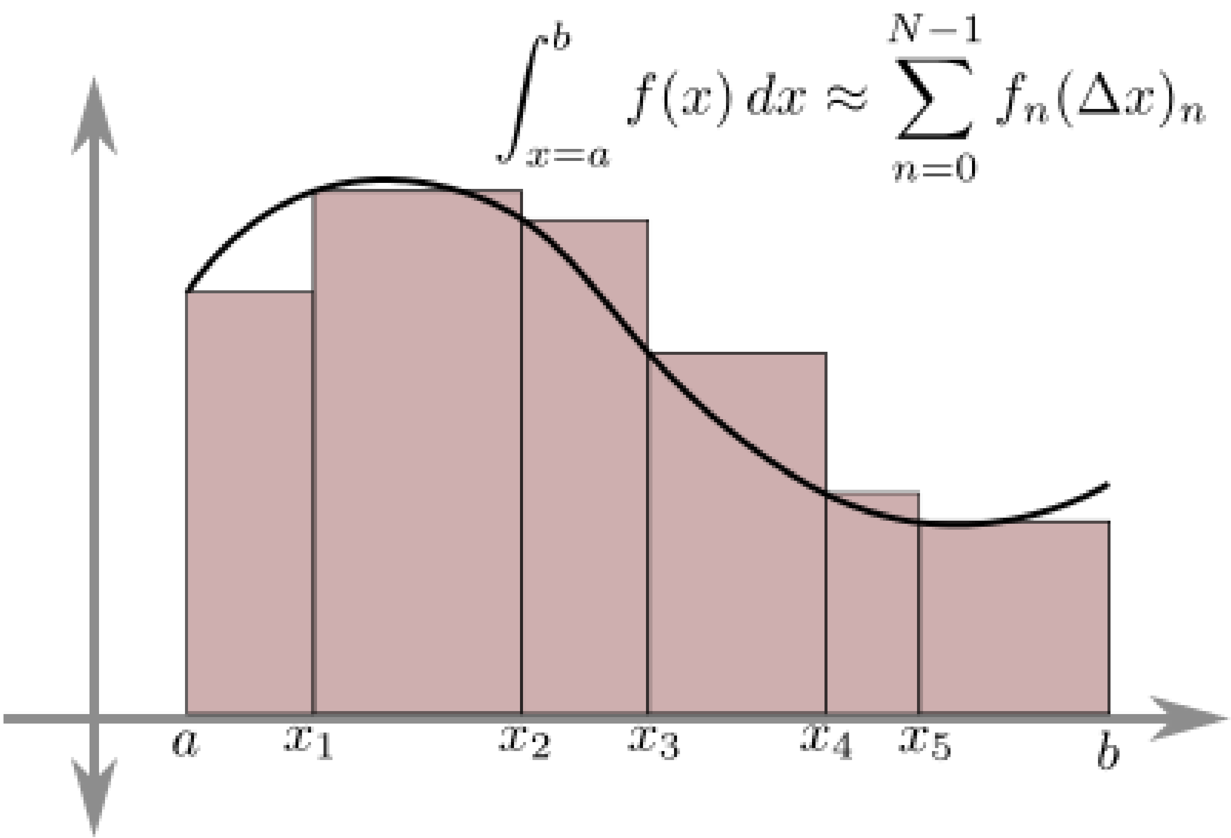
Integral Calculus



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Advice for students:

1. Consider this accelerated class. We will typically cover new material fairly quickly. Once we have gone over the key fundamentals with fairly rote problems, we will often apply the new material to more challenging non-routine problems. These can be difficult, and one of my main goals this semester is to help you develop important problem-solving skills. In the face of challenges, try to be persistent, patient, and creative. Be proactive about seeking help when you need it. We do not expect everyone to get all homework problems correct.

2. In sports, you have practices. In performing arts, you have rehearsals. In academics, you have homework. Homework in this class is designed to help you cement your understanding of the material by practicing straightforward problems and develop your problem-solving skills. Do your best to trying all the homework questions. If you have difficulty, come to the next class with specific questions that will help you advance your understanding.

3. Every problem set in this book comes with answers. They are at the end of each individual problem set. Checking your answers is essential. We recommend that you check answers after every few problems to make sure you are on the right track. We all make mistakes; we are no exception. Please let us know if you think you have found an error in our answers. I'm pretty sure that there are at least a few in this book!

4. HELP!!! Get help when you need it. Some places (in no particular order):

-Classmates
-Parents (?)
-Your teacher
-Internet (believe it or not, Google can help you find great explanations and practice problems)
-Khan Academy videos: goto www.khanacademy.org and scroll down to Calculus

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Anti-Derivative Examples

#1: $\int (x^3 - 3x + 1)dx$

This means find a function whose derivative (with respect to *x*) is $x^3 - 3x + 1$. Since the derivative of $x^n = nx^{n-1}$ the anti-derivative of x^n must be $\frac{1}{n+1}x^{n+1}$, as long as *n* is not -1. So the answer is $\frac{x^4}{4} - 3\cdot\frac{x^2}{2} + x + C$. We need the C because the derivative of a constant is zero so we don't know what constant it is.

#2: $\int (6x-1)^7 dx$

We could expand this, but should try to avoid that. It looks like the answer might be $\frac{(6x-1)^8}{8} + C$, but if we take the derivative of that we get $(6x-1)^7(6)$, since the Chain Rule tells us we need the inside derivative.

In general, $\int u^n du = \frac{u^{n+1}}{n+1}$ if *u* is a function of *x* and *n* is not -1. So try to write it in the form $\int u^n du$ by putting a 6 in front of the dx and multiplying by 1/6 outside of the integral:

$\int (6x-1)^7 dx = \frac{1}{6}\int (6x-1)^7(6dx)$.

Now it is in the form $\int u^n du$ so we can integrate and get $\frac{1}{6}\cdot\frac{(6x-1)^8}{8} + C$

#3: $\int \frac{xdx}{\sqrt[3]{x^2-7}}$

Try to write is as $\int u^n du$. So $\int \frac{xdx}{\sqrt[3]{x^2-7}} = \int (x^2-7)^{-1/3}(xdx) = \frac{1}{2}\int (x^2-7)^{-1/3}(2xdx)$.

Now the integral is in the form $\int u^n du$ so we get $\frac{1}{2}\cdot\frac{(x^2-7)^{2/3}}{2/3} + C = \frac{3}{4}(x^2-7)^{2/3} + C$

#4: $\int xe^{x^2}dx$: Can be written in form $\int e^u du$ where $u = x^2$ *and* $du = 2xdx$

$$0.5\int e^{x^2}(2x)dx = 0.5\int e^u du = 0.5e^u + C = 0.5e^{x^2} + C$$

#5: $\int e^{x^2}dx$ cannot be done because there is no *x* inside the integral and since it is not a constant we cannot insert one in front of the dx and compensate by multiplying by 1/x in front.

Unit 1 Handout #1 Anti-Derivatives

Find the following anti-derivatives, also known as ***indefinite integrals***. Numbers 1-5 involve using the power rule backwards; most of the remainder use the chain rule backwards.

1. $\int (x^2 + 2)dx$

2. $\int \left(x^{-2} + 3x + 4\right)dx$

3. $\int \frac{1}{x}dx$

4. $\int \left(6x\sqrt{x} - 4 + \frac{1}{2 \cdot \sqrt[3]{x}}\right)dx$

5. $\int \frac{7x^2 - 3\sqrt{x} + 4}{2x}dx$ (split the fraction up)

6. $\int \cos x dx$

7. $\int 2\cos(2x)dx$

8. $\int \cos(7x)dx$

9. $\int (x+1)^7 dx$

10. $\int (x^2+1)^7 \cdot 2x dx$

11. $\int \frac{8}{(2x-3)^7} dx$

12. $\int x\sqrt{9-x^2}\, dx$

13. $\int x\sin(5x^2) dx$

14. $\int \frac{1}{x+3} dx$ (think ln)

15. $\int \frac{x^2-2x-5}{x+2} dx$ hint: polynomial division

16. $\int xe^{-x^2+1} dx$

17. $\int \frac{2x-6}{x^2-6x+2} dx$

18. $\int \cos^3 x \sin x dx$

19. $\int \tan x dx$ (hint: rewrite)

20. If $f'(x) = 3x + 1$ *and* $f(4) = 12$, find $f(x)$. This is called a ***differential equation***. You are given a derivative and want to find the function. You need to have one point on the function given (here it is (4,12)) so that you can find the value of "c".

Answers

1. $\frac{x^3}{3} + 2x + C$ 2. $-x^{-1} + \frac{3x^2}{2} + 4x + C$ 3 $\ln|x| + C$ 4. $\frac{12}{5}x^{5/2} - 4x + \frac{3}{4}x^{2/3} + C$

5. $\frac{7}{4}x^2 - 3x^{1/2} + 2\ln|x| + C$ 6. $\sin x + C$ 7. $\sin(2x) + c$ 8. $\frac{1}{7}\sin(7x) + C$

9. $\frac{(x+1)^8}{8} + C$ 10. $\frac{(x^2+1)^8}{8} + C$ 11. $-\frac{2}{3}(2x-3)^{-6} + C$ 12. $-\frac{1}{3}(9-x^2)^{3/2} + C$

13. $-\frac{1}{10}\cos(5x^2) + C$ 14. $\ln|x+3| + C$ 15. $\frac{x^2}{2} - 4x + 3\ln|x+2| + C$

16. $-0.5e^{-x^2+1} + C$ 17. $\ln|x^2 - 6x + 2| + C$ 18. $\frac{-(\cos x)^4}{4} + C$

19. $-\ln|\cos x| + C$ *or* $\ln|\sec x| + C$ 20. $f(x) = 1.5x^2 + x + C$ and thus $f(x) = 1.5x^2 + x - 16$

Unit 1 Handout #2: More Integrals

Evaluate the following indefinite integrals

1. $\int x^3 dx$

2. $\int (2x+5)dx$

3. $\int \frac{2}{x^2} dx$

4. $\int \left(x - \frac{3}{x^2} \right) dx$

5. $\int \sqrt{x} dx$

6. $\int \left(x + x^{-3} \right) dx$

7. $\int x^{3/2} dx$

8. $\int 2dx$

9. $\int \pi \, dx$

10. $\int \frac{5dx}{\sqrt[3]{x}}$

11. $\int \left(\frac{2x^2 - 6x^3}{4x} \right) dx$

12. $\int \sin x dx$

Evaluate the following indefinite integrals. Some may be impossible. Others may just be difficult!

13. $\int \sqrt{2x-1}dx$

14. $\int (x^2+1)^7 xdx$

15. $\int (x^2+1)^2 dx$

16. $\int (e^{2x}+2x^e)dx$

17. $\int \sin(3x)dx$

18. $\int \sqrt[3]{(2x-1)^2}dx$

19. $\int x\cos(2x^2)dx$

20. $\int 3xe^{-x^2+5}dx$

21. $\int e^{x^2}dx$

22. $\int \frac{1}{3x+5}dx$

23. $\int \frac{3x^2+5}{x}dx$

24. $\int \frac{x}{3x^2+5}dx$

25. $\int \frac{x+5}{x}dx$

26. $\int \frac{x}{x+5}dx$

27. $\int \frac{xdx}{(3x^2+5)^7}$

28. $\int \sin^2 xdx$

29. $\int \sin^2 x\cos xdx$

30. $\int \frac{2}{e^{3x}}dx$

31. Milo and Charlie are both trying to find $\int (x+4)dx$.

Milo does this: $\int (x+4)dx = \int xdx + \int 4dx = 0.5x^2 + 4x + C$

Charlie does this: $\int (x+4)dx = \int (x+4)^1 dx = \int udu = 0.5u^2 + C = \dfrac{(x+4)^2}{2} + C$

Each is convinced that he is the only correct one. Help them resolve their disagreement.

32. Given that $f'(x) = 3x^2 - 6x + 1$ *and* $f(0) = 8$, find $f(x)$ *and* $f(2)$.

33. Given that $f'(x) = x - \dfrac{1}{x}$ and $f(1) = 7$, find $f(x)$.

34a. Given that $f''(x) = -9.8$ and $f'(0) = 20$, find $f'(x)$.

b. Given $f'(x)$ from part a above and the fact that $f(0) = 30$, find $f(x)$.

c. Where have you seen an equation like this before?

35a. In your own words, describe why $e^{\ln 6} = 6$.

b. Evaluate $\int 4^x dx$ by writing 4 as $e^{\ln 4}$.

36. An object moves back and forth along a track. Its initial position is at $s(0) = 5$ and its initial velocity is -4. Its acceleration on the interval [0,8] is given by the function $s''(t) = 5 - 2t$.

a. Find its velocity function and its position function $s(t)$.

b. What are its position, velocity, and acceleration at t=3?

c. When is its velocity the highest? [remember: for extreme on an interval you must check end points and critical points!]

d. What was the farthest right and left it got, and at what times did it reach these points?

e. How far did it travel in total over the 8 seconds?

Answers

1. $\frac{x^4}{4}+C$ 2. x^2+5x+C 3. $-2x^{-1}+C$ 4. $\frac{x^2}{2}+\frac{3}{x}+C$ 5. $\frac{2}{3}x^{3/2}+C$

6. $\frac{x^2}{2}-\frac{1}{2x^2}+C$ 7. $\frac{2}{5}x^{5/2}+C$ 8. $2x+C$ 9. $\pi x+C$ 10. $\frac{15}{2}x^{2/3}+C$

11. $\frac{x^2}{4}-\frac{x^3}{2}+C$ 12. $-\cos x+C$ 13. $\frac{1}{3}(2x-1)^{3/2}+C$ 14. $\frac{1}{16}(x^2+1)^8+C$

15. $\frac{x^5}{5}+\frac{2x^3}{3}+x+C$ 16. $0.5e^{2x}+\frac{2x^{e+1}}{e+1}+C$ 17. $\frac{-\cos(3x)}{3}+C$ 18. $\frac{3}{10}(2x-1)^{5/3}+C$

19. $0.25\sin(2x^2)+C$ 20. $-1.5e^{-x^2+5}+C$ 21. Can't do easily 22. $\frac{\ln|3x+5|}{3}+C$

23. $\frac{3x^2}{2}+5\ln|x|+C$ 24. $\frac{\ln(3x^2+5)}{6}+C$ (no absolute value needed since $3x^2+5>0$)

25. $x+5\ln|x|+C$ 26. tough—divide and write as $\int(1-\frac{5}{x+5})dx = x-5\ln|x+5|+C$

27. $\frac{-1}{36}(3x^2+5)^{-6}+C$ 28. Can't easily do (could rewrite using $\cos(2x)=1-2\sin^2 x$)

29. $\frac{\sin^3 x}{3}+C$ 30. This is $\int 2e^{-3x}dx = -\frac{2}{3}e^{-3x}+C$

31. both are; they just have different C's since Charlie has some constant in his antiderivative

32. $f(x)=x^3-3x^2+x+C$; given $f(0)=8$ C=8 so $f(x)=x^3-3x^2+x+8$ *and* $f(2)=6$

33. $f(x)=0.5x^2-\ln|x|+6.5$

34a. $f'(x)=-9.8x+20$ b. $f(x)=-4.9x^2+20x+30$ c. $h(t)=-4.9t^2+v_0t+h_0$

35a. the inverse property of logs; or take the natural log of both sides and they are equal; or "ln6 means what number do you raise *e* to in order to get 6—so when you raise *e* to that number you get 6."

b. $\int 4^x dx = \int\left(e^{\ln 4}\right)^x dx = \int e^{x\ln 4}dx = \frac{1}{\ln 4}\int e^{x\ln 4}(\ln 4)dx = \frac{1}{\ln 4}e^{x\ln 4} = \frac{1}{\ln 4}4^x = \frac{4^x}{\ln 4}$

