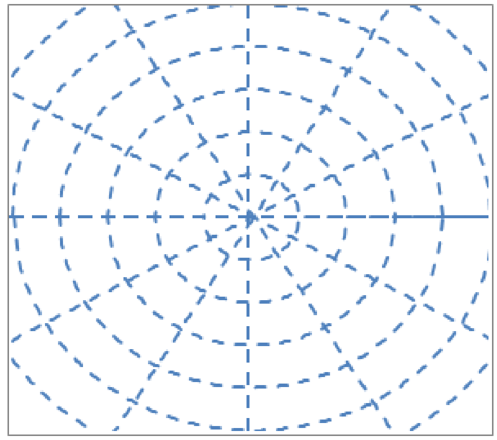
3. Give all possible values of *r* and θ that yield the polar point $(6,\pi/5)$

4. Without using your calculator, fill in the table of values below for the function $r = 3\cos(\theta) + 3$.

r	θ
	π/2
	2π/3
	11π/6
0	

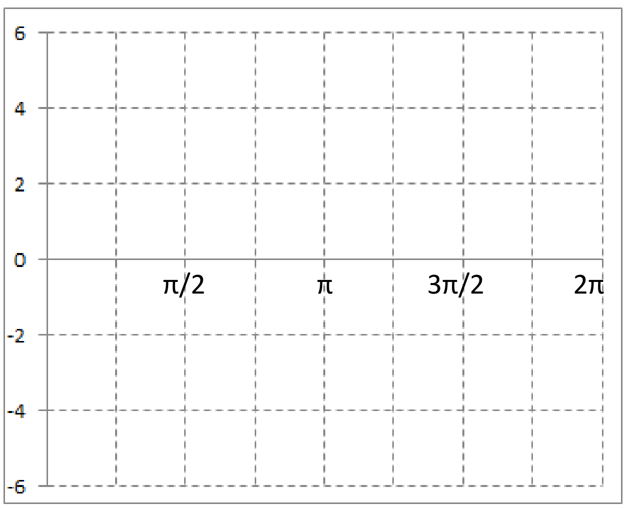
5. Here is a graph (in the xy plane) showing how r relates to θ. Take some points there and show how the graph would look when drawn on polar axes.

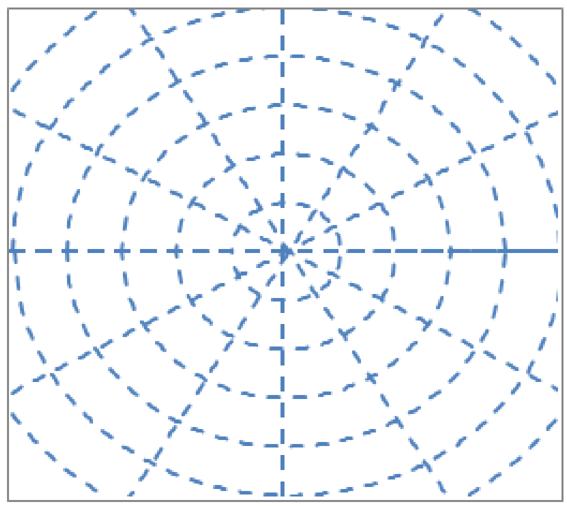




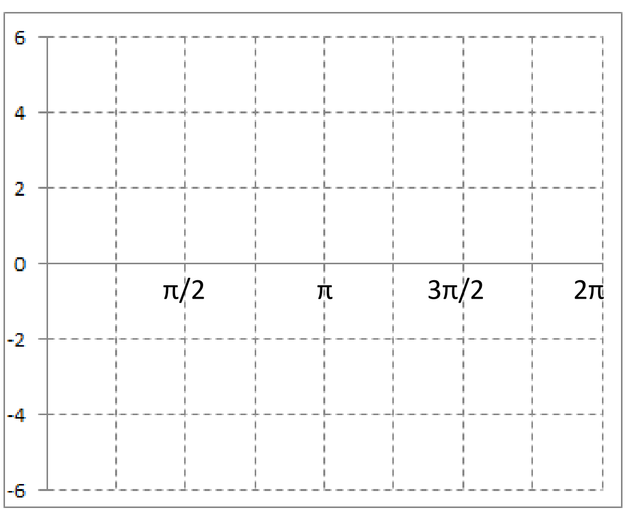
6. Graph some polar by sketching the in xy-plane first. Use $0 \le \theta \le 2\pi$.

