Climbing Stairs with Fibonacci and Pascal

As a teacher, I try to make my students more aware of mathematics all around them in their daily lives. I also like to help them recognize patterns and gain intuition as to why those patterns exist. The following is a problem I have used in class that usually satisfies these goals:

> Assume that someone climbs a set of steps, and in each stride they may either ascend one step or two steps. How many different ways can this person climb a set of *n* steps?

If *n* is one, there is only one way to climb the step: take it in a single stride. If there are two steps (*n*=2), the climber may take them one at a time or both at once. Call this set of possible outcomes {11, 2}.

My students quickly find that the set of possible ways to climb three steps is {111, 12, 21}, four steps is {1111, 211, 121, 112, 22}, and five steps is {11111, 2111, 1211, 1121, 1112, 221, 212, 122}. Thus the number of different ways to climb a set of *n*=1 through 5 steps is 1, 2, 3, 5, and 8.

My students may need to list and count the ways to climb six or seven steps to recognize that the number of ways to ascend the steps follows the Fibonacci Sequence, where each term is the sum of the two prior terms.

Sometimes students wonder where the other "1" at the start of this sequence is, correctly defining the start of the Fibonacci Sequence as 1, 1, 2, 3, 5, 8. This usually leads to a semantic discussion about how many ways one can "climb" a set of zero steps. When we use combinations below, the idea that there is one way to do so seems reasonable.

Developing the intuition as to why this pattern emerges is more challenging. If the students are struggling a bit, I might suggest that for each length staircase they separately count the number of ways to ascend where the initial stride covers one step and where it covers two steps. For example, in the set of eight ways to climb five steps {11111, 2111, 1211, 1121, 1112, 221, 212, 122}, there are five elements using a one-step initial stride and three using a two-step initial stride. The students notice the repetition of the Fibonacci Sequence here. Often when they step back from the numbers, they can see what drives the pattern. If the first stride covers one step, there are *n*-1 steps remaining, so the number of ways to ascend these is the previous term in the sequence. If the first stride covers two steps, there are *n*-2 steps remaining and the number of ways to climb them is the term before that one. Since these two subsets are mutually exclusive, the total number of ways to climb n steps is the sum of the prior two terms, the ways to climb *n*-1 steps and *n*-2 steps.

To confirm their grasp, I might ask them how many ways one can ascend a set of *n* steps if one is able to climb one, two, ***or three*** steps in a single stride. The students quickly see that, in this sequence, each term is the sum of the prior three terms.

Some students try to solve the original problem using combinatorics. Consider the case of climbing three steps taking one or two steps per stride. The climber may use three strides, each covering one step. There is only one way to climb three steps with three strides. It is the number of ways to order the digits of "111". From a combinatorics perspective, one may consider this is case of choosing zero two-step strides from a set of three strides, or ${}_3C_0$. If the climber uses two strides to climb this set of three steps, then one must cover two steps and one must cover one step. So the number of ways is the number of ways to order the digits of "12", which is ${}_2C_1$: two digits, one of which is a two. Hence the number of ways to ascend three steps is ${}_3C_0 + {}_2C_1$.

The number of ways to climb four steps is the number of ways to do it in 4 strides (orders of "1111") plus the number of ways to do it in 3 strides (orders of "112") plus the number of ways to do it in 2 strides (orders of "22"). This is ${}_4C_0 + {}_3C_1 + {}_2C_2$

A set of five steps may be ascended in five strides (order of "11111"), four strides (orders of "2111"), or three strides (orders of "221"). So the total number of ways is ${}_5C_0 + {}_4C_1 + {}_3C_2$

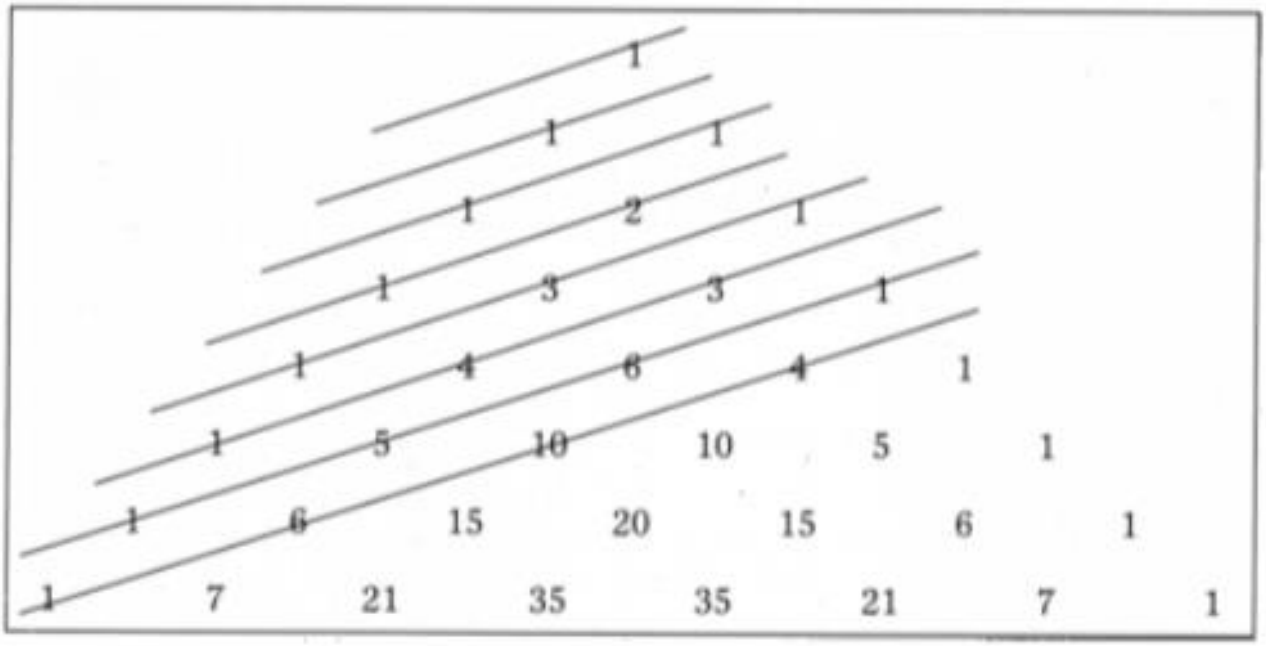
A set of six steps may be climbed using six, five, four, or three strides, for a total number of ways of ${}_6C_0 + {}_5C_1 + {}_4C_2 + {}_3C_3$.