36a. velocity is $s'(t)=5t-t^2+C$ *so* $5t-t^2-4$ position is $s(t)=\frac{-t^3}{3}+2.5t^2-4t+5$

b. position is 6.5, velocity=2, acceleration=-1

c. max $s'(t)$ *when* $s''(t)=0$ *so* $t=2.5\sec$ (this is a max and is higher than the end points of interval)

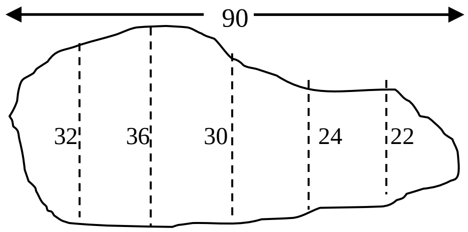
d. max or min of $s(t)$ is where $s'(t)=0$ or at end of interval… so check t=1, 4

$s(0)=5$ $s(1)=3.17$ $s(4)=7.67$ $s(8)=-37.67$ so most right at t=4 (7.67) and left at t=8 (-37.67)

e. left until t=1 so covered 1.83 then right until t=4 so covered 4.5 then left until t=8 and covered 45.33 → so total distance covered is 51.67

Unit 1 Handout #3: Areas

1. The surface of a pond is drawn below. The pond is 90 feet long, and the width varies. It is measured at each of the dashed lines, and they are drawn 15 feet apart from each other and the ends of the pond.



a. Approximate the area of the pond's surface

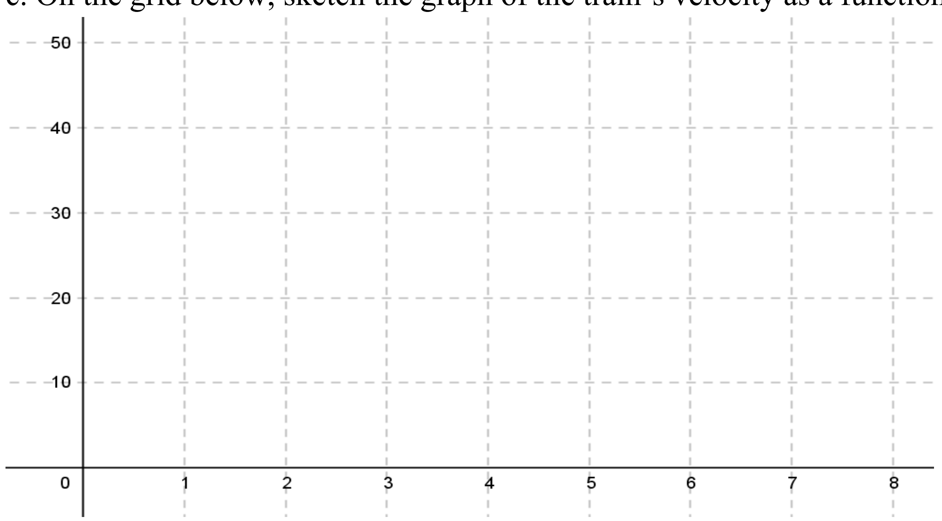
b. What is its average width of the pond?

2. Starting at noon, a train travels a constant speed of 40 mph from noon to 6 pm.

a. How far will it travel over two hours?

b. How far will it travel over six hours?

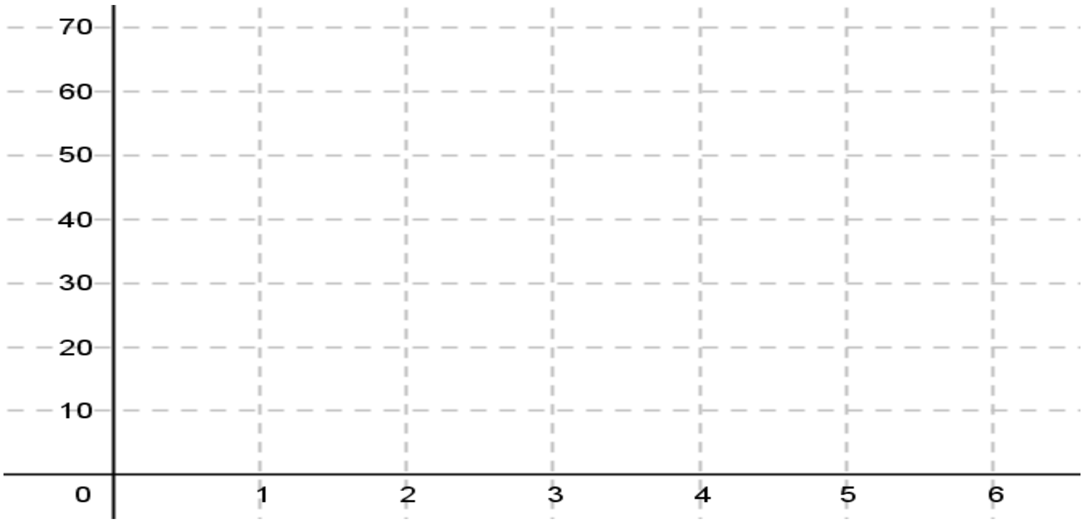
c. On the grid below, sketch the graph of the train's velocity as a function of the hours since noon.



d. Where on the graph can you see the distance the train travels in two hours? In six hours? Hint: watch the units!

3. On a different day, the same train leaves Boston at noon. Its velocity (in mph, as a function of hours since departure) is given by the function $v(t) = \begin{cases} 10|t-2|+10 & 0 \le t \le 5 \\ 40 & 5 < t \le 6 \end{cases}$

a. Sketch the graph of the train's velocity versus hours since noon.



b. What is the train's acceleration at t=4? t=5? Include units.

c. Where on the graph is the total distance traveled between noon and 1 pm shown? Calculate this distance geometrically.

d. What is the total distance that the train covers between noon and 6 pm? Where on the velocity graph is this represented?

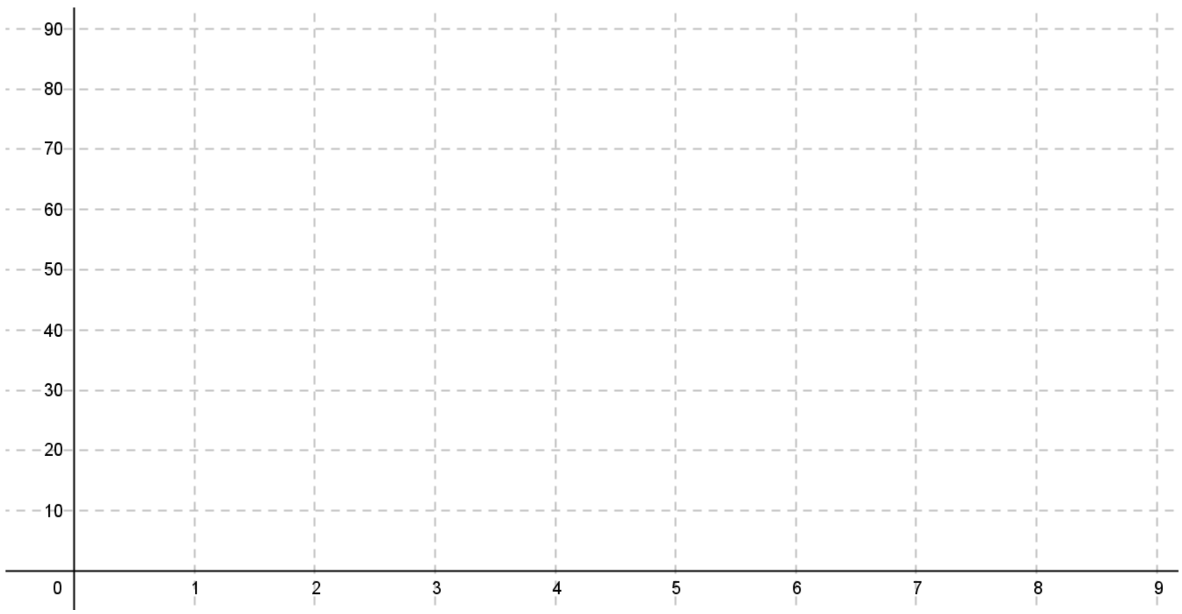
e. What is the average velocity?

4. For Spring Break, a few seniors decide to take a train trip to Cleveland. They are interested in keeping track of the distance that the train has traveled, but they don't know how far from Boston any of the cities they stop at are. They do know how far apart telephone poles are, and they have a stopwatch. Every hour, on the hour, they calculate the train's speed (in miles per hour) by using telephone poles to measure the distance it covers in one minute. The table below shows the speed for each hour. The train ride takes exactly 8 hours. The last measurement they took was near the end of the 8th hour.

Train's speed (miles per hour) at the start of each hour:

Time	1	2	3	4	5	6	7	8	9 (end-8)
Speed	0	40	70	50	60	30	70	70	20

a. Plot these points on the graph below. Again, velocity is on the *y*-axis and time on the *x*-axis. Note that these are just points. These seniors know nothing about the speed of the train at any time besides for these 9 one-minute intervals.



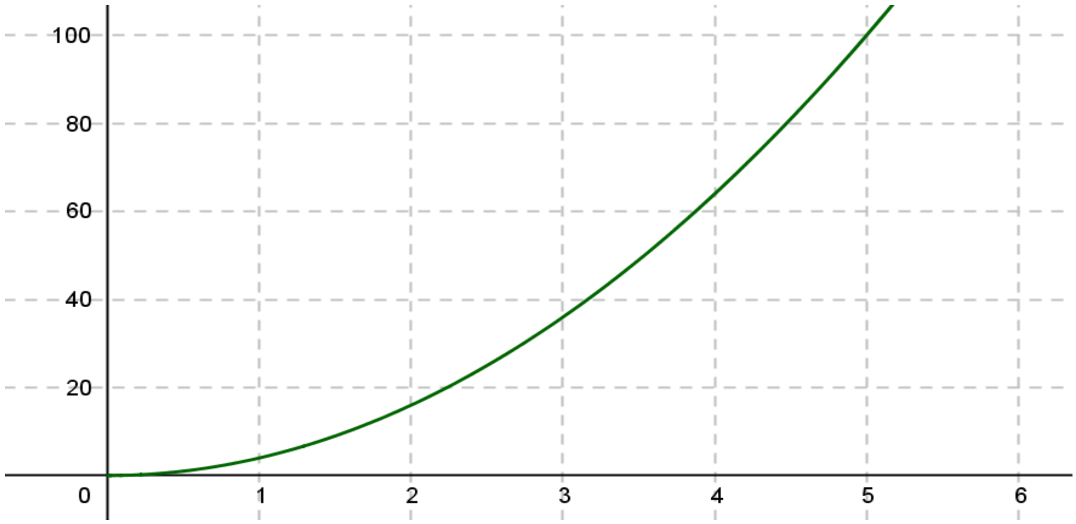
b. Assume that the train's speed that the seniors measured at the start of the hour was the actual speed of the train for the entire hour (this is unrealistic, in that it would have to somehow accelerate or decelerate to the next hour's speed, but assume it for simplicity's sake). On the graph above, draw the piecewise function that shows the train's velocity at each time in this interval. Then calculate the total distance the train traveled.

c. In the previous part you assumed that the train's speed each hour was the speed at the start of the hour. Instead, assume that each hour's speed was constant, but at the speed the seniors measured at the end of that hour (ie, the start of the next hour). In a different color, draw the piecewise function that shows the train's velocity at each time in this interval. Then calculate the total distance the train traveled under these assumptions.

d. In a different color, draw a more reasonable function showing the train's velocity at each point in time. Using this function, how far did the train travel in this 8-hour interval?

e. Which estimate is best, and why? Is it possible to know the total distance the train traveled using these data only? What data would you like to be able to make a more accurate estimate?

5. Water is flowing into an empty hot tub. The rate of inflow (in gallons per hour) given by the graph below; its equation is $y = 4x^2$.

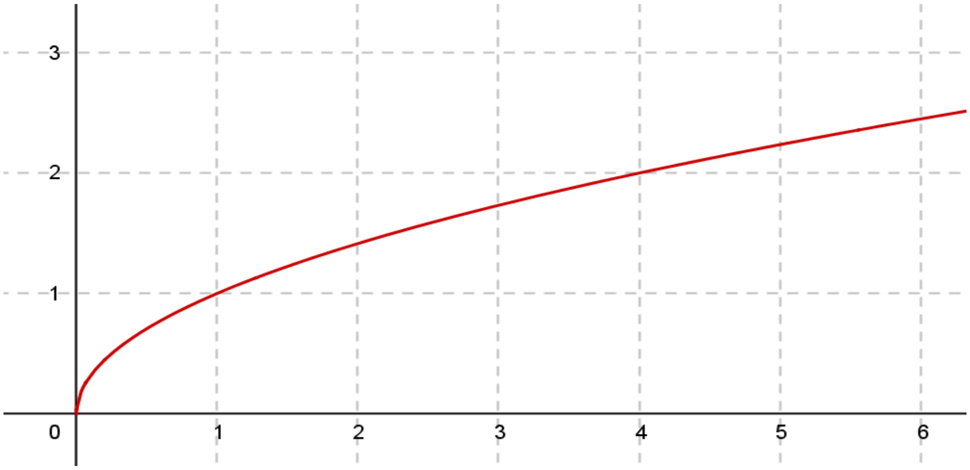


a. Someone looking quickly at the graph notes that the inflow rate is initially zero and increases to 100 at hour five. Thus she concludes that the average inflow rate is 50 gallons per hour, so there should be 250 gallons in the tub at the end of five hours. Is this a good estimate, or is it too high or too low? Explain briefly.

b. Use geometry to get a reasonable estimate of the amount of water in the tub after five hours. Use no more than 5 regions.

c. Describe a process that you could use to get an extremely accurate estimate of the amount of water in it at any point in time.

6. Estimate the area bounded by the graph of $y = \sqrt{x}$ and the x-axis, between x=0 and x=4. Use four rectangles. Estimate it in the following ways: left sums, right sums, and middle sums. By left sums, I mean use the height at the left end of the interval for the rectangle's height. Right sums uses the height at the right end-point, and middle sums uses the height in the middle (***not*** the average height).



a. Left sums:

Rectangle	Base	Height	Area
1			
2			
3			
4			

b. Right sums:

Rectangle	Base	Height	Area
1			
2			
3			
4			

c. Middle sums:

Rectangle	Base	Height	Area
1			
2			
3			
4			

d. For each left, right, and middle sums: is the approximated area too small, too large, or undetermined? Explain.

e. Using your work from the problem above, what do you think the area bounded by the graph of $y = \sqrt{x} + 2$ and the x-axis is, also between x=0 and x=4? Explain your reasoning.

7. Use the LMRRAM program in your calculator to estimate the area under $f(x) = \sqrt{x}$ in the ways below. To run it you need to put set Y1 equal to $\sqrt{x}$. Then run the program (hit $\boxed{\text{PRGM}}$ and then, while EXEC is highlighted, select LMRRAM and hit $\boxed{\text{ENTER}}$) . Since we are looking for the area between x=0 and x=4, you need to set A=0 and B=4. It will prompt you for the number of sub-intervals to use. Enter the number of rectangles you want to use (it assumes that they share the distance from 0 to 4 equally).