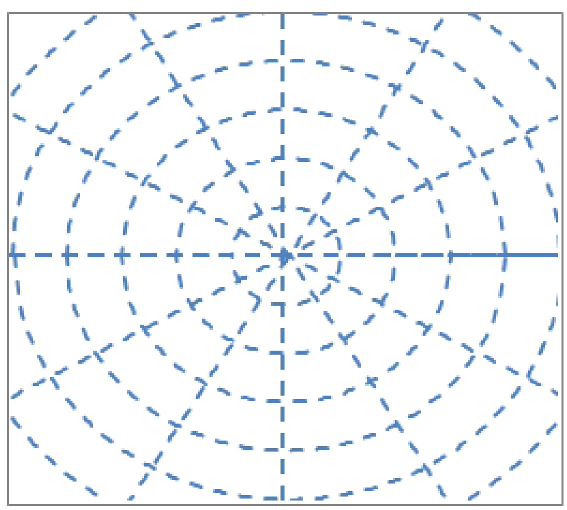
a. $r = 3\cos\theta$



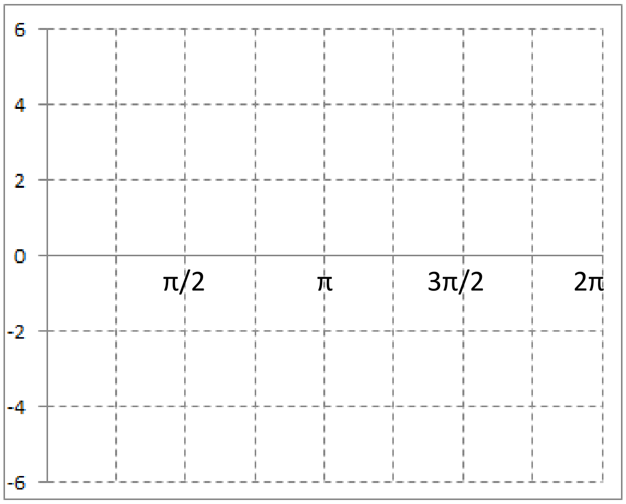


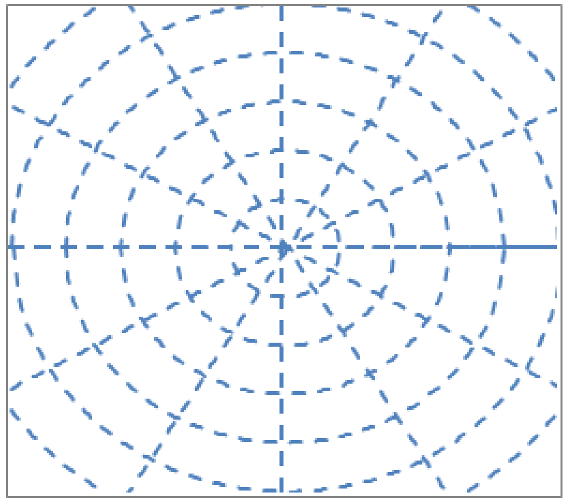
b. $r = 3\cos\theta + 3$



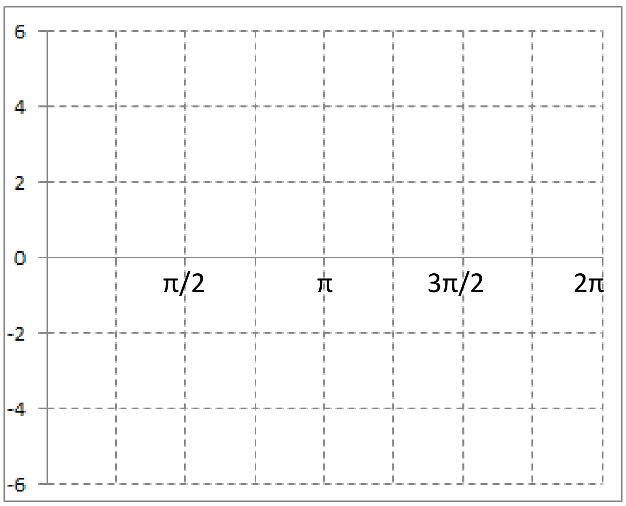


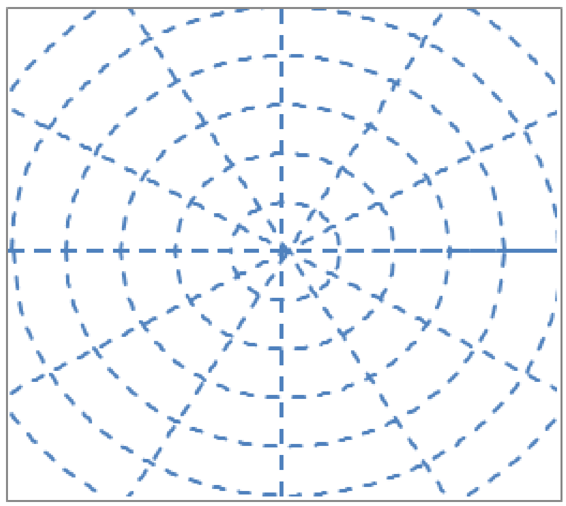
c. $r = 3\cos\theta + 2$



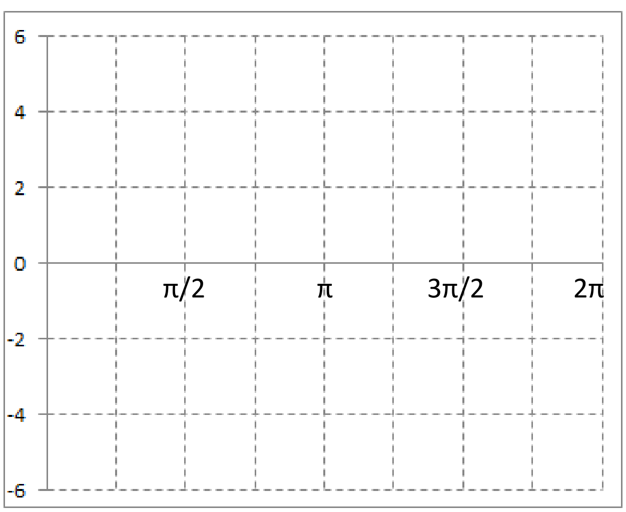


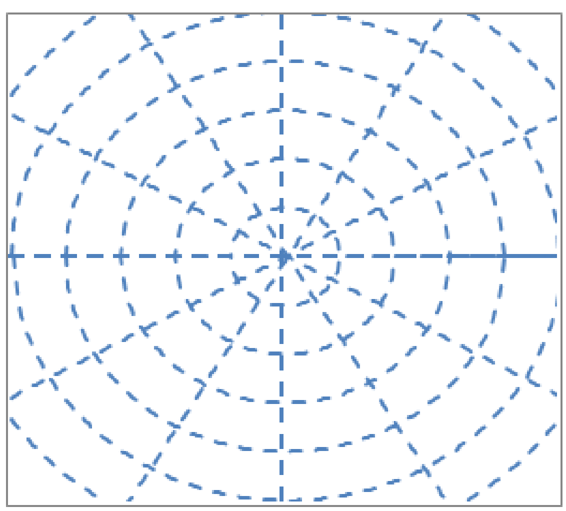
d. $r = 4\sin(2\theta)$



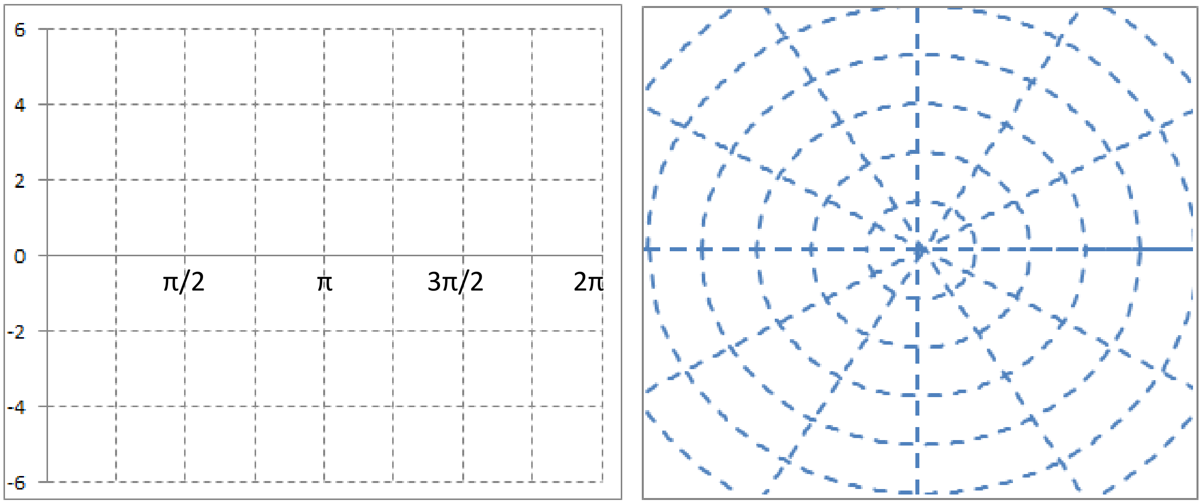


e. $r = 4\cos(2\theta)$

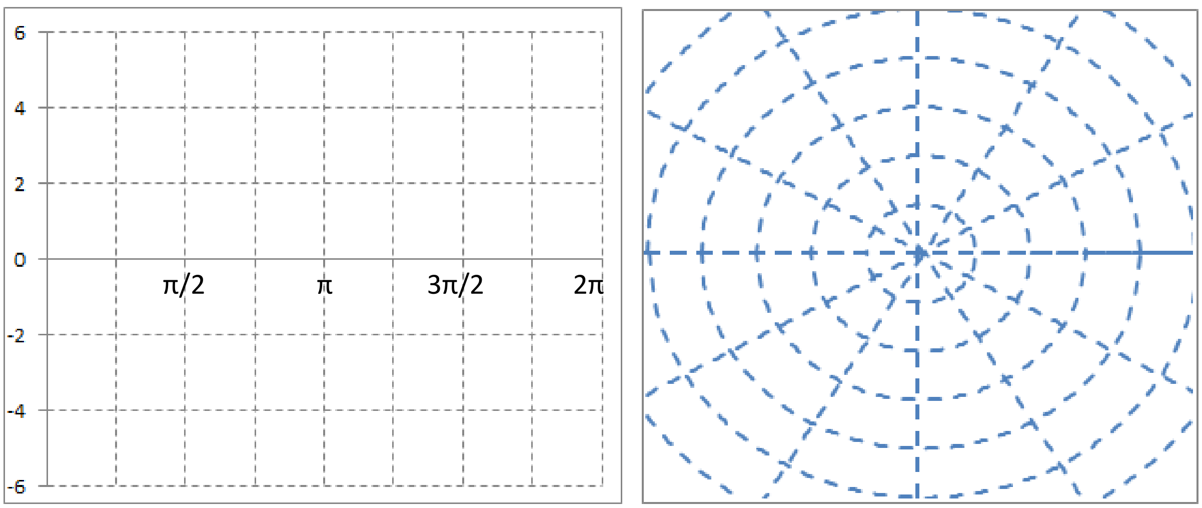