Generalizing this pattern, one can see that the number of ways to ascend a set of *n* steps can represented as

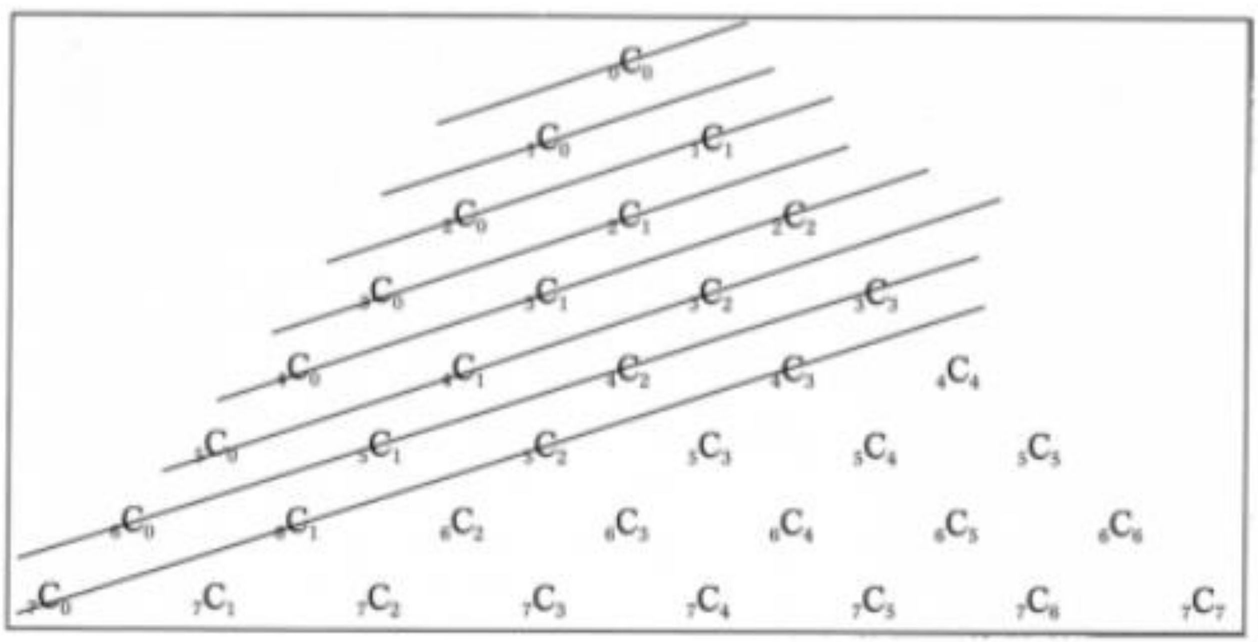
${}_nC_0 + {}_{n-1}C_1 + {}_{n-2}C_2 + {}_{n-3}C_3 + \ldots{}_{n-i}C_i + \ldots$ for all $i \le n - i \quad or \ i \le 0.5n$

Taking this approach, the students find a nice method to count the ways to climb a set of n steps without needing to use any prior terms. In the vocabulary of sequences, this approach generates an explicit formula for the *n*-step problem rather than a recursive formula using the Fibonacci pattern.

Interestingly, this representation of the Fibonacci Sequence also arises in Pascal's Triangle. Parallel lines drawn at a certain slope through Pascal's Triangle intersect entries that sum to the terms of the Fibonacci Sequence.



Writing the elements of Pascal's Triangle using combinations, it is not coincidence that the entries intersected by each line are the combinations used above to count the number of ways to climb *n* steps.



I have found that the students have an easy time relating to the Fibonacci Sequence in this context (easier than rabbit populations with some assumptions of gender and sexual maturity!). This example also nicely relates to the interesting way that Pascal's Triangle can be used to generate Fibonacci numbers. It may even cause them to remember high school math as they climb stairs in the future!