# Rectangles	Left Sums	Right Sums	Middle Sums
4			
8			
10			
20			
50			
100			

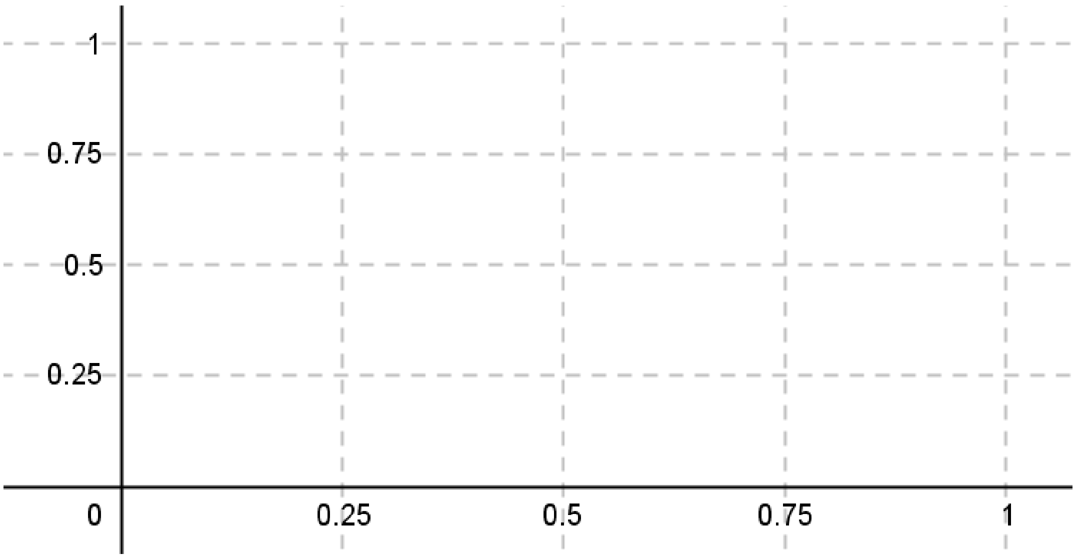
a. What do you think the area actually is?

b. Why do left, right, and middle sums get closer as the number of rectangles increases?

8. The graphs of $f(x) = x^3$ and $g(x) = \sqrt{x}$ intersect at the origin and at (1,1). Sketch them below roughly. Two friends try to estimate the area above the cubic and below the square-root function:

a. Charlie thinks the best way to do this is to first approximate the area under the square-root function and then subtract the approximate area under the cubic. Do this, using left sums with four rectangles.

b. Milo thinks that Charlie's method is correct but inefficient. He thinks it would be easier just to use four rectangles to approximate the funny-shaped area. He feels that the idea of rectangular approximation is to assume some vertical distance remains fixed for a bit of horizontal distance and feels that he can use that idea directly. What approximation does Milo get using left sums?



c. Determine how you can use LMRRAM to match your area with 4 rectangles. Then use the program to estimate the area with 30 rectangles. What is the area?

9. A baked potato is at room temperature, 70° Fahrenheit, when it is put in a hot oven and its temperature increases. The function $R(t)$ shows the ***rate of change*** in the temperature, in degrees per minute. It is a differentiable function and $R'(t) < 0$; selected values are shown in the table below. The potato is taken out of the oven at *t*=50 minutes.

t (min)	0	10	20	30	40	50
R(t)	6	4	3	2.6	2	1.2

a. What is represented by the area of the region bound by $R(t)$, the *t*-axis, and the lines *t*=10 and *t*=30? Think units! No numbers needed!

b. At *t*=20, what are the highest and lowest possible temperatures that the potato can be?

c. Why is $R'(t) < 0$ if the potato is getting warmer?

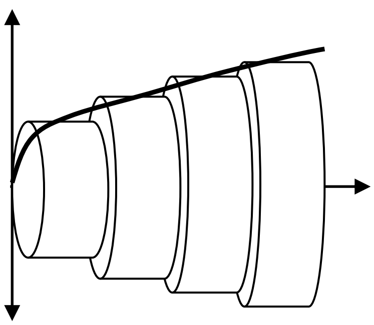
d. Must there be a time when the potato's temperature is increasing by π degrees per minute? If so, approximately when? Can it happen more than once? Justify your answer.

e. Use linearization to approximate the temperature of the potato at *t*=4 minutes. Is your approximation too high, too low, or could it be either?

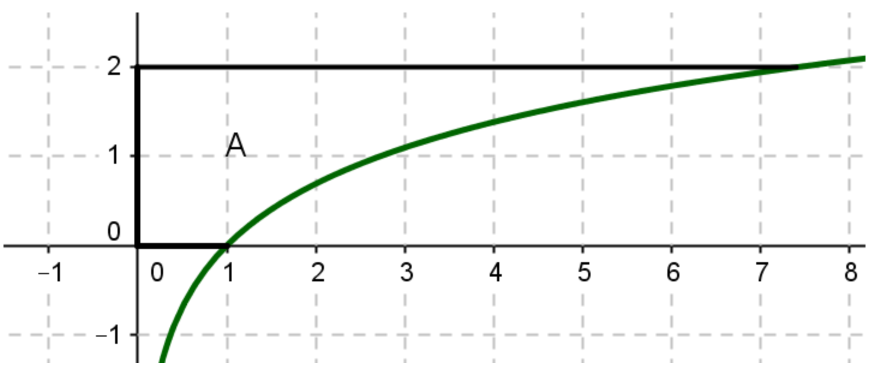
f. Sketch a rough graph showing a possible relationship between time and the potato's temperature.

10. We can approximate volumes with a similar method to that we've used to approximate area. Instead of adding up rectangles, we will add up discs (cylinders) that are to be turned sideways—see diagram below. The height of the graph becomes the radius of the cylinder and the part of the *x*-axis that was the base of the rectangle becomes the height of the cylinder.

a. Let's say we revolved region bound by the graphs of $f(x)=\sqrt{x}$, x=4, and y=0 around the *x*-axis. This would form a parabolic dish. Use middle sums with 4 discs to approximate the resulting volume.



b. Region A below is bound by the graphs of $y=\ln x$, x=0, y=0, and y=2. It is revolved around the ***y***-axis. Approximate the volume of the resulting solid using 4 cylinders with middle sums. You may need to find an inverse function.



11. Answer the following questions about the graph of $y = 4 - x$

a. Use the LMRRAM program to approximate the area under the graph of $y = 4 - x$ from x=0 to x=4. Then find the exact area geometrically.

b. Now use the LMRRAM program to approximate the area "under" the graph from x=0 to x=8. What do you find and why do you find it?

12. You want to estimate the area between $y = \sin(x)$ and the x-axis on the interval [0,2π]. Sketch a graph and then use the LMRRAM to approximate the area. What is the total area enclosed by the two parts of the graph?

13. Why is area under the x-axis treated as it is by the LMRRAM program? Hint: think of the trains and velocity, time, and distance.

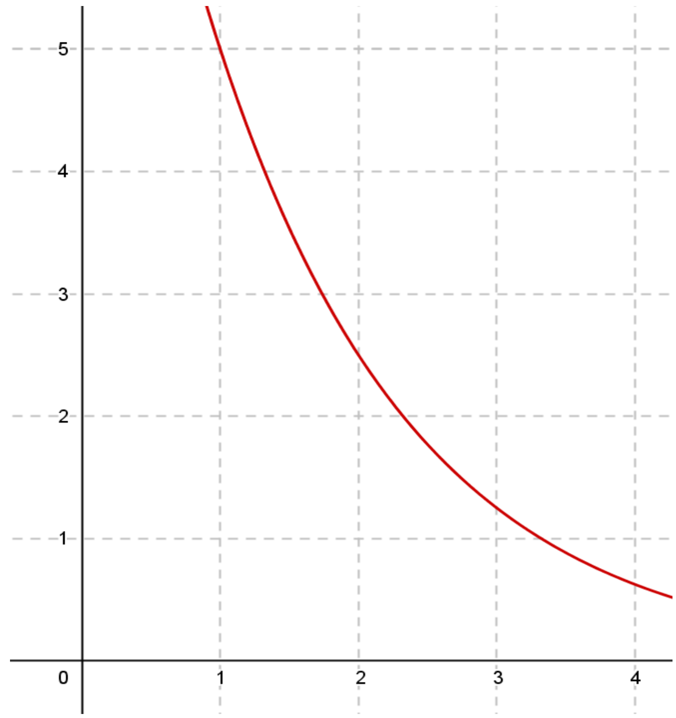
14. Given the function below, people are trying to estimate the area between the curve and the *x*-axis on the interval [1,4]. They try four different methods, using three shapes that are each one unit wide

i. Left sums

ii. Right sums

iii. Middle sums

iv. Trapezoids



a. Rank the four methods from high to low by the approximated area they result in.

b. Can we know where the actual area is in the ranking of areas?

Answers

1a. use trapezoids (triangles at the ends):

Get $0.5\cdot 32\cdot 15+34\cdot 15+33\cdot 15+27\cdot 15+23\cdot 15+0.5\cdot 22\cdot 15$ = 15(144)=2160 square feet

b. 2160/90=24 feet

2a. 80 miles b. 240 miles d. area "under the curve" – between curve and x-axis; miles per hour times hours equals miles

3b. at 4 pm: 10 miles per hour squared; at 5 pm undefined c. 25 miles d. area under curve is 155 miles e. about 25.83 miles per hour

4b. 390 miles c. 410 miles d. trapezoids: 400 miles e. I like d but can't know exactly from these data—more frequent observations would help (instead of just hourly)

5a. less since if it was the segment connecting (0,0) to (5,100) it would be 250 gallons and this is less

b. trapezoids gives 2+10+26+50+82 = 170 gallons c. slice it into very many trapezoids

6a. total is 4.14 b. 6.14 c. 5.38 d. a is too small; b is too large; c looks reasonable e. 13.38

7a. representative: at 30 get 5.19, 5.34, 5.46 at 100 get 5.29, 5.33, 5.37 → 5.33 looks good (it actually is 16/3) b. rectangles are so narrow that left, right, and middle heights are almost identical

8a. under sq root us 0.518; under cubic is 0.141 so area≈0.377

b. $\frac{\sqrt{0}-0^3}{4}+\frac{\sqrt{0.25}-0.25^3}{4}+\frac{\sqrt{0.5}-0.5^3}{4}+\frac{\sqrt{0.75}-0.75^3}{4}\approx 0.378$. Should give the exact same number (rounding may make it a bit off)

c. make Y1 the difference: $\sqrt{x}-x^3$ and for four rectangles left sums is 0.3777; for 30 rectangles get 0.415 (left), 0.417 (middle), and 0.415 (right). The left and right will always be the same for this problem because at both ends of the interval the graphs meet

9a. the total increase in temperature between minutes 10 and 30

b. lowest is right sums is 70 + 40 + 30 = 140°; highest is left sums: 70+60+40=170°

c. gets warmer at decreasing rate (1st derive of temp function is positive and decreasing so 2nd derive of temperature function is negative)

d. the Intermediate Value Theorem says sometime between t=10 and t=20 this must occur

e. 70+4(6)=94°; this is too high because it assumes the rate of increase is 6°/min for all four minutes when we are told that $R(t)$ is strictly decreasing.

f. increasing at decreasing rate: like an exponential function below its asymptote and approaching the asymptote as t → ∞

10a. $\pi\cdot\left(\sqrt{0.5}\right)^2+\pi\cdot\left(\sqrt{1.5}\right)^2+\pi\cdot\left(\sqrt{2.5}\right)^2+\pi\cdot\left(\sqrt{3.5}\right)^2=8\pi$

b. sum of $\pi r^2 h$ where h=0.5 and r=e^x so $0.5\pi(e^{0.25})^2+0.5\pi(e^{0.75})^2+0.5\pi(e^{1.25})^2+0.5\pi(e^{1.75})^2\approx 80.78$

11a. 8 b. 0→ there is "negative area" 12. 4 13. Train is going backwards

14a. left sums, trapezoids, middle sums, right sums.

b. left sums and trapezoids definitely over-estimate and right sums under-estimates. Middle sums also under-estimates it because the left part is more above the middle than the right part is below the middle.

Unit 1 Handout #4: Writing Σ's

1. Assume the function $f(x)$ is continuous on the interval [0,6] and, for all x on this interval $f(x) \geq 0$. You want to approximate the area of the region above the x-axis and below the graph of $f(x)$ on the interval [0,6].

a. You decide to use 3 rectangles and middle sums. What is an expression for this area?

b. You decide to use 6 rectangles and right sums. What is an expression for the area?

c. Write your answer to part *b* using Σ notation.

d. You decide to use 12 rectangles and right sums. What is the area of the 4^{th} one? The *i*th one? Write an expression for the area? (use Σ)

e. What if you decide to use *n* rectangles and right sums. What is an expression (using Σ notation) for this area?

f. Using your answer to part *e* above, write a limit showing the ***exact*** area of this region?

g. Write a limit showing the exact area of the region under $f(x)$ between x=1 and x=7. Think about how to compute the area of the i^{th} rectangle now—what is the x-coordinate of the end-point?

2. You are approximating the area between the graph of $f(x) = 2x^2 + 1$ and the x-axis on the interval [0,8] using rectangles (it is technically called using "Reimann Sums") with the right sums.

a. If you use 16 rectangles, what is the area of the 5$^{\text{th}}$ one?

b. If you use 16 rectangles, what is the area of the i^{th} one? (Where $1 \le i \le 16$)

c. If you use n rectangles, what is the area of the 3$^{\text{rd}}$ one? The ith one?

d. Write a limit showing the exact area of this region.

3. The area of the region between the x-axis and the graph of $y = \sqrt{x}$ between x=1 and x=4 is approximated using right sums and 12 rectangles. Which expression below gives the value of this approximation?

a. $\sum_{i=1}^{12}\left(\frac{1}{4}\right)\sqrt{\frac{i}{4}}$ b. $\sum_{i=1}^{12}\left(\frac{1}{3}\right)\sqrt{\frac{i}{3}}$ c. $\sum_{i=1}^{12}\left(\frac{1}{4}\right)\sqrt{1+\frac{i}{4}}$ d. $\sum_{i=1}^{12}\left(\frac{1}{3}\right)\sqrt{1+\frac{i}{3}}$ e. $\sum_{i=1}^{12}\left(\frac{1}{4}\right)\sqrt{1+\frac{i}{3}}$

4. (Continuation of prior problem): Write a sigma showing the area if it is computed with 30 rectangles instead of 12. One way to check yourself is to make sure the input of the last rectangle is reasonable (ie, the right end of your interval).

5. Write an expression showing the exact area of this region. It should contain a limit.

6. In each part below, the sum approximates the area of a region using right sums. What region, and using how many rectangles? In parts *a* and *b*, your answer will be in terms of the generic function $f(x)$; in the other parts you should be able to recognize a function (there may be multiple correct answers to some of the later ones).

a. $0.5f(1)+0.5f(1.5)+0.5f(2)+0.5f(2.5)$

b. $\frac{1}{10}f(5.1)+\frac{1}{10}f(5.2)+........\frac{1}{10}f(8)$

c. $\frac{(0.25)^2}{4}+\frac{(0.5)^2}{4}+\frac{(0.75)^2}{4}+\frac{(1)^2}{4}+\frac{(1.25)^2}{4}+\frac{(1.5)^2}{4}+\frac{(1.75)^2}{4}+\frac{(2)^2}{4}$

d. $\sum_{i=1}^{30}\frac{1}{3}\left(\frac{i}{3}\right)^3$

e. $\sum_{i=1}^{60}\frac{1}{3}\left(4+\frac{i}{3}\right)^3$

f. $\frac{\log 2.05}{20}+\frac{\log 2.1}{20}+\frac{\log 2.15}{20}+....+\frac{\log 12}{20}$

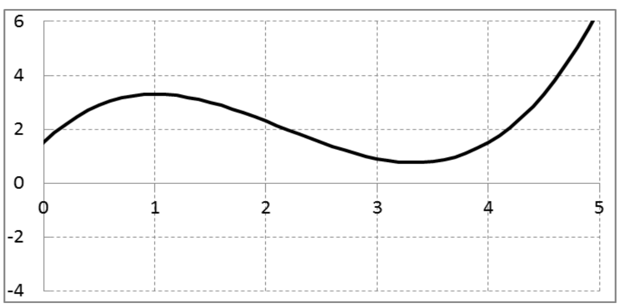
g. $\sum_{i=1}^{100}\frac{1}{5}\cdot 2^{3+0.2i}$

7. For the function $f(x) = x^2 + 3x$ do the following:

a. Write a limit showing the area between $f(x)$ and the *x*-axis on the interval [0,6].

b. Write a limit showing the area between $f(x)$ and the *x*-axis on the interval [2,8].

