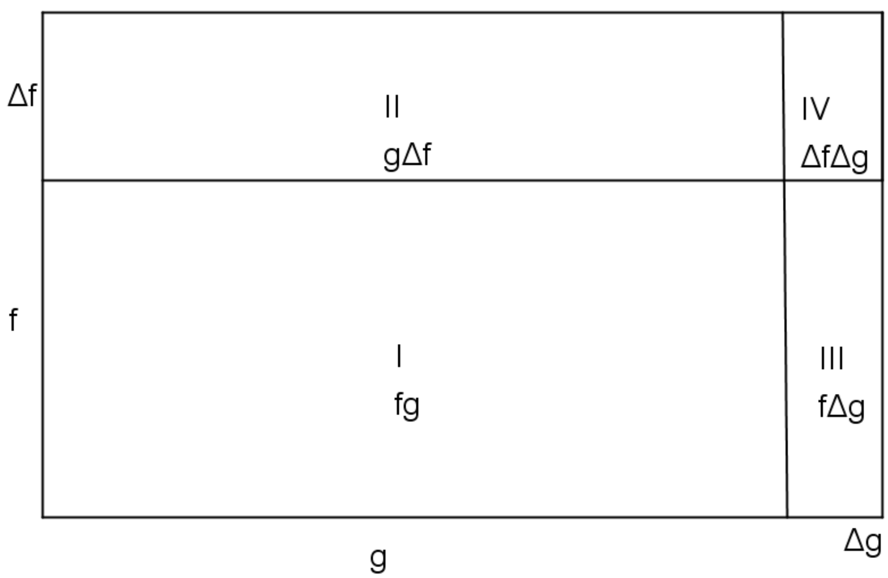
Some intuition with the product and quotient rules

As I teacher I try to balance rigor, problem-solving and intuition, as I want students to be able to explain where the rules come from even if they cannot always prove them rigorously on their own. What follows are some ways that I have tried to help the product and quotient rules resonate with my students.

Product rule



The derivative of the product of f and g is the rate of change in the area of a rectangle as both the length and width change. The change is represented by rectangles II, III, and IV above. As we make it a rate of change over an infinitesimal changes of *x*, the area of rectangle IV approaches zero while the areas of rectangles II and III do not necessarily do so. In other words, the change over small changes in x is two line segments (II and III) and one point (IV) and the point's lower dimensionality means that it becomes negligible. Therefore the rate of change in the product is gf' + fg'.

While we also prove the product rule using the limit definition, this diagram seems to be the one that resonates with the students. When I subsequently ask them why the product rule works as it does, they inevitably draw this.

Years ago, one student, upon seeing this diagram, asked if it could be extended to three dimensions. The product of three functions f, g, and h, can be thought of as a rectangular prism, and he posited that the change in this product, as the changes in x approached zero, could be represented as three rectangles, three line segments, and one point. Since the line segments and point become negligible over small intervals given their lower dimensionality, the rate of change in the product should be the area of the three rectangles, namely $(fg)h'+(fh)g'+(gh)f'$. We confirmed this was the case by applying the product rule repeatedly to find the derivative of fgh.

While it is not difficult to find some intuition in the product rule, the quotient rule seems to defy intuition. Sure, one can prove it with limits, or derive it by using the product and chain rules. But year after year, it is my students' least favorite differentiation technique, as it simply does not resonate deeply. Students wonder, as I do, why it looks so different from the product rule when products and quotients are inverses of each other.

Playing around with some problems years ago, I found some symmetry between the product and quotient rules that my students appreciate. It involves looking at the rate of change in a quantity relative to the value of the quantity, similar to a percentage change. If a population is growing at a certain rate, say 100 bacteria per minute, it means different things if this growth is generated by a population of 300 versus 50000. Likewise, an investment whose value is increasing at the rate of $100 per month is thought of differently if its value is $1000 or $10000. Thus percent changes are commonly used.

Call f'/f the "percentage rate of change".

What is the percentage rate of change in the product of two functions?

$$\frac{(fg)'}{fg}=\frac{gf'+fg'}{fg}=\frac{f'}{f}+\frac{g'}{g}$$

The percentage rate of change in the product is simply the sum of the percentage rate of changes of the two functions.

And the percentage rate of change of the quotient of two functions?

$$\frac{(f/g)'}{f/g}=\frac{(gf'-fg')/g^2}{f/g}=\frac{(gf'-fg')}{fg}=\frac{f'}{f}-\frac{g'}{g}$$

The percentage rate of change in the quotient is simply the difference of percentage rates of change of the numerator and denominator!

The students appreciate this symmetry between the product and quotient rules. And many of them use this to derive the quotient rule when they need to refresh their memory.

While intuition is not a substitute for rigor, I find that combining the two can enhance my students' understanding and appreciation of the material.