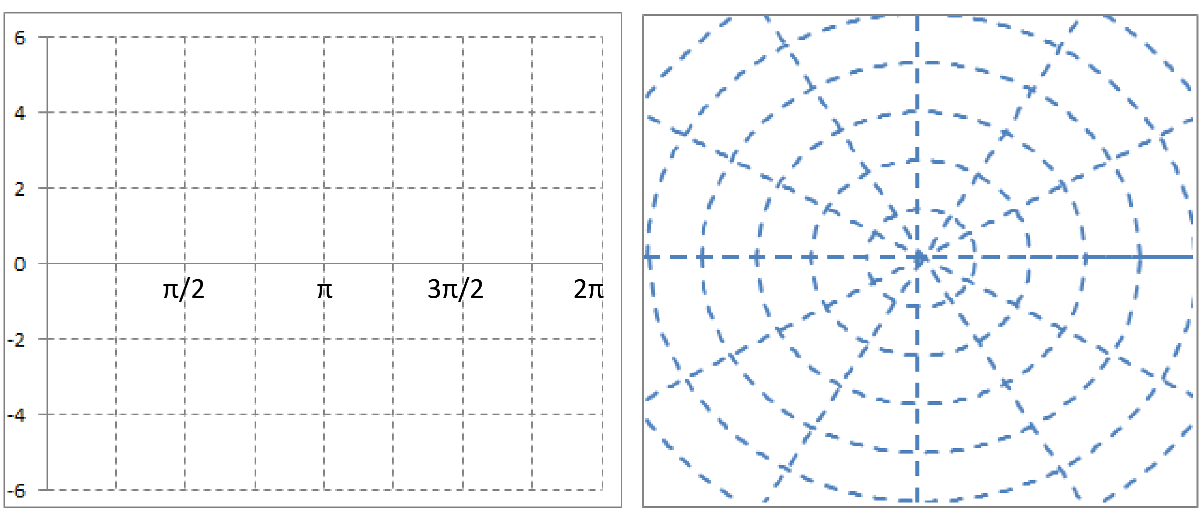
f. $r=\theta$



g. $r=5\sin(3\theta)$



h. $r=5\sin(3\theta)+2$



7. Solve the following equations for x is 0 to 2π. No calculator allowed.

a. $3\sin(x)-1=2$

b. $-4\sec(2x)+1=-3$

c. $\cos^2 x=3/4$

d. $(2\sin(2x)-1)(\cos x)=0$

e. $\tan^4 x-4\tan^2 x+3=0$

f. $\csc^2 x=4$

8. Write each of these equations in rectangular coordinates. Try substituting $\sin\theta=\dfrac{y}{r}$ *and* $\cos\theta=\dfrac{x}{r}$ and then $r=\sqrt{x^2+y^2}$.