8. In the graph below, the region bound by $f(x)$, the *x*-axis, and the lines *x*=0 and *x*=5 is revolved around the *x*-axis creating a solid. The goal of this problem is to approximate the volume of this solid by slicing it into cylindrical disks.



a. Using 10 disks with right sums, what is the volume of the 3rd disk? The i^{th} disk?

b. Write a Σ to approximate the volume using 10 disks with right sums.

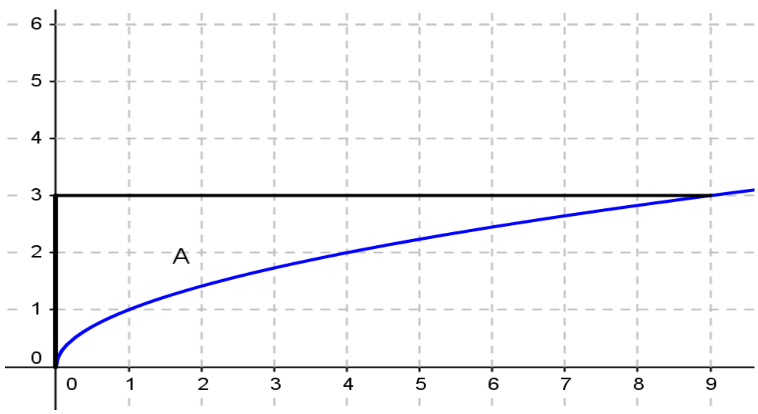
c. Write an expression involving a Σ to give the exact volume of this solid.

9. The region in the first quadrant below the graph of $y = \sqrt[3]{x}$ between x=0 and x=3 is revolved around the x-axis.

a. Write an expression that approximates the volume of the solid created using 10 disks with right sums.

b. Write a limit that shows the exact volume of the solid.

10. Region A in the graph below, above $y = \sqrt{x}$ and below y=3 from x=0 to x=9 is revolved around the line y=3 to create a solid. Which expression gives the volume of the solid?



a. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\pi\frac{3}{n}\left(\sqrt{\frac{3i}{n}}\right)^2$ b. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\pi\frac{9}{n}\left(\sqrt{\frac{9i}{n}}\right)^2$ c. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\pi\frac{9}{n}\left(3-\sqrt{\frac{9i}{n}}\right)^2$ d. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\pi\frac{9}{n}\left(\sqrt{3-\frac{9i}{n}}\right)^2$

11. The region in the first quadrant bound by the y-axis, the graph of $y = x^2$ and the lines $y = 10$ and $x = 3$ is revolved around the line $y = 10$. Write an expression giving the volume of the solid created.

Keeping old material fresh!

12. Find the following anti-derivatives:

a. $\int \sqrt{8-2x^2} \cdot x dx$ b. $\int \left(e^{4x} + e\right) dx$ c. $\int \left(\theta - \frac{5}{\sec\theta})\right) d\theta$ d. $\int \frac{\cot\theta}{\sin\theta} d\theta$

13. Given that $f'(x) = \frac{x^2-2}{x+1}$ *and* $f(0) = 4$, *find* $f(x)$ *and* $f(1)$. Hint: divide!

Answers

1a. $2f(1) + 2f(3) + 2f(5)$ b. $f(1) + f(2) + f(3) + f(4) + f(5) + f(6)$ c. $\sum_{i=1}^{6} f(i)$

d. 4th one is $\frac{1}{2} f(2)$; ith one is $\frac{1}{2} f\left(\frac{i}{2}\right)$; area= $\sum_{i=1}^{12} \frac{1}{2} f\left(\frac{i}{2}\right)$

e. $\sum_{i=1}^{n} \frac{6}{n} f\left(\frac{6i}{n}\right)$ f. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} f\left(\frac{6i}{n}\right)$ g. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} f\left(1 + \frac{6i}{n}\right)$

2a. $0.5\,f(2.5)=6.75$ b. $(0.5)\,f\left(\frac{i}{2}\right)=0.5(0.5i^2+1)$

c. 3rd is $\frac{8}{n}f\left(\frac{24}{n}\right)=\frac{8}{n}\left(\frac{1152}{n^2}+1\right)$; ith is $\frac{8}{n}f\left(\frac{8i}{n}\right)=\frac{8}{n}\left(\frac{128i^2}{n^2}+1\right)$ d. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{8}{n}\left(\frac{128i^2}{n^2}+1\right)$

3. c is correct: the rectangles have width ¼ .

4. $\sum_{i=1}^{30}\left(\frac{1}{10}\right)\sqrt{1+\frac{i}{10}}$ 5. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\right)\sqrt{1+\frac{3i}{n}}$

6a. $f(x)$ on [0.5,2.5] using 4 rectangles

b. $f(x)$ on [5,8] using 30 rectangles (width of interval is 3 units and each rectangle is 1/10)

c. $f(x)=x^2$ on [0,2] using 8 rectangles

d. $f(x)=x^3$ on [0,10] using 30 rectangles e. $f(x)=x^3$ on [4,24] using 60 rectangles

f. $f(x)=\log x$ on [2,12] using 200 rectangles g. $f(x)=2^x$ on [3,23] using 100 rectangles

7a. $\lim_{n\to\infty}\sum_{i=1}^{n}bh=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}f\left(\frac{6i}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(\frac{36i^2}{n^2}+\frac{18i}{n}\right)$

b. $\lim_{n\to\infty}\sum_{i=1}^{n}bh=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}f\left(2+\frac{6i}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(4+\frac{24i}{n}+\frac{36i^2}{n^2}+6+\frac{18i}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\left(10+\frac{42i}{n}+\frac{36i^2}{n^2}\right)$

8a. 3rd: $\pi r^2h=\pi\left[f(1.5)\right]^2\cdot 0.5$ ith: $\pi r^2h=\pi\left[f(0.5i)\right]^2\cdot 0.5$

b. $\sum_{i=1}^{10}\pi r^2h=\sum_{i=1}^{10}\pi[f(0.5i)]^2(0.5)$ c. $\lim_{n\to\infty}\sum_{i=1}^{n}\pi\cdot\left(\frac{5}{n}\right)\left[f\left(\frac{5i}{n}\right)\right]^2$

9a. $\sum_{i=1}^{10}\pi r^2h=\sum_{i=1}^{10}\pi\left(\sqrt[3]{\frac{3i}{10}}\right)^2\left(\frac{3}{10}\right)$ b. $\lim_{n\to\infty}\sum_{i=1}^{n}\pi\left(\sqrt[3]{\frac{3i}{n}}\right)^2\left(\frac{3}{n}\right)$

10. c is correct: the radius of each disk is $3-f(x)$

11. $\sum\pi r^2h=\sum\pi(10-y)^2\frac{3}{n}$ so Volume= $\lim_{n\to\infty}\sum_{i=1}^{n}\pi\frac{3}{n}\left(10-\left(\frac{3i}{n}\right)^2\right)^2$. [There are two squares because we are adding up πr^2 and the radius is the vertical distance from $y=x^2$ *to* $y=10$.]

12a. $\left(\frac{-1}{6}\right)(8-2x^2)^{3/2}+C$ b. $\frac{1}{4}e^{4x}+ex+C$ c. $0.5\theta^2-5\sin\theta+C$

d. $\int\frac{\cos\theta}{\sin^2\theta}d\theta=\int(\sin\theta)^{-2}\cos\theta d\theta=-(\sin\theta)^{-1}+C$

13. $f'(x)=\frac{x^2-1-1}{x+1}=x-1-\frac{1}{x+1}$ *so* $f(x)=0.5x^2-x-\ln|x+1|+C$ *and* $f(0)=4$ *so* $C=4$

Unit 1 Handout #5: Evaluating Σ's

Formulas: $\sum_{i=1}^{n} c = cn$ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

1. Use the above formulas for sums to evaluate the expressions below:

a. $\sum_{i=1}^{n} i^2 - 3$ b. $\sum_{i=1}^{n} 2i^2 + 3i$ c. $\sum_{i=1}^{n} 3(i+1)^2$

2a. If k is a constant, explain why $\sum_{i=1}^{n} kf(x) = k\sum_{i=1}^{n} f(x)$.

b. Can we do the same thing with an n? In other words, is $\sum_{i=1}^{n} \frac{4}{n} i^2 = \frac{4}{n}\sum_{i=1}^{n} i^2$? Explain your answer.

3. Use the formulas at the top of this tape to write each expression below in terms of n (with no i's):

a. $\sum_{i=1}^{n} \frac{6}{n}(i-2)^2$ b. $\sum_{i=1}^{n} \left(\frac{5}{n}\right)\left(\frac{5i}{n}\right)^2$ c. $\sum_{i=1}^{n} \left(\frac{3}{n}\right)\left(1+\frac{3i}{n}\right)^2$

4. For parts *b* and *c* of question #3 above, evaluate the limit as *n* approaches infinity of the expression you wrote.

5. You want to find the area between the graph of $f(x) = 2x^2 + 3$ and the *x*-axis on the interval [0,5].

a. Write a limit that shows the exact area of this region.

b. Evaluate this limit using the formulas above to find the area.

6. You want to find the area between the graph of $f(x) = x^3$ and the *x*-axis on the interval [0,3].

a. Write a limit that shows the exact area of this region.

b. Evaluate this limit using the formulas above to find the area.

7. You want to find the area between the graph of $f(x) = x^2 + 2x$ and the *x*-axis on the interval [1,4].

a. Write a limit that shows the exact area of this region.

b. Evaluate this limit using the formulas above to find the area.

8. You want to find the area between the graph of $f(x) = x^2 - 3x$ and the *x*-axis on the interval [4,8].

a. Write a limit that shows the exact area of this region.

b. Evaluate this limit using the formulas above to find the area.

9. We have been using right sums in writing Σ. Does it matter? To find the area between the x-axis and the graph of $y = x^2$ on the interval [0,3], do the following.

a. Write a limit showing the exact area using right sums; then evaluate it.

b. Write a limit showing the exact area using left sums; then evaluate it.

c. How does the difference between left and right sums appear in the computations?

Answers

1a. $\frac{n(n+1)(2n+1)}{6}-3n$ b. $\frac{2n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}$ c. $\frac{3n(n+1)(2n+1)}{6}+\frac{6n(n+1)}{2}+3n$

2a. It is the same in each term, so you can factor it out and write it in front. b. yes.

3a. $\frac{6}{n}\left[\frac{n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+4n\right]$ b. $\frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

c. $\frac{3}{n}(n)+\frac{18}{n^2}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

4b. 125/3 c. 21

5a. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\left(2\left(\frac{5i}{n}\right)^2+3\right)$ b. $=\lim\limits_{n\to\infty}\frac{250}{n^3}\sum_{i=1}^{n}i^2+\frac{5}{n}\sum_{i=1}^{n}3=\lim\limits_{n\to\infty}\frac{250}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{5}{n}\cdot 3n$

Which simplifies to $\lim\limits_{n\to\infty}\frac{250n(n+1)(2n+1)}{6n^3}+15$; which is $\frac{500}{6}+15=98.3333$

6a. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{3i}{n}\right)^3$ b. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{3i}{n}\right)^3=\lim\limits_{n\to\infty}\frac{81}{n^4}\sum_{i=1}^{n}i^3=\lim\limits_{n\to\infty}\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2$ = 81/4

7a. with n rectangles the base=3/n and height of i*th* rectangle is $f(1+\frac{3i}{n})$

So Area = $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}\right)$ *or* $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(3+\frac{12i}{n}+\frac{9i^2}{n^2}\right)$

b. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(3+\frac{12i}{n}+\frac{9i^2}{n^2}\right)=\lim\limits_{n\to\infty}\frac{3}{n}(3n)+\frac{36}{n^2}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=9+18+9=36$

8a. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}\right)\left(16+\frac{32i}{n}+\frac{16i^2}{n^2}-12-\frac{12i}{n}\right)$ *or* $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}\right)\left(4+\frac{20i}{n}+\frac{16i^2}{n^2}\right)$

b. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}\right)\left(4+\frac{20i}{n}+\frac{16i^2}{n^2}\right)=\lim\limits_{n\to\infty}\frac{4}{n}(4n)+\frac{80}{n^2}\cdot\frac{n(n+1)}{2}+\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=16+40+64/3=\frac{232}{3}$

9a. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{3i}{n}\right)=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{3i}{n}\right)^2=\lim\limits_{n\to\infty}\frac{27}{n^3}\sum_{i=1}^{n}i^2=\lim\limits_{n\to\infty}\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=9$

b. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{3i-3}{n}\right)=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{3i-3}{n}\right)^2=\lim\limits_{n\to\infty}\frac{27}{n^3}\sum_{i=1}^{n}(i-1)^2=\lim\limits_{n\to\infty}\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{54}{n^3}\cdot\frac{n(n+1)}{2}+\frac{27}{n^3}\cdot n$

this limit is 9 + 0 + 0 = 9.

c. The difference is in terms that "disappear" as n → ∞. As we saw earlier, the more rectangles one uses, the less difference the choice of left, middle, and right makes. At an infinite number of rectangles, it makes no difference at all!

Unit 1 Handout #6: Definite Integrals

Key idea: $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i = \int_a^b f(x)dx$ → A ***definite integral*** is the sum of an infinite number of terms and can represent area "under" a curve and above the *x*-axis. The area can be interpreted as the sum of an infinite number of very narrow rectangles. A volume of revolution can sometimes be interpreted as the sum of an infinite number of very thin disks (there are other techniques we will use later).

1. Write definite integrals to express the area described in each part below. A sketch may help.

a. The area bound by the graph of $y = x^2$ and the *x*-axis between *x*=1 and *x*=5.

b. The area bound by the graph of $y = x^2$ and the line *y*=16.

c. The area bound by the *y*-axis, the graph of $y = \sqrt[3]{x}$ and the line *y*=2.

d. The area bound by the graphs of $y = x^2$ and $y = \sqrt{x}$ in the first quadrant.

e. The area of the region bound by the graphs of $y = 3x + 6$ and $y = x^2 + 2x$.

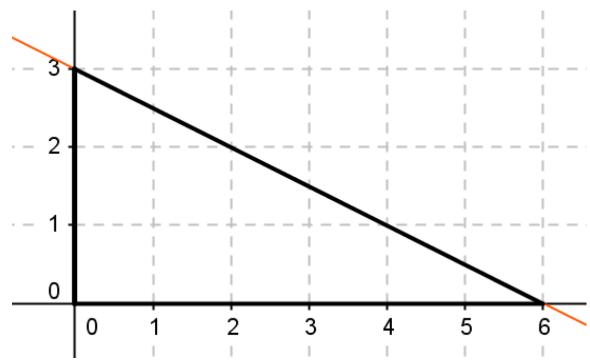
f. The area of the region bound by the graphs of $y = 3x$ and $y = 10 - x^2$.

2. Assume that $a < b < c$ and $f(x) > 0$ for all $a < x < c$. Is it true that $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$? Explain your answer.

3. Two friends are trying to find the area of the region below bound by the axes and the graph of $f(x) = 3 - 0.5x$. Of course they could use the formula for the area of a triangle, but they just learned about definite integrals and are excited to use them!

Mort thinks of the area as $\int h \cdot b$, or $\int y dx$, and says the area is $\int_0^6 (3 - 0.5x)dx$.

Dash decides to slice the area horizontally. Using $\int x dy$ he says the area is $\int_0^3 (6 - 2y)dy$. What do you think of his answer? And where did the $6 - 2y$ come from?



4. Sketch a graph of $f(x) = -(x-2)^3 + 8$ (you may want to start with $y = x^3$ and transform it.) Find the axis intercepts. You want to find the area in the first quadrant bound by the axes and the graph of $f(x)$.

a. If you slice it vertically (as we usually do), you get an integral with a *dx*, which represents the very small width of each of the infinite number of rectangles. Write the integral.

b. If you slice it horizontally, you get an integral with a *dy*, which represents the very small height of each of the infinite number of rectangles. Write the integral.

5. Sketch the graph of $f(x) = x^2 + 2.$ Imagine that the region bounded by the *x*-axis, $f(x)$, and the vertical lines *x*=0 and *x*=3 is revolved around the *x*-axis, creating a funny-shaped solid. Which of the following integrals represents the volume of this solid?

a. $\int_{-3}^{3} (x^2+2)dx$ b. $\int_{0}^{3} (x^2+2)^2 dx$ c. $\int_{-3}^{3} \pi(x^2+2)^2 dx$ d. $\int_{0}^{3} \pi(x^2+2)^2 dx$ e. $\int_{0}^{3} 2\pi(x^2+2)dx$

6. For each part below, write a definite integral showing the volume of the solid described.

a. The region bound by $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ *and* $x = 4$ revolved around the x-axis.

b. The region bound by $y = e^x$, the x-axis and the lines $x = -1$ *and* $x = 3$ revolved around the x-axis.

c. The region under $f(x) = \sin x$ between x=0 and x=π being rotated around the x-axis.

d. The region bound by $y = \frac{4}{x}$ and the lines $x = 2$ *and* $y = 1$ revolved around the line y=1.

e. The region bound by the *y*-axis, the graph of $y = \sqrt{x}$ *and* $y = 4$ revolved around the line *y*=4.

f. The region bound by the *y*-axis, the graph of $y = \sqrt{x}$ *and* $y = 4$ revolved around the ***y***-axis.

g. The region bound by the *y*-axis, the graph of $y = e^x - 1$ *and* $y = 7$ revolved around the ***y***-axis.

7. Use the graph of $f(x)$ below to find the following:

a. $\int_{2}^{4} f(x)dx$

b. $\int_{0}^{5} f(x)dx$

c. $\int_{7}^{10} f(x)dx$

d. $\int_{0}^{10} f(x)dx$

e. If $\int_{10}^{15} f(x)dx = 14$ then what is $\int_{0}^{15} f(x)dx$?

8. Given that $f(x)$ is an ***even*** function and $\int_{0}^{9} f(x)dx = 279$, find the following.

a. $\int_{-9}^{9} f(x)dx$

b. $\int_{0}^{9} (f(x)+2)dx$

c. $\int_{0}^{9} (f(x)-4)dx$

d. $\int_{0}^{9} (f(x)-11)dx$

e. $\int_{0}^{9} (f(x)-x)dx$ (use geometry)

9. Write an integral that shows the area enclosed by the graphs of $y = x^2 - 3x + 7$ and $y = x + 12$.

10. The function $I(t)$ shows the rate of water flowing into an empty bathtub (in liters per minute) t minutes after the water was turned on.

a. What is the meaning of $\int_0^{10} I(t)dt$, and what are its units?

b. Write an integral describing the amount of water in the tub after 25 minutes.

c. Let the function $O(t)$ show the rate of water leaving the bathtub t minutes after the water was first turned on. Write an integral (or two) describing the amount of water in the tub after 25 minutes.

d. What is going on when $\int_0^{n} O(t)dt = \int_0^{n} I(t)dt$?

e. Can $\int_0^{n} O(t)dt > \int_0^{n} I(t)dt$? Explain.

11. When people talk about "the area between two curves" they typically mean the positive area of all regions enclosed by the two curves.

a. On your calculator, graph $f(x) = x^3 - 2x^2 - 7x + 3$ and $g(x) = x + 3$. Algebraically find the x-coordinates of the three points of intersection.

b. Write an integral (or two) that describes the area bounded by the two curves. Would it help to use absolute value signs?

12. Given that $\int_0^3 x^2 dx = 9$ *and* $\int_0^9 x^2 dx = 243$, find the following: (you may need to use some geometry).

Note that x^2 is an even function.

a. $\int_0^3 (x^2 + 5)dx$

b. $\int_3^9 x^2 dx$

c. $\int_{-3}^9 x^2 dx$

d. $\int_0^9 (x^2 + x)dx$

e. $\int_0^3 (2x^2 - 4)dx$

13. Sketch the graph of a **linear** function $f(x)$ such that $\int_0^2 f(x)dx = 5$ and $\int_0^4 f(x)dx = 0$. Then write its equation and use it to evaluate $\int_0^6 f(x)dx$.

14. Given that $\int_0^7 f(x)dx = 30$ *and* $\int_0^8 g(x)dx = 15$ *and* $\int_0^{10} g(x)dx = 7$ and $f(x)$ is odd and $g(x)$ is even, evaluate the following:

a. $\int_{-7}^{0} f(x)dx$

b. $\int_0^7 3f(x)dx$

c. $\int_0^7 (2f(x)+1)dx$

d. $\int_{-8}^{8} g(x)dx$

e. $\int_8^{10} g(x)dx$

15. Graph the function $y = \sqrt{4 - x^2}$ (if it does not look familiar square both sides—but watch the range!)

a. Use your graph and knowledge of geometry to evaluate $\int_0^2 \sqrt{4 - x^2}\, dx$.

b. Use your graph and knowledge of trigonometry (but not your calculator) to evaluate $\int_0^1 \sqrt{4 - x^2}\, dx$.

Hint: make it a sector plus a right triangle.

16. Use geometry to evaluate $\int_{-1}^{5} |x - 2|\, dx$

17. The rate that a concrete truck is pumping concrete (in cubic meters per minute) is given by the function $C(t)=\begin{cases}0.4t & 0\le t<10\\ 4 & 10\le t<40\\ 44-t & 40\le t<44\end{cases}$.

a. How much concrete is pumped over these 44 minutes? Use geometry!

b. At what time has half of the concrete been pumped?

c. What is the meaning of $\frac{1}{44}\int_0^{44} C(t)dt$? What is its numerical value?

d. Hard: For some k<20, $\int_k^{k+10} C(t)dt = 36$. Find k and describe what this means.

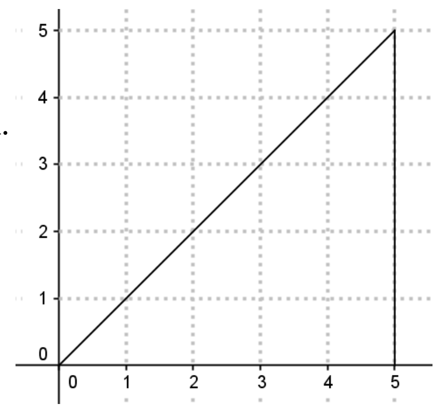
18. The table below shows selected values of $f(x)$, an increasing and differentiable function.

x	0	2	5	8	14
$f(x)$	9	14	16	20	24

a. What is the smallest possible value of $\int_0^{14} f(x)dx$? The largest possible value?

b. What is a reasonable estimate for $f'(11)$?

19. While it may seem difficult to evaluate $\int_0^5 x^2 dx$ geometrically, there actually is a way. It involves the region in the first quadrant bound by $y = x$, the x-axis, and x=5.



a. When this region is revolved around the x-axis, it creates a solid. Write an integral that shows the volume of this solid.

b. What is the shape of this solid? Find its volume geometrically. Look up the volume formula if you need it.

c. Given two ways of expressing the volume in parts *a* and *b*, what must be true of the integral?

d. Use a similar approach to show that $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$. Geometrically, it will involve a "truncated cone" – a large cone with a smaller one removed from the "top". This shape is technically called the ***frustum*** of a cone.

Answers

1a. $\int_1^5 x^2 dx$ b. $\int_{-4}^4 \left(16 - x^2\right) dx$ c. $\int_0^8 \left(2 - \sqrt[3]{x}\right) dx$ d. $\int_0^1 \left(\sqrt{x} - x^2\right) dx$ e. $\int_{-2}^3 \left((3x+6) - (x^2 + 2x\right) dx$

f. $\int_{-5}^2 \left(10 - x^2 - 3x\right) dx$ 2. yes; we are adding contiguous areas

3. Both are OK. Since Dash is using *dy* and not *dx*, his limits of integration are *y*=0 to *y*=3. The 6-2*y* is an inverse function; he got it by solving *y*=3-0.5*x* for *x*.

4a. $\int_0^4 \left(-(x-2)^3 + 8\right) dx$ $\int_0^{16} \left(2 + \sqrt[3]{8-y}\right) dy$ 5. $\int_0^3 \pi r^2 h = \int_0^3 \pi y^2 dx$ so choice *d* is correct

6a. $\int_1^4 \pi r^2 h = \int_1^4 \pi y^2 dx = \int_1^4 \frac{\pi}{x^2} dx$ b. $\int_{-1}^3 \pi r^2 h = \int_{-1}^3 \pi y^2 dx = \int_{-1}^3 \pi e^{2x} dx$ c. $\int_0^\pi \pi r^2 h = \int_0^\pi \pi y^2 dx = \int_0^\pi \pi \sin^2 x dx$

d. $\int_2^4 \pi r^2 h = \int_2^4 \pi (y-1)^2 dx = \int_2^4 \pi \left(\frac{4}{x} - 1\right)^2 dx$ e. $\int_0^{16} \pi r^2 h = \int_0^{16} \pi (4-y)^2 dx = \int_0^{16} \pi \left(4 - \sqrt{x}\right)^2 dx$

f. $\int_0^4 \pi r^2 h = \int_0^4 \pi x^2 dy = \int_0^4 \pi \left(y^2\right)^2 dy = \int_0^4 \pi y^4 dy$ g. $\int_0^7 \pi r^2 h = \int_0^7 \pi x^2 dy = \int_0^7 \pi \left(\ln(y+1)\right)^2 dy$

7a. 14 b. 29.5 c. -4 d. 28.5 e. 42.5

8a. 558 b. 297 c. 243 d. 180 e. 279-area of triangle below y=x so area is 238.5

9. $\int_{-1}^5 (x+12-(x^2-3x+7))dx$ (line is above parabola)

10a. total amount of water that has flowed into tub (in liters) in first 10 minutes. Since the tub was empty initially this is also the amount of water in the tub 10 minutes after the water was turned on.

b. $\int_0^{25} I(t)dt$ c. total in minus total out so $\int_0^{25} I(t)dt$ - $\int_0^{25} O(t)dt$ or $\int_0^{25} (I(t)-O(t))dt$

d. tub is empty at time n since total inflow equals total outflow

e. no, since tub was empty initially there can't be more water flowed out than in

11a. they meet at -2, 0, and 4 b. $\int_{-2}^0 (x^3-2x^2-8x)dx + \int_0^4 (-x^3+2x^2+8x)dx$

12a. 24 b. 234 c. 252 d. 283.5 e. 6

13. $f(x) = 5-2.5x$ and $\int_0^6 f(x)dx = -15$ 14a. -30 b. 90 c. 67 d. 30 e. -8

15a. a quarter circle, so π b. sector plus a right triangle: right triangle's area is $\sqrt{3}/2$ and central angle of sector is $\frac{\pi}{2} - \tan^{-1}\sqrt{3} = \frac{\pi}{6}$ so sector's area is $\frac{\pi}{3}$ *so* $\int = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ 16. 9

17a. 148 cubic meters b. after 23.5 min c. avg rate concrete is pumped, which is 37/11 cubic m/min

d. let w be minutes before 20: concrete pumped in 10 minutes starting at w is rectangle minus triangle, or $40-0.5(w)(0.4w)$ so $40-0.2w^2 = 36$ and $w = 2\sqrt{5}$ so $k = 10-2\sqrt{5} \approx 5.53$. So between t=5.53 and t=15.53, 36 cubic meters were pumped.

18a. small: (9*2)+(14*3)+(16*3)+(20*6) = 228 ; large: (14*2)+(16*3)+(20*3)+(24*6) = 280 b. 2/3

19a. $\int_0^5 \pi x^2 dx$ b. a cone; $V = \frac{1}{3}\pi r^2 h = \frac{125\pi}{3}$ c. $V = \frac{125\pi}{3} = \int_0^5 \pi x^2 dx$ so $\int_0^5 x^2 dx = \frac{125}{3}$

d. revolve the region bound by *x*=*a*, *x*=*b*, the *x*-axis, and *y*=*x* around the *x*-axis. It makes a truncated cone whose volume is $\frac{\pi b^3}{3} - \frac{\pi a^3}{3}$

Unit 1 Handout #7: Unit Review Questions

Formulas: $\sum_{i=1}^{n} c = cn$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

1. Find the following anti-derivatives:

a. $\int (3x^2 - x + 1)dx$

b. $\int \frac{2}{\sqrt{3-5x}} dx$

c. $\int \frac{2x}{x^2 - 3} dx$

d. $\int xe^{-x^2} dx$

e. $\int \frac{4}{e^{4x}} dx$

f. $\int (x^2 - 1)^7 x dx$

g. $\int (x^3 - 1)^2 dx$

h. $\int 4\sin(3x)dx$

i. $\int \sqrt{\sin x} \cos x dx$

j. $\int \frac{x^2 dx}{x+3}$ k. $\int \frac{5}{\sqrt[3]{6x+5}} dx$ l. $\int \frac{x^2 - 4x + 5}{x\sqrt{x}} dx$

2. Solve the following differential equations:

a. $f'(x) = x^2 - \frac{1}{x}$ and $f(1) = 3$ b. $f'(x) = 2 + \sqrt{3x+1}$ and $f(0) = -7$

3. Water is flowing through a pipe at the rate of $r(t)$ gallons per hour, where t is hours since noon.

a. Given that $r(2) = 6$ *and* $r'(2) = 4$, what is going on?

b. If $\int r(t)dt = s(t)$, then what are the units of $s(t)$?

4. The table below shows the rate of people entering a Wal-Mart store at various times of the day. The rate is in people per minute. Estimate the total number of people who entered between 9 am and 4 pm.

Time	9-9:02	10-10:01	11-11:01	12-1201	1-1:01	2-2:02	3-3:02	3:59-4:00
Ppl/min	12	15	18	16	20	20	16	10

5. Answer the following questions about the region in the first quadrant between the *x*-axis and the graph of $f(x) = 2x - x^2$.

a. Write a definite integral showing its area.

b. Approximate the area with 4 rectangles using middle sums.

c. Write a Σ showing the approximate area using 40 rectangles and right sums.

d. Write a Σ showing the exact area.

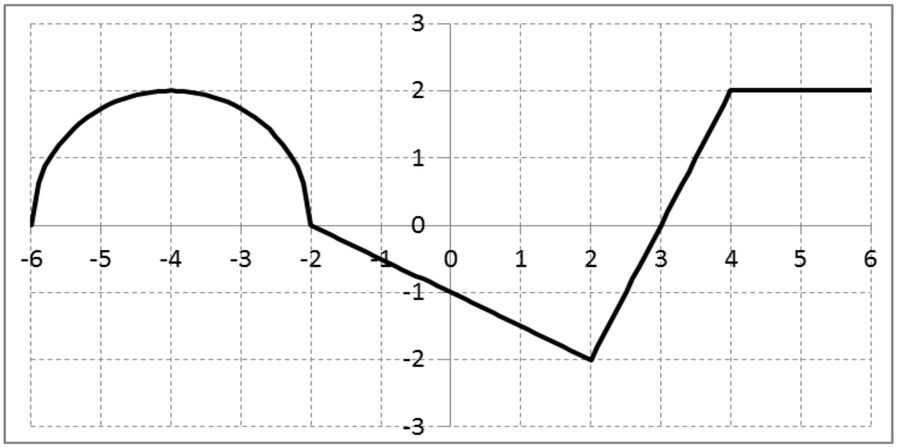
e. Use the formulas to evaluate it and get the exact area.