a. $r=4\sec\theta$

b. $r=5$

c. $r=2\cos\theta+2\sin\theta$

d. $r = 3\cos\theta$

e. $\theta = \pi/3$

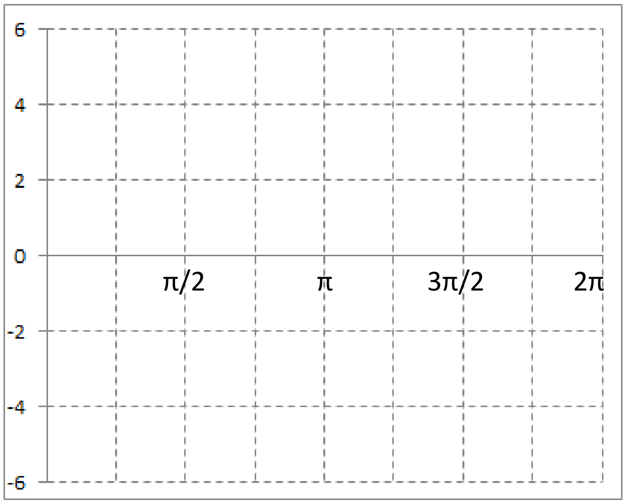
f. $r = \dfrac{2}{1+\sin\theta}$

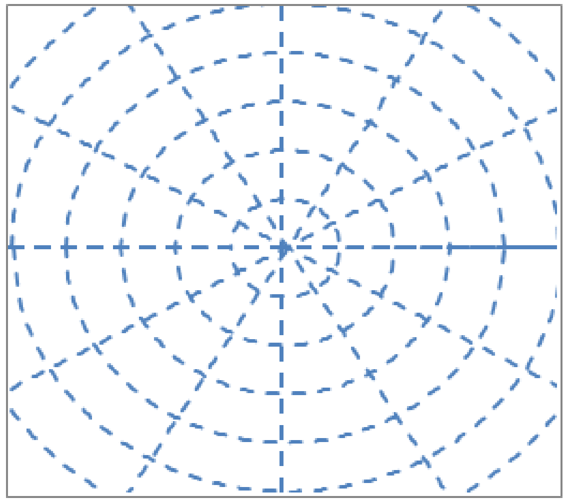
g. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

h. $r^2 \sin 2\theta = 2$ (hint: $\sin 2\theta =$ ____)

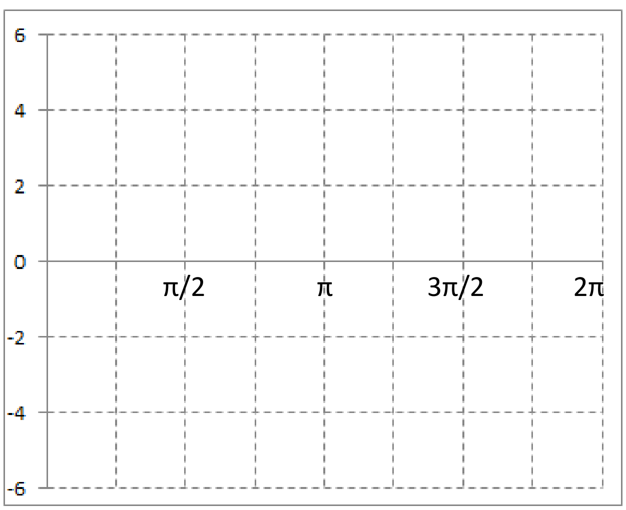
9. Sketch the graphs of $y = 1 + \cos x$ and $y = \sin x - 2$ in the xy-plane. Are there any points of intersection? Now graph $r = 1 + \cos\theta$ *and* $r = \sin\theta - 2$ on your calculator (change mode to POL). Do the graphs meet at all? What accounts for the difference in intersections between the representations?

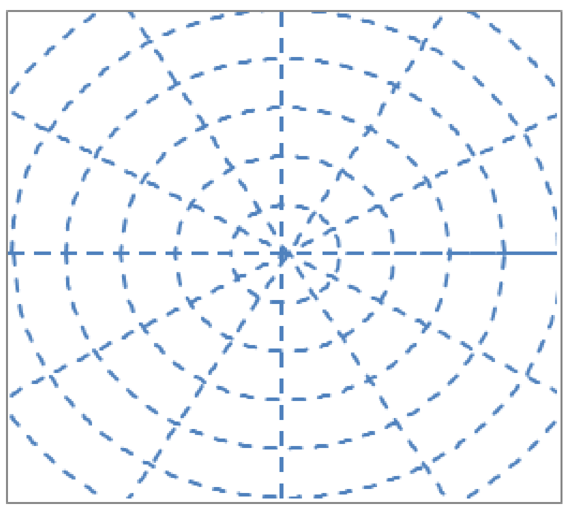
10. Graph the limacon $r = 2\cos\theta + 1$. Find the θ values associated with the points on the "inner loop"





11. Sketch a graph (in polar) of $r = 3\sin(2\theta)$ for $0 \le \theta \le 2\pi$. One petal of the rose is in the second quadrant. What values of θ correspond to that petal.





12. Is the slope of a polar graph $\frac{dr}{d\theta}$? Why or why not?

13. To find areas in the *xy*-plane, we typically sliced a region vertically into very thin rectangles. How should we compute areas in polar coordinates?

14. Interesting graphs: Graph $r = 8\sin(2\theta)$ for $0 \le \theta \le 2\pi$ on your calculator. Then, for each of the following, think about how they differ from this basic graph and why. Check yourself by graphing them.

a. $r = 8\sin(3\theta)$ *from* $0 \le \theta \le \pi$

b. $r = 8\sin(3\theta)$ *from* $0 \le \theta \le 2\pi$

c. $r = 8\sin(4\theta)$ *from* $0 \le \theta \le \pi$

d. $r = 8\sin(4\theta)$ *from* $0 \le \theta \le 2\pi$

e. $r = 8\sin(4\theta)$ *from* $0 \le \theta \le 4\pi$

f. $r = 8\sin(3.5\theta)$ *from* $0 \le \theta \le 2\pi$

g. $r = 8\sin(3.5\theta)$ *from* $0 \le \theta \le 4\pi$

h. $r = 8\sin(3.2\theta)$ *from* $0 \le \theta \le 4\pi$

i. $r = 8\sin(3.2\theta)$ *from* $0 \le \theta \le 8\pi$

j. $r = 8\sin(3.2\theta)$ *from* $0 \le \theta \le 10\pi$

15. More interesting graphs: Graph $r = 5\sin(4\theta)$ *from* $0 \le \theta \le 2\pi$. Now try these:

a. $r = 5\sin(4\theta) + 3$ *from* $0 \le \theta \le 2\pi$. Why are petals alternating sizes?