6. Evaluate the following geometrically:

a. $\int_0^6 (2x+1)dx$

b. $\int_0^5 \left(-|x-3|+5\right)dx$

7. The graph of $f(x)$ below consists of a semi-circle and some line segments. Use it to evaluate the integrals below:



a. $\int_{-6}^{-2} f(x)dx$

b. $\int_{0}^{3} f(x)dx$

c. $\int_{-2}^{6} f(x)dx$

d. $\int_{3}^{5} (2f(x)+3)dx$

e. $\int_{-6}^{6} |f(x)|dx$

f. If $\int_{2}^{k} f(x)dx = 1$, then what is k?

8. Answer the following about the area below $y = 9 - x^2$ and above the x-axis in the first quadrant.

a. Write an integral that describes that area.

b. The area described above is revolved around the x-axis, making a solid dish. Write an integral that gives its volume.

9. An object moves back and forth along a horizontal line. Its initial position (t=0) is x=8 cm. Its *velocity* is given by the following piece-wise function: (measured in cm per second)

$$v(t) = \begin{cases} 2t & 0 \le t \le 4 \\ 8 & 4 < t < 6 \\ 14 - t & 6 \le t \le 20 \end{cases}$$

a. Where is the object at t=5?

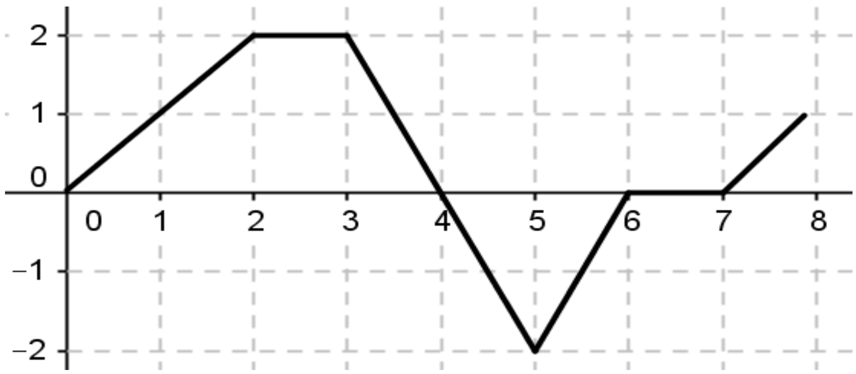
b. What is the average velocity of the object in the first 10 seconds?

c. When was the object furthest right (in the 20 seconds) and what was its position?

d. What is the total distance the object travels in these 20 seconds?

e. (a little tougher): When, if ever, is the object's position first equal to 18 cm?

10. The chart below shows the velocity of a particle over time, measured in km/minute, on the interval [0,8]. Assume its initial position is zero. Use the graph only- do not define this function symbolically!



a. Where is the particle at *t*=6?

b. When, if ever, does the particle change direction?

c. When is the particle furthest right, and what is its position.

d. Sketch graphs showing the particle's position and acceleration.

e. What are the average velocity and speed over this interval [0,8]?

f. When, if ever, is its position 4.25?

11. Given $\int_0^a \left(x^2 - 3x\right)dx$, answer the following questions; a is a constant and $0 < a \le 6$.

a. What value of a maximizes the definite integral?

b. What value of a minimizes the definite integral?

12. Find the following anti-derivatives:

a. $\int \frac{t^3 + 3t^2 - 4}{2t^2} dt$

b. $\int (x + \frac{1}{x})dx$

c. $\int \left(2\sqrt{x} + x\sqrt{2}\right)dx$

d. $\int e^{5x} dx$

e. $\int (x^3 + 2)^{11} \cdot 3x^2 dx$

f. $\int \frac{dx}{x+3}$

g. $\int \frac{2}{\sqrt[3]{x}}\,dx$

h. $\int \frac{2x}{\sqrt[3]{x}}\,dx$

i. $\int \frac{1}{\sqrt[3]{2x}}\,dx$

j. $\int xe^{-x^2}\,dx$

k. $\int (8-2x)^{15}\,dx$

l. $\int \sin^2(3x)\cdot\cos(3x)\,dx$

m. $\int x\cos(\pi x^2+1)\,dx$

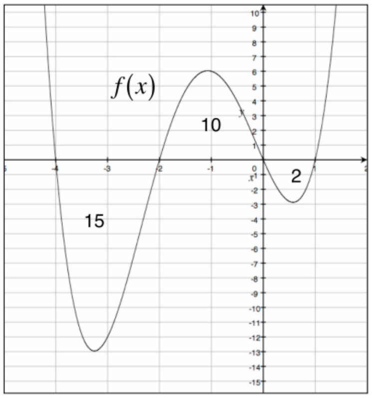
n. $\int \tan x\,dx$ (hint: rewrite it in terms of other trig functions)

13. Solve the following differential equation: $f'(x)=e^{0.1x}$ $f(0)=20$.

14. The area bounded by the graphs of $y=0,\ x=6, x=1,\ and\ \ y=\sqrt{x}$ can be written as:

a. $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{\infty}\frac{6}{n}\sqrt{\frac{6i}{n}}$ b. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{5}{n}\sqrt{\frac{5i}{n}}$ c. $\lim\limits_{n\to\infty}\sum\limits_{n=1}^{\infty}\frac{6}{n}\sqrt{1+\frac{6i}{n}}$ d. $\lim\limits_{i\to\infty}\sum\limits_{n=1}^{\infty}\frac{5}{n}\sqrt{1+\frac{5i}{n}}$ e. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{5}{n}\sqrt{1+\frac{5i}{n}}$

15. (From http://mastermathmentor.com/mmm/jeopardy/jeopardy.html). The graph on the right shows the area of three regions bound by the x-axis and the graph of $f(x)$. Use this graph to find $\int_{-3}^{2}\left(f(x-1)-1\right)dx$ or explain why it is not possible.



Answers

1a. $x^3 - 0.5x^2 + x + C$ b. $\frac{-4}{5}(3-5x)^{1/2} + C$ c. $\ln|x^2 - 3| + C$

d. $-0.5e^{-x^2} + C$ e. $-e^{-4x} + C$ f. $\frac{(x^2-1)^8}{16} + C$ g. $\frac{x^7}{7} - \frac{x^4}{2} + x + C$

h. $-\frac{4}{3}\cos(3x) + C$ i. $\frac{2}{3}(\sin x)^{1.5} + C$ j. $0.5x^2 - 3x + 9\ln|x+3| + C$

k. $1.25(6x+5)^{2/3} + C$ l. $\frac{2}{3}x^{3/2} - 8x^{1/2} - 10x^{-1/2} + C$

2a. $f(x) = \frac{x^3}{3} - \ln x + \frac{8}{3}$ b. $f(x) = 2x + \frac{2}{9}(3x+1)^{3/2} - \frac{65}{9}$

3a. at 2 pm water flow is 6 gal/hr and increasing at rate of 4 gal/hr-squared 10b. gallons (gal/hr*hrs)

4. using trapezoids I get 6960 (be sure to multiply by 60 to b/c rate is people per minute)

5a. $\int_0^2 \left(2x - x^2\right)dx$ b. $\frac{1}{2}\left[f(0.25) + f(0.75) + f(1.25) + f(1.75)\right] = 1.375$

c. $\sum_{i=1}^{40} \frac{1}{20} f\left(\frac{i}{20}\right) = \sum_{i=1}^{40} \frac{1}{20}\left(\frac{2i}{20} - \frac{i^2}{400}\right)$ d. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{4i}{n} - \frac{4i^2}{n^2}\right)$ e. 4/3

6a. 42 b. 18.5 7a. 2π b. -4 c. 0 d. 12 e. 2π+10 f. 4.5 8a. $\int_0^3 (9 - x^2)dx$ b. $\int_0^3 \pi(9 - x^2)^2 dx$

9a. 32 b. at t=0 at 8; t=4 at 24; t=6 at 40 ; t=10 at 64 so 56/10=5.6 cm/seconds

c. when v(t)=0 so t=14 and position = 72 d. from t=0 to 14 goes 64 then 18 so 82 cm

e. t such that $0.5(t)(2t) = 10$ so $t = \sqrt{10}$ OR use anti-derivs… $X(t) = t^2 + 8$ for first 4 seconds and this is 18 when $t = \sqrt{10}$

10a. 3 b. t=4 and t=6(??) c. at t=4 it is at 5 e. velocity is 7/16 and speed is 15/16 f. t=4-$\sqrt{0.75}$

11a. 6 b. 3 12a. $\frac{t^2}{4} + \frac{3t}{2} + 2t^{-1} + C$ b. $\frac{x^2}{2} + \ln x + C$ c. $\frac{4}{3}x^{3/2} + \frac{\sqrt{2}x^2}{2} + C$ d. $\frac{e^{5x}}{5} + C$

e. $\frac{(x^3+2)^{12}}{12} + C$ f. $\ln|x+3| + C$ g. $3x^{2/3} + C$ h. $\frac{6}{5}x^{5/3} + C$ i. $\frac{3}{2\cdot\sqrt[3]{2}}x^{2/3} + C$ or $\frac{3}{4}(2x)^{2/3} + C$

j. $-0.5e^{-x^2} + C$ k. $\frac{-1}{32}(8-2x)^{16} + C$ l. $\frac{1}{9}\sin^3(3x) + C$ m. $\frac{1}{2\pi}\sin(\pi x^2 + 1) + C$

n. $-\ln|\cos x| + C$ *or* $\ln|\sec x| + C$ 13. $f(x) = 10e^{0.1x} + 10$ 14. E

15. -12 (from -3 to 2 of f(x-1) moves graph right so it is the same as intergral from -4 to 1 of f(x), or -7)

Unit 2 Handout #1: Evaluating Definite Integrals with Anti-Derivatives

1. Evaluate the following:

a. $\int_{-1}^{3} (x^2 - 4x)dx$

b. $\int_{1}^{4} 3\sqrt{x}dx$

c. $\int_{2}^{8} \frac{1}{x^2} dx$

d. $\int_{1}^{5} e^2 dx$

2. Evaluate the following:

a. $\int_{1}^{2} (x^2 + \frac{3}{x})dx$

b. $\int_{0}^{\pi/2} \sin(2x)dx$

c. $\int_{1}^{4} \frac{1}{\sqrt{x}} dx$

d. $\int_0^1 e^x dx$

e. $\int_0^1 e^{-x} dx$

f. $\int_1^3 (x-1)^2 dx$

g. $\int_1^2 (x^2-3)^3 x dx$

h. $\int_1^3 x\sqrt{2x^2-2} dx$

i. $\int_0^{3.5} \frac{3dx}{\sqrt[3]{2x+1}}$

3. Evaluate the following:

a. $\int_1^3 \frac{2x^2+1}{x} dx$

b. $\int_1^3 \frac{x}{x^2+1} dx$

c. $\int_1^3 \frac{x^2+3x}{x+1}dx$

d. $\int_0^1 (2x^2-3)^2\,dx$

4. Find the areas bounded by the following sets of functions. You must find the points of intersection algebraically. Write definite integrals and then evaluate them.

a. $y = x^2 - 3x + 3$
$y = x + 8$

b. $y = \sqrt{x-2} + 3$
$y = 4$
$x = 11$

c. $y = \sin x \quad [0, 2\pi]$
$y = 0.5$

5. Answer these questions about the region above the x-axis and below the graph of $f(x) = 4x - x^2$.

a. Write an integral describing its area and evaluate it with anti-derivatives.

b. This area is now revolved around the x-axis. Write an integral describing the volume of revolution and evaluate it. Leave your answer in terms of π.

6. Given that $f(x) = \begin{cases} 2x & 0 \le x \le 4 \\ 12 - x & 4 \le x \le 6 \end{cases}$ find the following. Feel free to use geometry sometimes!

a. $\int_3^6 f(x)dx$ (split it into 2 integrals)

b. $\int_0^6 f(x)dx$

c. If $\int_0^k f(x)dx = 8$ then what is k?

d. If $\int_4^k f(x)dx = 8$ then what is k?

e. If $\int_0^k f(x)dx = 18$ then what is k?

f. What is the volume when the region bounded by the x-axis, x=6, and the graph of $f(x)$ is revolved around the x-axis?

7. The velocity of an object is given by the function $v(t) = -2|t-20| + 60$ on the interval [0,50], where $v(t)$ is measured in meters per second.

a. What is the total distance it travels?

b. When does it reach its half-way point?

8. When you revolve the semi-circle defined by the equation $y = \sqrt{1-x^2}$ around the x-axis, you get a sphere of radius one. Write an integral describing this volume and evaluate it. Verify that is matches the answer you get from the formula $V = \frac{4}{3}\pi r^3$.

9. Two friends are trying to evaluate the following integral: $\int_3^0 x^2 dx$. Ernie says it is $\frac{x^3}{3}\Big|_{x=3}^{x=0}$ which is -9. Bert says it cannot be negative since it represents the area between the graph of $y = x^2$ and the x-axis, and the graph is never negative. Who is correct?

10. Find the following:

a. $\int \frac{1}{\sqrt[3]{4-x}}dx$

b. $\int \cos x\sqrt{\sin x}dx$

c. $\int \frac{x+1}{x^2+2x+5}dx$

d. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}dy$

e. $\int x^2(1+x)^2 dx$

f. $\int \frac{x}{1+x^2}dx$

g. $\int \frac{x}{(1+x^2)^2}dx$ h* (for hard). $\int \cos^3 x dx$ i* (for ?). $\int \frac{1}{x\ln x}dx$

ANSWERS

1a. -20/3 b. 14 c. 3/8 d. $4e^2$ 2a. $\frac{7}{3}+3\ln 2$ b. 1 c. 2 d. e-1 e. $1-\frac{1}{e}$ f. 8/3 g. -15/8

2h. 32/3 i. 27/4 3a. $8+\ln 3$ b. $0.5\ln 10-0.5\ln 2=\ln\sqrt{5}$ c. $8-2\ln 2$ d. 29/5

4a. 36 b. 28/3 c. $\int_{\pi/6}^{5\pi/6}(\sin x-0.5)dx = \sqrt{3}-\frac{\pi}{3}$ 5a. $\int_0^4 (4x-x^2)dx = 32/3$ b. $\int_0^4 \pi(4x-x^2)^2 dx = \frac{512\pi}{15}$

6a. $\int_3^4 f(x)dx + \int_4^6 f(x)dx = 7+14=21$ b. 30 c. $k^2-0^2=8$ *so* $2\sqrt{2}$

d. $(12k-0.5k^2)-(48-8)=8$ so $x^2-24x=-96$ and completing the square, $x=12-4\sqrt{3}$

e. $\int_0^4 f(x)dx + \int_4^k f(x)dx = 18$ so $\int_4^k f(x)dx = 2 \rightarrow x = 12-2\sqrt{15}$ f. $\int_0^4 \pi(2x)^2 dx + \int_4^6 \pi(12-x)^2 dx = 184\pi$

7. a. 1700 meters b. gets to 850 when $\int_t^{50}(100-2x)dx = 850$

So $2500-(100t-t^2)=850$ and $t^2-100t+2500=850$ and $(t-50)^2=850$ so $t=50-5\sqrt{34}$

8. $V(x)=\int_{-1}^{1}\pi y^2 dx = \pi x-(\pi/3)x^3\Big|_{-1}^{1} = 4\pi/3$; Cool!

9. Ernie is right; the integral is negative. You can think of adding up positive y's but the dx is negative since you are going "from 3 to 0".

10a. $\frac{-3}{2}(4-x)^{2/3}+C$ b. $\frac{2}{3}(\sin x)^{3/2}+C$ c. $\frac{1}{2}\ln(x^2+2x+5)+C$ d. $2e^{\sqrt{y}}+C$

e. FOIL and get $\frac{x^5}{5}+\frac{x^4}{2}+\frac{x^3}{3}+C$ f. $\frac{1}{2}\ln(x^2+1)+C$ g. $\frac{-1}{2}(1+x^2)^{-1}+C$

h. substitute $\cos^2 x = 1-\sin^2 x$ *and get* $\sin x - \frac{\sin^3 x}{3}+C$ i. $\int(\ln x)^{-1}\left(\frac{dx}{x}\right)$ *and get* $\ln|\ln x|+C$

Unit 2 Handout #2: Definite Integrals and Average Values

1. Evaluate the following:

a. $\int_0^3 (2x+1)^2\,dx$

b. $\int_0^{\pi/2} \cos x\,dx$

c. $\int_1^e \frac{1}{x}\,dx$

d. $\int_1^3 \frac{dx}{5-x}$

2. For $f(x) = \begin{cases} x^2 & x < 0 \\ x^3 & x \geq 0 \end{cases}$ find $\int_{-2}^{2} f(x)\,dx$

3. Evaluate $\int_{-2}^{5} \left| x^2 - 4 \right| dx$. You may want to split it into pieces.

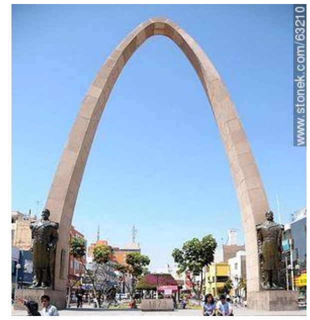
4. Find the average values of the given functions over the specified intervals.

a. $f(x) = x^2$ [0,2] b. $f(x) = \dfrac{1}{x^2}$ [1,4] c. $f(x) = |x|$ [−1,3]

5. The average value of the function $f(x) = 3x^2 + 2x + 4$ on the interval [1,*k*] is 41. What is *k*? You can do this without your calculator, but I'd understand if you decided to use it…

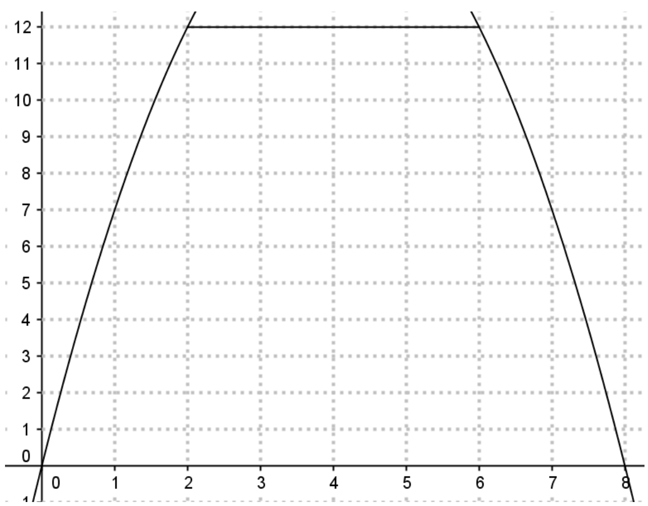
6. Instead, what if the average value of the same function on the interval [1,*k*] were *w*. What is the value of *k*, in terms of *w*?

7. The average height of a parabolic arch is what percent of the maximum height? The question implies that it is the same for all arches, so make up any one and see!



8. The graph below is part of the parabola $y = 8x - x^2$, but has the horizontal line $y = 12$ on the interval [2,6] instead of the top of the parabolic arch. [It can be written as $y = \min(12, 8x - x^2)$, where the min means to take the minimum of those two outputs for each input.]

The average value of $y = 8x - x^2$ on the interval [0,8] is $32/3$. Find the average value of the function $y = \min(12, 8x - x^2)$.



9. A helicopter starts on the ground and rises straight up into the air. Its velocity t seconds after it starts is given by the equation $v(t) = 2t + 1$. (in meters per second).

a. Write an integral describing how much it rises in the first 3 seconds. Then evaluate it.

b. Write an integral describing how much it rises in the first 8 seconds and evaluate it.

c. What is its average velocity over the first 4 seconds?

d. What is its average height over the first 4 seconds? Hint: first find the height function.

10. A train's velocity (in kilometers per minute) t minutes after it leaves the station is given by the piecewise function $V(t)=\begin{cases}0.01t^2 & t<10\\ 0.5+0.05t & t\geq 10\end{cases}$. Note that its velocity is always positive.

a. How far did the train go in its first 7 minutes?

b. How far did it go in its first 30 minutes?

c. What was its average velocity in its first 30 minutes?

11. For a continuous and differential function $f(x)$, the line $y=3x-7$ is tangent to $f(x)$ at x=2 and the line $y=-0.5x+12$ is tangent to $f(x)$ at x=6.

a. Find the average value of $f'(x)$ on the interval [2,6] or explain why it cannot be known.

b. Find the average value of $f''(x)$ on the interval [2,6] or explain why it cannot be known.

c. Find the average value of $f(x)$ on the interval [2,6] or explain why it cannot be known.

12. A population of bacteria is initially 300. At any time *t* (in hours), the instantaneous increase in the population is given by $30e^{0.5t}$.

a. Write an integral describing the total increase in population in the first four hours. Evaluate it with anti-derivatives.

b. What is the actual population of bacteria at time *t*?

c. Determine the average population in the first 4 hours.

13. An object moves back and forth along a track. Its initial position is $s(0)=5$ and its initial velocity is -4. Its acceleration on the interval [0,8] is given by the function $s''(t)=5-2t$.

a. Find its position function $s(t)$.

b. What is its average velocity on the interval [0,6]?

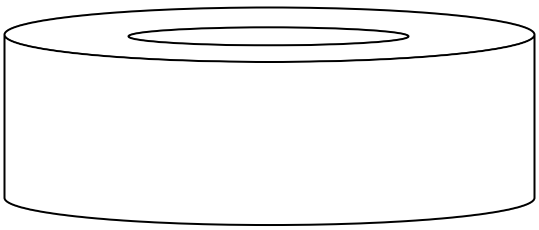
c. What are its position, velocity, and acceleration at t=3?

d. When is its velocity the highest?

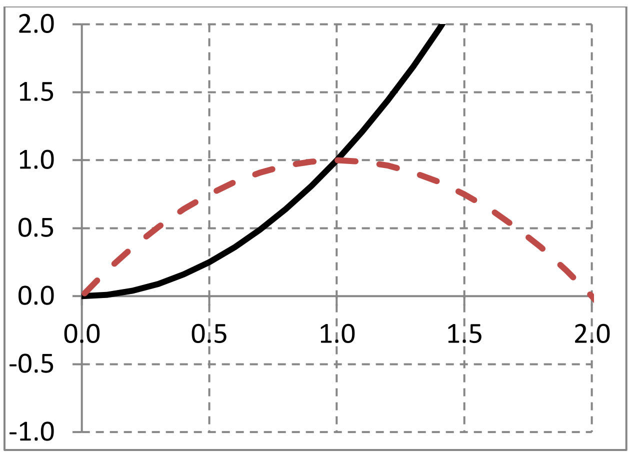
e. What was the farthest right and left it got, and at what times did it reach these points?

f. How far did it travel in total over the 8 seconds?

14. What is the volume of the object below? It is a cylinder with radius 10 and height 7 with a cylindrical hole of radius 5 drilled through the center. It is called a washer. We will use them in finding volumes of revolution sometimes, as disks do not always apply.



15. The graph below shows $f(x)$ (solid) and $g(x)$ (dashed). The region between the two curves on the interval [0,1] is called *K*. Answer the following three questions using this graph:



15-1. When *K* is revolved around the *x*-axis, a solid is created. Slicing the solid perpendicular to the *x*-axis, each cross-section is a washer. The volume of this solid is given by which integral below?

a. $\int_0^1 \pi\left[f(x)-g(x)\right]dx$

b. $\int_0^1 \pi\left[f(x)-g(x)\right]^2 dx$

c. $\int_0^1 \pi\left[\left(f(x)\right)^2-\left(g(x)\right)^2\right]dx$

d. $\int_0^1 \pi\left[\left(g(x)\right)^2-\left(f(x)\right)^2\right]dx$

e. $\int_0^1 \pi\left[g(x)-f(x)\right]^2 dx$

15-2. When *K* is revolved around the line *y*=-1, a solid is created. Slicing the solid perpendicular to the *x*-axis, each cross-section is a washer. The volume of this solid is given by which integral below?

a. $\int_0^1 \pi\left(\left[g(x)-f(x)\right]^2+1\right)dx$

b. $\int_0^1 \pi\left[\left(g(x)\right)^2-\left(f(x)\right)^2+1\right]dx$

c. $\int_0^1 \pi\left[g(x)-f(x)+1\right]^2 dx$

d. $\int_0^1 \pi\left[\left(g(x)-1\right)^2-\left(f(x)-1\right)^2\right]dx$

e. $\int_0^1 \pi\left[\left(g(x)+1\right)^2-\left(f(x)+1\right)^2\right]dx$

15-3. Write an integral showing the volume when *K* is revolved around the line *y*=2.

16. Answer the following questions about the region in the first quadrant bounded by the curve $y = (x-3)^2 + 5$ and the lines $y = 1,\ x = 2,\ and\ x = 6.$

a. What is its area?

b. What is the volume of the solid that results from this region being revolved around the *x*-axis? Write an integral, but you do not have to evaluate it.

c. What is the volume of the solid that results from this region being revolved around the line *y*=1? Write an integral, but you do not have to evaluate it.

d. What is the volume of the solid that results from this region being revolved around the line *y*=-1? Write an integral, but you do not have to evaluate it.

17. The function $N(t)$ shows the number of people at South Beach t hours after 9 am. It is twice differentiable, meaning its first and second derivatives are defined everywhere in its domain. The table below shows selected values of $N(t)$

t	0	2	3	5	7	8
$N(t)$	20	100	300	700	450	200

a. What is a reasonable approximation for $N'(4)$ and what does it mean in the context of the problem?

b. What is the meaning of $\frac{1}{8}\int_0^8 N(t)dt$? Give an approximate value.

c. The Mean Value Theorem says that at least one time between t=2 and t=7 what must occur?

d. Do you think $N''(2)$ is positive or negative? Justify your answer.

e. What are the units of $\int N'(t)dt$? What is the value and meaning of $\int_0^5 N'(t)dt$?

18. A cup of hot coffee is initially 180° Fahrenheit and is in a room where the temperature is 70°. After five minutes, the temperature is 160°.

a. Given that Newton's Law of Cooling says the temperature of an object over time is given by the equation $y = ae^{bx} + k$, find *a*, *b*, and *k* that describe this situation (hint: find *k* first, then *a*, then *b*).

b. Exactly when does the coffee's temperature hit 85 degrees?

c. What is the average temperature of the coffee in the first 30 minutes?

d. The secant line to the graph of the temperature function on the interval [*w*,*w*+10] has a slope of -1. What does this mean in the context of the problem?

19. Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2}$. This answer may seem a bit surprising, why? (Hint: what is the range of the function $f(x) = \frac{1}{x^2}$?) What, if anything, can we learn from this example? A graph may help!

20. Show that for $y = \sin^{-1}(x)$ the derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. Hint: rewrite as $x = \sin y$, use implicit differentiation, and then write $\frac{dy}{dx}$ terms of *x*.

21. Using a similar technique to the one you used in the prior problem, find the derivative of $f(x) = \tan^{-1}(x)$. What is the derivative of $f(x) = \tan^{-1}(u)$, where u is some function of x?

22. Three friends are trying to find $\int 2\sin x \cos x dx$. Here's how each approaches it:

-Larry thinks of this as $2\int (\sin x)^1 \cos x dx$ and, using $u = \sin x$ and $du = \cos x dx$, he gets $\sin^2 x + C$.

-Curly thinks of this as $2\int (\cos x)^1 \sin x dx$. Using $u = \cos x$ and $du = -\sin x dx$, he gets $-\cos^2 x + C$

-Moe knows that $\sin(2x) = 2\sin x \cos x$ so he writes it as $\int \sin(2x) dx$ and gets $-\frac{1}{2}\cos(2x) + C$.

As they discuss their answers, each is convinced that he is right and his friends are wrong. Help them resolve their dispute. Who is correct?

23. Find $\int \sin^2 x dx$ by using the identity $\cos(2x) = 1 - 2\sin^2 x$.

24. A circle has a radius of 4. What is the average length of all possible chords of the circle? There's actually no calculus required!

25. If $f(x)$ is an even function and $\int_0^5 f(x)dx = a$ *and* $\int_0^8 f(x)dx = b$, then what is $\int_{-8}^5 f(x)dx$?

26. Given that $\int_a^b f(x)dx = 2a - 3b$, what is the value of $\int_a^b (2f(x) + 3)dx$?

27. Given that $\int_0^4 f(x)dx = 10$, what is $\int_0^2 f(2x)dx$?

28. A train moves on a track leading directly away from Boston. Its distance from Boston *t* hours after noon is given by the function $P(t)$. What is its average velocity between 2 and 4 pm?

a. $\frac{1}{2}\int_2^4 P(t)dt$ b. $\frac{1}{2}\int_2^4 P'(t)dt$ c. $\frac{P(4)-P(2)}{2}$ d. $\frac{P'(4)-P'(2)}{2}$ e. More than one a-d is correct

Answers

1a. 57 b. 1 c. 1 d. $\ln 2$ 2. 20/3 3. 113/3 4a. $\frac{1}{2}\cdot\int_0^2 x^2 dx = \frac{4}{3}$ b. $\frac{1}{3}\cdot\int_1^4 \frac{1}{x^2}dx = \frac{1}{4}$ c. 1.25

5. I get $\frac{k^3+k^2+4k-6}{k-1} = 41$ and the left side divides, giving k=-7 or 5, so answer is 5.

6. $k = -1 + \sqrt{w-5}$ by completing the square…. 7. 2/3 (pick any parabola)

8. 28/3; one way is to remove the part above the line y=12… its area is 32/3.. divide this by 8 and get 4/3 so removing the top reduces the avg value by 4/3

9a. $\int_0^3 (2t+1)dt = 12$ b. $\int_0^8 (2t+1)dt = 72$ c. $\frac{1}{4}\cdot\int_0^4 (2t+1)dt = 5$ d. hgt is t^2+t so $\frac{1}{4}\cdot\int_0^4 (t^2+t)dt = 22/3$ m

10a. $\int_0^7 0.01t^2 = 1.143km$ b. 33.33 km c. 1.11 km/min

11a. $\frac{1}{4}\int_2^6 f'(x) = \frac{f(6)-f(2)}{4} = \frac{9-(-1)}{4} = 2.5$ b. $\frac{1}{4}\int_2^6 f''(x) = \frac{f'(6)-f'(2)}{4} = \frac{-0.5-3}{4} = -\frac{7}{8}$

c. unknowable.. we only know the value of f(x) at two points.