b. $r = 5\sin(4\theta) + 1$ *from* $0 \le \theta \le 2\pi$

c. $r = 5\sin(4\theta) + 7$ *from* $0 \le \theta \le 2\pi$

d. $r = 5\sin(6\theta) + 1$ *from* $0 \le \theta \le 2\pi$. To make it less pointy, try a smaller θstep

Selected Answers

1. $(2, -2\sqrt{3})$ 2. (5,-0.927) (-5,π-0.927) are some… 3. $\left(6, \frac{\pi}{5} + 2\pi n\right)$ *and* $\left(-6, \frac{6\pi}{5} + 2\pi n\right)$

4. 3, 1.5, $3 + 1.5\sqrt{3}$, π (and others for this last one)

7a.π/2 b. 0, π, 2π c. π/6, 5π/6, 7π/6, 11π/6 d. π/2, 3π/2, π/12, 5π/12, 13π/12, 17π/12
e. π/4, 3π/4, 5π/4, 7π/4, π/3, 2π/3, 4π/3, 5π/3 f. π/6, 5π/6, 7π/6, 11π/6

8a. x=4 b. $x^2 + y^2 = 25$ c. $x^2 + y^2 = 2x + 2y$ *or* $(x-1)^2 + (y-1)^2 = 2$
d. $x^2 + y^2 = 3x$ *or* $(x-1.5)^2 + y^2 = 1.5^2$ e. $y = x\sqrt{3}$ f. $y = -0.25x^2 + 1$
g. y=2x + 5 h. xy=1

9. the old "-r vs r trick"
10. when 2cos(theta)+1<0 so between 2π/3 and 4π/3
11. 3π/2 to 2π
12. that reflects how fast the radius is increasing as theta increases—not the slope of the tangent line
13. sectors of circles.

Unit 4 Handout #6: The Calculus of Polar Coordinates

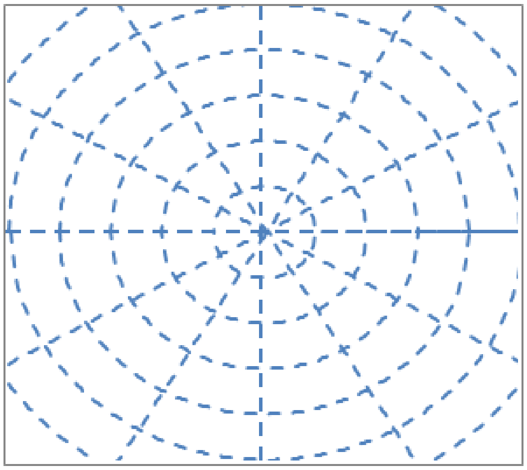
In #1-7, find the areas of the specified regions. Try to sketch the graphs by hand and determine the limits of integration by hand. But you may use fnInt to evaluate the integrals. (Some of these are from George Best's Calculus book)

1. $r = 2 + \cos\theta$

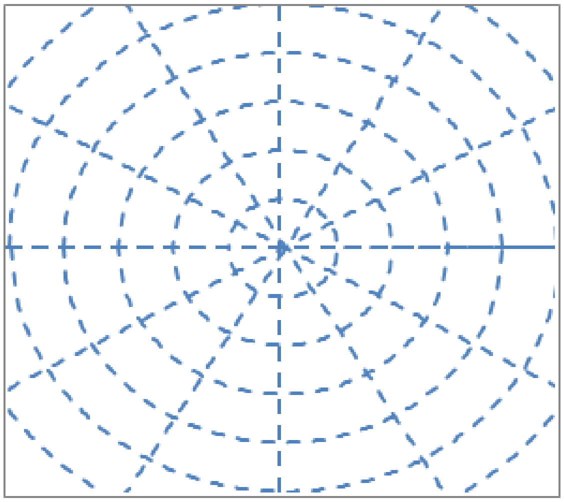
2. $r = \sqrt{\sin\theta}$ (no fnInt)

3. $r = \cos(3\theta)$

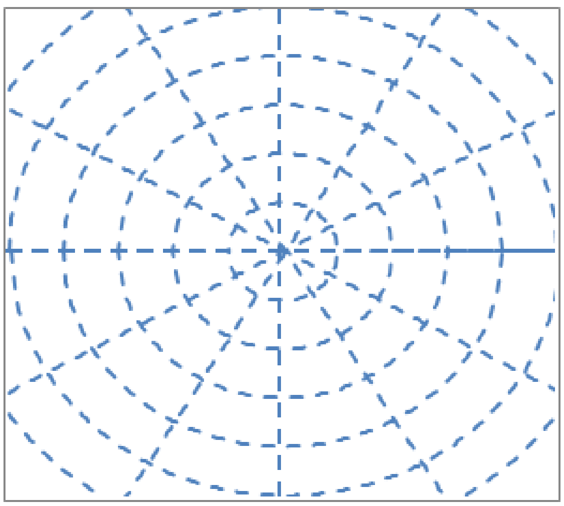
4. Bounded by $r = \theta \;\; for \;\; 0 \le \theta \le 2\pi$ and the positive x-axis (no fnInt). What part of the circle r=2π does this shape occupy?



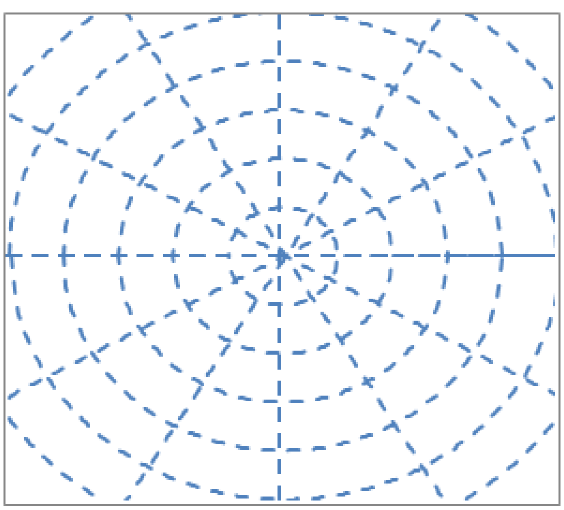
5. Inside $r = 2 + \cos(2\theta)$ and outside r=1.



6. Inside $r = 4\cos\theta$ and outside r=2. Find out where they meet!



7. Inside both $r = 2\cos\theta$ and $r = 2\sin\theta$



8. Find the slope of $r = 3\cos\theta \;\; at \;\; \theta = \pi/3$.

9. Find the slope of $r = \theta \;\; at \;\; \theta = \pi$.

10. Find the equation of the tangent line to the graph of $r = \sin(3\theta)$ where $\theta = \pi/6$. You should assume this to mean to use rectangular coordinates.

11. Where does $r = 3\cos\theta$ for $[0,\pi)$ have a tangent line that is horizontal? Vertical? Give points in (r,θ).

12. Answer the following questions about the cardioid $r = 1 + \sin\theta$ for $[0,2\pi]$

a. Write the equation of the tangent line when $\theta=\pi/3$.

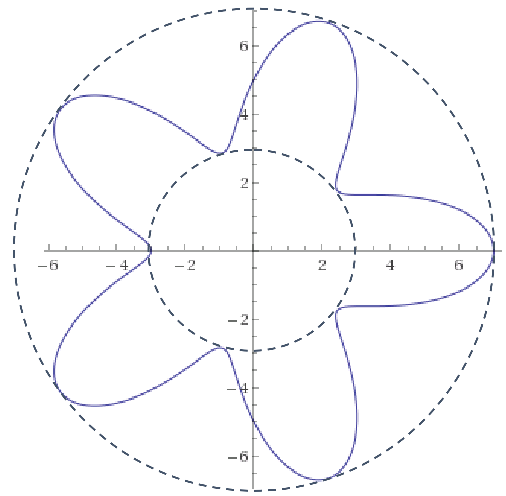
b. At what point or points (r,θ) are the tangent lines horizontal or vertical?

c. You want to create a viewing window (using x's and y's) that just fits the cardioid. In other words, you want the cardioid to be tangent to the four sides of the viewing window. What should the window be?

13. Find the length of the cardioid $r = 1 + \sin\theta$. Use fnInt.

14. Find the perimeter of one "leaf" of the graph of $r = \sin(3\theta)$. Use fnInt.

15. Answer the following questions about the "starfish" below.



a. Its equation is $r = a\cos(b\theta) + c$. Find the values of *a*, *b*, and *c*. Note: The graphs of *r*=3 and *r*=7 are also drawn.

b. Write the equation of the line tangent to the graph when $\Theta = \pi/2$.

c. Find the area enclosed by the graph of the starfish. (calculator OK but doable without)

d. Find the perimeter of the starfish (calculator required).

e. On the *xy*-plane, the domain of the graph is $w \le x \le 7$. What is the value of *w*? (calculator required)

Answers

1. $\int_0^{2\pi} (0.5)(2+\cos\theta)^2\,d\theta = 4.5\pi \approx 14.14$ 2. $\int_0^{\pi} 0.5\sin\theta d\theta = 1$

3. 3 petals each are $\int_{\pi/2}^{5\pi/6} (0.5)(\cos(3\theta))^2\,d\theta = 0.25\pi \approx 0.785$ 4. $\int_0^{2\pi} 0.5\theta^2 d\theta = \frac{8\pi^3}{3}$ one third.

5. four parts where each is $\int_0^{\pi/2} 0.5\left[(2+\cos(2\theta))^2 - 1^2\right]d\theta$ so 10.996 or 3.5π

6. meet at –π/3 and π/3 so $\int_{-\pi/3}^{\pi/3} 0.5\left[16\cos^2\theta - 4\right]d\theta \approx 7.653$

7. meet where tan=1 so π/4 so can write it as $2\int_0^{\pi/4} 0.5(2\sin\theta)^2\,d\theta \approx 0.571$

8. $1/\sqrt{3}$ 9. π 10. $y - 0.5 = -\sqrt{3}(x - \sqrt{3}/2)$

11. horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ *and* $\left(\frac{-3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ and vertical at (3,0) and (0,π/2)... this last point is the pole (anything with r=0 plots there, whatever the angle)

12a. $y - \left(\frac{2\sqrt{3}+3}{4}\right) = -1(x - \frac{\sqrt{3}+2}{4})$ b. horiz at $(2, \pi/2), (0.5, 7\pi/6), (0.5, 11\pi/6)$

Vertical at $(1.5, \pi/6), (1.5, 5\pi/6)$ and where $\theta = 1.5\pi$ we get 0/0 so try l'Hopitals and it is vertical, so vertical at the pole. c. $-\frac{3\sqrt{3}}{4} \le x \le \frac{3\sqrt{3}}{4}$ *and* $-0.25 \le y \le 2$

13. $\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\,d\theta = 8$

14. first loop is from 0 to π/3 (draw it in xy plane to see) so $\int_0^{\pi/3} \sqrt{(\sin 3\theta)^2 + (3\cos 3\theta)^2}\,d\theta \approx 2.227$

15a. $r = 5 + 2\cos(5\theta)$ b. $y = 2x + 5$ c. 27π d. 53.03 e. $\frac{dx}{d\theta} = 0$ so Θ=2.598 radians so w=-5.840

Unit 4 Handout #7: Practice Problems

1. Given the polar equation $r = 5\sin(3\theta)$.

   a. Sketch a graph (no calculator).

   b. Find the area enclosed by one leaf.

   c. Find the perimeter of one leaf.

   d. Without using your calculator, write an integral giving the area of the part any one leaf that lies outside the graph of $r = 2.5$.

   e. The leaf in the first quadrant has some horizontal and vertical tangent lines. Give their equations.

   f. Write the equation of the tangent line to the graph where $\theta = \pi / 6.$ (the tangent line's equation should be in terms of *x* and *y*, not *r* and θ).

2. Find the perimeter of the ellipse $x^2 + 4y^2 = 36$.

3. An object's position is given by the function $x(t) = 2\sin(\pi t)$ $\quad y(t) = t^2 - 4$ for $0 \le t \le 5$.

a. What is its acceleration vector at t=2?

b. What is its speed at t=3?

c. What is the equation of the tangent line to the graph at t=3?

d. What is the total distance traveled in these 5 units of time?

e. What is its average speed over these 5 seconds?

f. What is $\dfrac{d^2y}{dx^2}$ at t=1?

g. What is its speed at the point its y-coordinate is equal to 1?

4. The surface of a pond is the region in the first quadrant bound by the coordinate axes and the function $y = 3\cos x$ on the interval [0,0.5π].

a. What is the area of its surface? No calc.

b. What is the perimeter of its surface? Calc ok.

c. If its depth is a function of distance from the *y*-axis (*d*) given by depth=2*d*+1, then what is the volume of water it holds? No Calc.

d. Instead, if its depth is a function of the distance from the *x*-axis given by depth=2*d*+1, then what is the volume of water it holds? What is its average depth? Calc ok.

5. The region R is in the first quadrant, bound by the graphs of $y = x^2$ *and* $y = 2x + 3$.

a. What is the volume of the solid created when R is revolved around the line y=-2?

b. What is the volume of the solid created when R is revolved around the y-axis?

c. What is the perimeter of R?

Some AP free-response questions:

1. **(2008B #1- CALC OK)** A particle moving along a curve in the *xy*-plane has position $\left(x(t), y(t)\right)$ at time $t \geq 0$ with $\frac{dx}{dt} = \sqrt{3t}$ and $\frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right)$

The particle is at position (1,5) at time t=4.

a. Find the acceleration vector at time t=4.
b. Find the *y*-coordinate of the position of the particle at time t=0.
c. On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5?
d. Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

2. **(2003B #4- NO CALCULATOR)** A particle moves in the *xy*-plane so that position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

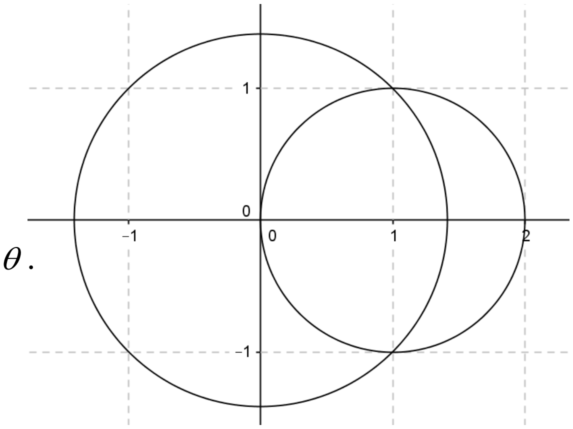
a. Find the velocity vector for the particle in terms of t, and find the speed of the particle at $t=0$.

b. Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t\to\infty} \frac{dy}{dx}$.

c. Find each value t at which the tangent to the path of the particle is horizontal, or explain why none exist.

d. Find each value t at which the tangent to the path of the particle is vertical, or explain why none exist.

3. **(2003B #2 NO CALCULATOR)** The figure below shows the graphs of $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points (1,1) and (1,-1). Let *R* be the shaded region in the first quadrant bound by the two circles and the *x*-axis.



a. Set up an expression involving one or more integrals with respect to *x* that represents the area of *R*.

b. Set up an expression involving one or more integrals with respect to *y* that represents the area of *R*.

c. The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of *R*.

4. (2019 #5: NO CALCULATOR)

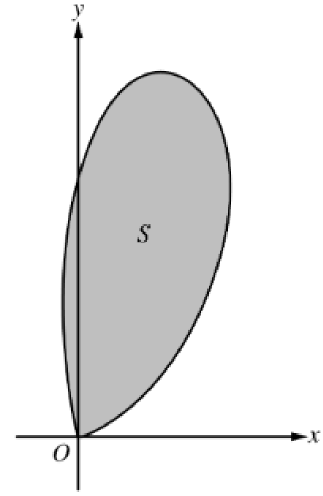
Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

5. (2019 #2 CALCULATOR OK)



Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

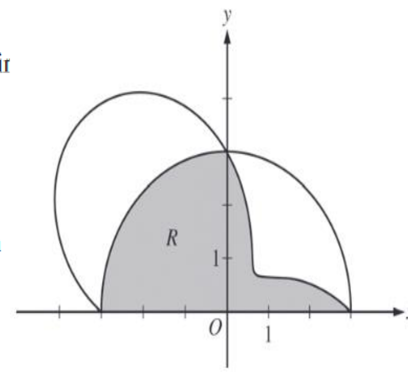
(a) Find the area of S.

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim\limits_{k\to\infty} A(k)$.

6. (2014 #2: Calculator OK)



The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Fir the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\dfrac{dx}{d\theta}$ at $\theta = \dfrac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \dfrac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \dfrac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\dfrac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\dfrac{dr}{dt}$ at $\theta = \dfrac{\pi}{6}$.

Answers

1a. 3 petals b. $\int_0^{\pi/3} 0.5r^2 d\theta = \int_0^{\pi/3} 0.5(5\sin(3\theta))^2 d\theta = 6.545$ = 25π/12

c. $\int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/3} \sqrt{(5\sin(3\theta))^2 + (15\cos(3\theta))^2}\, d\theta = 11.137$

d. meet where $3\theta = \pi/6 \ or \ 5\pi/6$ so $\int_{\pi/18}^{5\pi/18} \left[0.5(5\sin(3\theta))^2 - 0.5(2.5)^2\right] d\theta$

e. since $y = r\sin\theta = 5\sin(3\theta)\sin\theta,\ \frac{dy}{d\theta} = 5\sin(3\theta)\cos(\theta) + 15\cos(3\theta)\sin\theta$

$x = r\cos\theta = 5\sin(3\theta)\cos\theta,\ \frac{dx}{d\theta} = 5\sin(3\theta)(-\sin\theta) + 15\cos(3\theta)\cos\theta$

Vertical tangent is when $dx/d\theta = 0$ and using calculator we get θ=0.468 as only value for first leaf

When θ is this, x is equal to 4.40

Horizontal tangent when $dy/d\theta = 0$ and get θ=0.659 so y=2.812

f. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{5\sin(3\theta)\cos(\theta) + 15\cos(3\theta)\sin\theta}{5\sin(3\theta)(-\sin\theta) + 15\cos(3\theta)\cos\theta}$ and when θ=π/6 I get $-\sqrt{3}$; the point on the graph is $(2.5\sqrt{3}, 2.5)$ *so* $y - 2.5 = -\sqrt{3}(x - 2.5\sqrt{3})$

2. Solve for the positive y and get $y = \sqrt{9 - 0.25x^2}$; find perim in first quadrant and quadruple it…

Perimeter = $4\int_0^6 \sqrt{1 + (y')^2}\, dx = 4\int_0^6 \sqrt{1 + \left(-0.25x(9 - 0.25x^2)^{-0.5}\right)^2}\, dx$ which is 29.065

Reality check: should be very close to the circumference of a circle with a radius of 4.5 (average of where ellipse crosses the axes), which is 28.274.

3a. $\langle -2\pi^2\sin(\pi t),\ 2\rangle$ *so* $\langle 0, 2\rangle$ b. $\frac{dx}{dt} = 2\pi\cos(\pi t) = -2\pi$ $\frac{dy}{dt} = 2t = 6$ so $\sqrt{4\pi^2 + 36} = 8.688$

c. t=3 the point is (0,5) and slope is $\frac{2t}{2\pi\cos(\pi t)} = -3/\pi$ *so* $y - 5 = (-3/\pi)x$

d. $\int_0^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^5 \sqrt{(2\pi\cos(\pi t))^2 + (2t)^2}\, dt = 34.305$ e. 34.305/5=6.861

f. it is $\frac{dy'/dt}{dx/dt} = \frac{\frac{4\pi\cos(\pi t) + 4t\pi^2\sin(\pi t)}{[2\pi\cos(\pi t)]^2}}{2\pi\cos(\pi t)}$ which happens to be $\frac{1}{2\pi^2}$

g. y is equal to 1 when $t = \sqrt{5}$ so its speed is $\sqrt{4\pi^2\cos^2(\pi\sqrt{5}) + 20} = 6.439$

4a. $\int_0^{0.5\pi} 3\cos x dx = 3\sin(0.5\pi) - 3\sin 0 = 3$

b. along the x-axis is 0.5π; curved is $\int_0^{0.5\pi} \sqrt{1+(-3\sin x)^2}\, dx = 3.494$; along y-axis is 3 → so total is 8.064

c. volume is $\int_0^{0.5\pi} (base)(depth)dx = \int_0^{0.5\pi} (3\cos x)(2x+1)dx = 6.425$

d. volume is $\int_0^3 (base)(depth)dy = \int_0^3 (x)(2y+1)dy = \int_0^3 (\cos^{-1}(y/3))(2y+1)dy = 10.069$

the average depth is this volume divided by the area of the surface (3), so it is 3.356

5a. washers: $\int_0^3 \pi[R^2 - r^2]dx = \int_0^3 \pi[(2x+5)^2 - (x^2+2)^2]dx = 327.982$

b. shells $\int_0^3 2\pi rh dx = \int_0^3 2\pi x(2x+3-x^2)dx = 70.686$

c. left edge is 3; top is $\sqrt{45}$ and bottom/right is $\int_0^3 \sqrt{1+4x^2}\, dx$ =9.747 so the total is 19.455

AP Questions

1a. $\left(1.5(3t)^{-0.5}, -3t\sin(0.5t^2)\right) = (0.433, -11.872)$

b. at t=4 y was 5, so subtract the accumulated change from 0 to 4 and get $5 - \int_0^4 3\cos(0.5t^2)dt$ = 1.601

c. speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{3t + 9\cos^2(0.5t^2)}$; graphing it, it first reaches 3.5 when t=2.226

d. $\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^4 \sqrt{3t + 9\cos^2(0.5t^2)}dt = 13.182$

2a. velocity vector is $\left(6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t}\right)$ at t=0 this is (-1,11) so speed is $\sqrt{(-1)^2 + 11^2} \approx 11.045$

b. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$ divide all terms by e^{3t} to get $\frac{9 + 2e^{-5t}}{6 - 7e^{-10t}}$ as t→∞, this approaches 1.5

c. horizontal when slope=0 so dy/dt=0 (and dx/dt is not also 0) so $9e^{3t} + 2e^{-2t} = 0$ and thus

$e^{-2t}(9e^{5t} + 2) = 0$ and since the range of an exponential function is all positives, this has no solutions

d. vertical when slope is undefined so dx/dt=0 and $6e^{3t} - 7e^{-7t} = 0 = e^{-7t}(6e^{10t} - 7) = 0$

this is zero when $e^{10t} = 7/6$ *and* $e = 0.1\ln(7/6) \approx 0.0154$

3a. $\int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx + \int_0^1 \sqrt{1-(x-1)^2}\,dx$

b. $\int_0^1 \left(\sqrt{2-y^2} - \left(1-\sqrt{1-y^2}\right)\right)dy$ **note 1- radical and not 1+radical because it is the “lower” semi-circle.

Test a point or two to confirm.. (ie, when y=0 you should get sqtr(2)-0)

c. since they meet at (1,1), the angle where the r changes is 45 degrees or π/4

$\int_0^{\pi/4} 0.5(\sqrt{2})^2\,d\theta + \int_{\pi/4}^{\pi/2} 0.5(2\cos\theta)^2\,d\theta$

4a. $1/\sqrt{3}$ b. $\frac{1}{6}\ln 0.5$ *or* $-\frac{1}{6}\ln 2$ c. diverges

5a. $\int_0^{\sqrt{\pi}} \frac{9\theta\sin^2(\theta^2)}{2}\,d\theta \approx 3.534$ b. $\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} 3\sqrt{\theta}\sin(\theta^2)\,d\theta \approx 1.580$

c. $\int_0^{\tan^{-1} m} \frac{9\theta\sin^2(\theta^2)}{2}\,d\theta = \frac{1}{2}\int_0^{\sqrt{\pi}} \frac{9\theta\sin^2(\theta^2)}{2}\,d\theta$ d. part in q1 so $\int_0^{0.5\pi} \frac{9\theta\sin^2(\theta^2)}{2}\,d\theta \approx 3.324$

6a. quad1: $\int_0^{0.5\pi} \frac{(3-2\sin(2\theta))^2}{2}\,d\theta$ in quad2: $\frac{1}{4}\pi(3^2)$ so sum is about 9.708

b. $x = r\cos\theta = (3-2\sin(2\theta))\cos\theta$ so $\frac{dx}{d\theta} = (3-2\sin(2\theta))(-\sin\theta) + (-4\cos(2\theta))\cos\theta$

which is $\frac{-3-\sqrt{3}}{2} \approx -2.366$

c. difference in radii which is $D = 2\sin(2\theta)$ so $\frac{dD}{d\theta} = 4\cos(2\theta) = -2$

d. $r = 3-2\sin(2\theta)$ *so* $\frac{dr}{dt} = -4\cos(2\theta)\cdot\frac{d\theta}{dt}$ so dr/dt = -6