12a. $\int_0^4 30e^{0.5t}dt = 60e^2 - 60 \; or \; 383.34$ b. $P(t) = 60e^{0.5t} + C \; and \; C = 240$ so $P(t) = 60e^{0.5t} + 240$

c. $(1/4)\cdot\int_0^4 \left[60e^{0.5t} + 240\right]dt = (0.25)(120e^2 - 120) + 240 \; or \; 30e^2 + 210 = 431.67$

13a. $s(t)=\frac{-t^3}{3}+2.5t^2-4t+5$ b. $\frac{s(6)-s(0)}{6}=-1$ c. position is 6.5, veloc=2, accel=-1

d. max $s'(t)$ $when\ s''(t)=0\ so\ t=2.5\text{sec}$ (this is a max and is higher than the end points of interval)

e. max or min of $s(t)$ is where $s'(t)=0$ or at end of interval… so check t=1, 4

$s(0)=5$ $s(1)=3.17$ $s(4)=7.67$ $s(8)=-37.67$ so most right at t=4 (7.67) and left at t=8 (-37.67)

f. left until t=1 so covered 1.83 then right until t=4 so covered 4.5 then left until t=8 and covered 45.33 → so total distance covered is 51.67

14. big circle minus little circle times height: $\left(\pi R^2-\pi r^2\right)h=525\pi$

15-1. d; it is $\int\pi\left(R^2-r^2\right)dx$ where R is the large/outer radius and r is the smaller one

15-2. e; the big radius is g+1; the small/inner radius is f+1

15-3. $\int_0^1\pi\left[(2-f(x))^2-(2-g(x))^2\right]dx$; R is (2-f) and r=(2-g) 16a. $\int_2^6((x-3)^2+4)dx=76/3$ 16b. $\int_2^6\pi(((x-3)^2+5)^2-1^2)dx$ c. $\int_2^6\pi((x-3)^2+4)^2dx$ d. $\int_2^6\pi(((x-3)^2+6)^2-2^2)dx$

17a. At 1 pm, the number of people on the beach is increasing at approximately 200 people per hour.

b. The average number of people on the beach over the 8-hour period. Using trapezoids, I get (1/8)*[(60*2)+ (200*1) + (500*2)+(575*2)+(325*1)] = 349.4

c. The number of people on the beach must be increasing at the rate of 70 people per hour

d. Positive; the function appears to be increasing at an increasing rate (so it appears to be concave up)

e. N' is ppl per hr so $\int N'(t)dt$ adds up people per hour times hrs and is people; the definite integral means the increase in people at the beach between 9 am and 2 pm and the answer is 680

18a. k=70 (long-term the temperature gets there asymptotically); y(0)=160 so a=110 and since y(5)=160 we know $160=110e^{5b}+70$ so $b=0.2\ln(90/110)\approx$-0.040

b. $85=110e^{-0.04t}+70$ so $t=-25\ln(15/110)\approx$49.8 min c. $\frac{1}{30}\int_0^{30}\left(110e^{-0.04t}+70\right)dt\approx 134°$

d. it cools by an average of 1 degree per minute over some 10-minute period:

19. -2, but the function's output is never negative! Integrals as the area under a curve depends on the curve being continuous over the interval.

21. $f'(x)=\frac{1}{x^2+1}$ *more generally* $f'(x)=\frac{1}{u^2+1}\cdot\frac{du}{dx}$

22. They are all right since $\cos(2x)=2\cos^2x-1=1-2\sin^2x$ (and it also is $\cos^2x-\sin^2x$). Cool!

23. $\frac{x}{2}-\frac{\sin(2x)}{4}+C$ 24. Think of all chords oriented in a given direction… the average of them will be the same as the average of all possible chords. The sum of them is the area of the circle and the "number" of them is the diameter, so the average is 16π/8 =2π

25. a+b 26. a-3b (which is 2*(2a-3b) + 3(b-a)) 27. 5 (same avg value but ½ the width)

28. both b and c are correct

Unit 2 Handout #3: More Integration Techniques

Major Themes for now:
1. Integrating inverse trig functions
2. Use of substitution for difficult integrals
3. Average value
4. “Rates problems”: word problems solved using integrals and derivatives
5. Area in the plane and volumes of revolution using disks and washers
6. Sketching a function from the graph of its derivative

1. Given that $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$, evaluate the following integrals:

a. $\int \frac{dx}{1+4x^2}$

b. $\int \frac{dx}{4+x^2}$

c. $\int \frac{dx}{x^2+4x+5}$ (hint: complete the square)

2. Find each of the following:

a. $\int \frac{e^x}{1+2e^x}\,dx$

b. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

c. $\int \frac{e^x}{1+e^{2x}}\,dx$

3. Integrate each of the following by making the substitution given. Your final answer should not contain any *u*'s.

a. $\int x(x-4)^5\,dx \quad u = x-4$

b. $\int x\sqrt{x-1}\,dx \quad u = x-1$

c. $\int x\sqrt{2x+1}\,dx \quad u = 2x+1$ be careful with *dx*!

d. $\int 2x^2\sqrt{x+3}\,dx \quad u = x+3$

e. $\int 3x\sqrt{2-x}\,dx \quad u = 2-x$

f. $\int \frac{x}{x+3}\,dx \quad u = x+3$ (or divide it!)

g. $\int \frac{x^2}{1+x^6}\,dx \quad u = x^3$

h. $\int \frac{x}{\sqrt{x-4}}\,dx \quad u = x-4$

i. $\int \frac{x+2}{\sqrt{x-3}}\,dx \quad u = x-3$

j. hard! $\int \sqrt{1-x^2}\,dx \quad x = \sin\theta$ (cal C preview!)

4. Use your understanding of geometry to find $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Now use that to find the area of the ellipse $x^2 + \frac{y^2}{9} = 1$ by solving for y and writing an integral. You can use this technique to find the equation for the area of the generic ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5. Write an integral that describes the area of the region in the first quadrant bounded by the curves $y = e^x$, y=1, and x=3. Call this region R. Evaluate your integral by hand (leave answer in terms of e).

6. (continuation) For each part below, find the volume of the solid that results where region R is revolved as specified.

a. Revolve R around the line y=1.

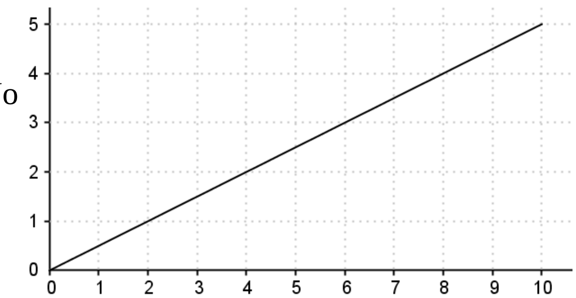
b. Revolve R around the line y=25. Write the integral but do not evaluate it.

c. Revolve R around the y-axis. Write the integral but do not evaluate it.

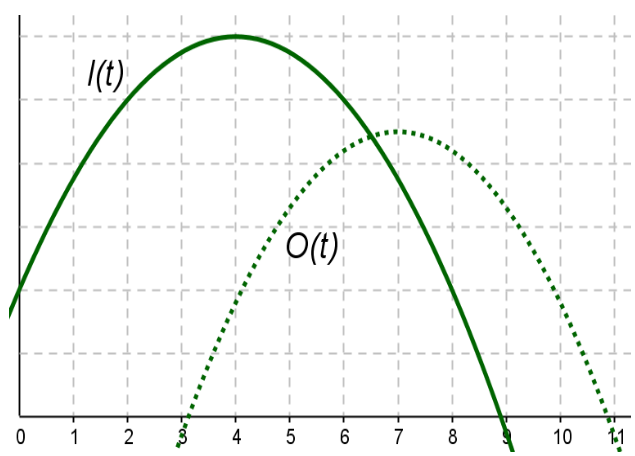
7. The graph on the top shows the rate of inflow of water into a previously empty tub on Monday. The graph on the bottom shows the rate of inflow of water into the same previously empty tub on Tuesday. The *x*-axis is minutes and the *y*-axis measures gallons per minute.

a. How does the average ***rate of inflow*** on Monday compare to the average rate of inflow on Tuesday, using the interval [0,10] for both? Explain. No computations required.

b. How does the average ***amount of water*** in the tub on the interval [0,10] on Monday compare to the average amount of water in the tub on the interval [0,10] on Tuesday? Explain. No computations required.



8. A bathtub was initially empty. Then water starting flowing in at the rate of $I(t)$ (in gallons per minute at minute *t*) and flowing out at the rate if $O(t)$. The graphs of $I(t)$ *and* $O(t)$ are below. Disregard the parts of the graphs below the *x*-axis.



a. Describe where on the graph we can see the amount of water in the tub at *t*=4.

b. When is $I'(t) = 0$? What is happening there?

c. When is the tub at its fullest? Show an area on the graph whose value is equal to the amount of water in the tub then. What is happening with $I(t)$ *and* $O(t)$?

d. When is the amount of water in the tub increasing the fastest?

9. An object moves in a straight line east away from CA. Its velocity (in meters per minute) is given by the equation $v(t) = 2 + 0.2t$, where t is minutes since noon. Its initial position is 40 meters east of CA. It may be useful to find the position function first. Note: velocity is always positive.

a. How far did it travel between 12:15 and 12:30?

b. How far east of CA was it at 12:20?

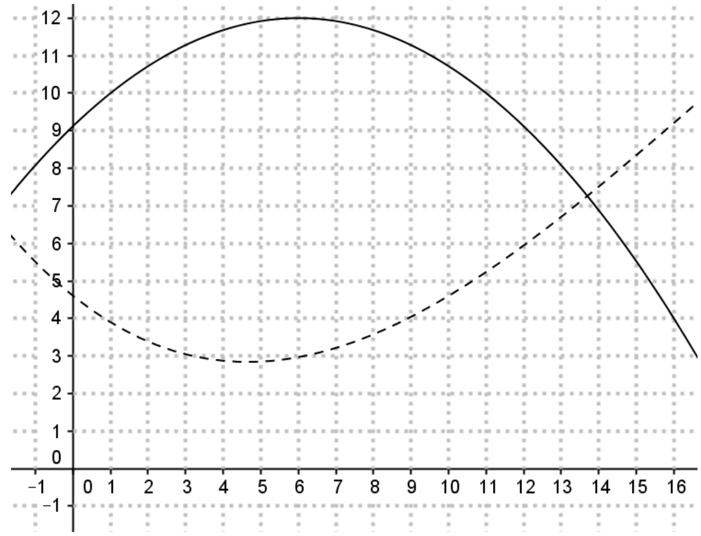
c. When will it be 500 meters away from CA (calculator OK to solve equation).

d. What was its average speed between 12:20 and 12:30?

e. On average, how far east of CA was it, between 12 and 12:30?

10. The graphs below show the rate that people enter a line at a ticket window (solid) and exit that line by being served (dashed). The t-axis is minutes and the y-axis is people per minute. The line initially has nobody in it. Disregard the parts of the graphs whose inputs are negative! Approximate the following:

a. When do people enter line the most quickly?



b. Approximate the area between the two curves on the interval [0,4]. What does this indicate?

c. Is the length of the line increasing or decreasing at t=15, and at approximately what rate?

d. Approximate the average rate that people entered the line on the interval [0,12].

e. When is the line the longest?

f. Are there more people in line at t=9 or t=16? Explain briefly.

g. Approximate the average number of people in line on the interval [0,4].

11. The instantaneous rate at which fans enter Fenway Park for a Red Sox game is given by the function $f(t) = -0.1t^2 + 6t + 210$ on the interval [0,80], where t is the number of minutes **after 6 pm** and the rate is people per minute. Assume that there were already 2,500 fans in the ballpark at 6 pm.

a. Write an integral showing how many fans entered between 6:10 pm and 6:20 pm and evaluate it.

b. What was the average rate that fans entered Fenway Park over the [0,80] interval, and what was the total number of fans entering the park during this 80-minute period?

c. Write a function $F(t)$ that shows how many fans were in the ballpark t minutes after 6 pm (assume no fans left the ballpark).

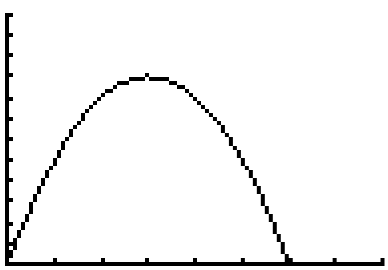
d. Exactly when did the 10,000$^{\text{th}}$ fan enter the ballpark? [You may use your calculator in any way that you like to answer this part.]

e. Use calculus to determine the time (in the interval [0,80]) at which fans were entering Fenway Park the fastest and find the fastest instantaneous rate in which fans were entering the park.

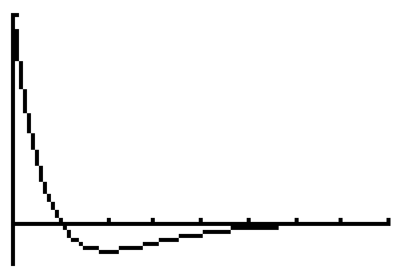
f. On average, how many fans were in the park between 6 and 7:20 pm?

12. For each graph below, sketch the graph of the derivative and the anti-derivative. For the anti-derivative, you may pick any value of C you like.

a.



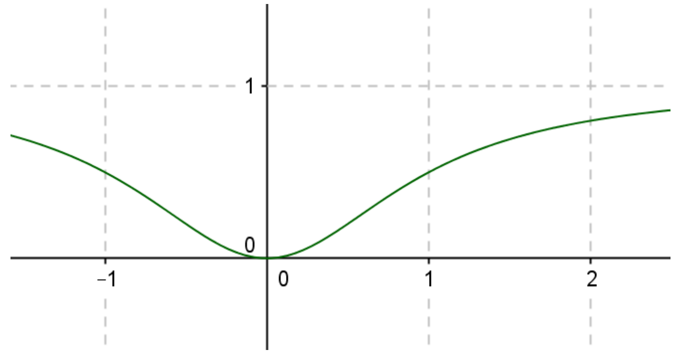
b.



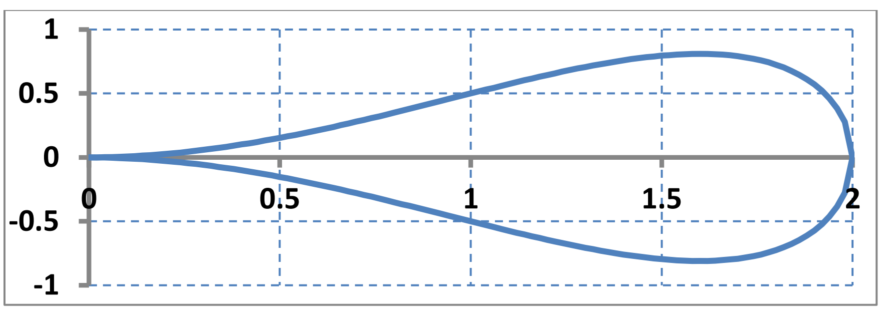
13. The graph on the right is that of $f(x)=\dfrac{x^2}{1+x^2}$.

a. Estimate $\int_0^1 f(x)dx$.

b. Compute $\int_0^1 f(x)dx$ using antiderivatives.



14. (Adapted from Larson's <u>Calculus</u> book) The cross-section of part of jet aircraft's wing is drawn below. The equations of the curves (in meters) are $y=\pm 0.5x^2\sqrt{2-x}$. What is the area of the wing's cross-section?



Answers

1a. $0.5\tan^{-1}(2x)+C$ b. $0.5\tan^{-1}(0.5x)+C$ c. $\tan^{-1}(x+2)+C$

2a. $0.5\ln(1+2e^{x})+C$ b. $2e^{\sqrt{x}}+C$ c. $\tan^{-1}(e^{x})+C$

3a. $\frac{(x-4)^7}{7}+\frac{2}{3}(x-4)^6+C$ b. $\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C$ c. $\frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C$

d. $\frac{4}{7}(x+3)^{7/2}-\frac{24}{5}(x+3)^{5/2}+12(x+3)^{3/2}+C$ e. $\frac{6}{5}(2-x)^{5/2}-4(2-x)^{3/2}+C$

f. $x+3-3\ln(x+3)+C$ g. $\frac{1}{3}\tan^{-1}(x^3)+C$ h. $\frac{2}{3}(x-4)^{3/2}+8(x-4)^{1/2}+C$

i. $\frac{2}{3}(x-3)^{1.5}+10\sqrt{x-3}+C$

j. $\int\sqrt{1-\sin^2\theta}\cos\theta d\theta=\int\cos^2\theta d\theta=\int\frac{1+\cos(2\theta)}{2}d\theta=\frac{\theta}{2}+\frac{\sin(2\theta)}{4}+C$

in terms of x: $\frac{\sin^{-1}x}{2}+\frac{2\sin(\theta)\cos\theta}{4}+C=\frac{\sin^{-1}x}{2}+\frac{2x\sqrt{1-x^2}}{4}+C$ …Phew! It checks with derivs!

4. integral is $\frac{\pi}{2}$ (semi-circle)…. Solving for y we get $y=\pm3\sqrt{1-x^2}$

Area of ellipse: top is $\int_{-1}^{1}3\sqrt{1-x^2}\,dx=1.5\pi$ so total is 3π; Area of ellipse is πab, so circle is a=b=r!

5. $\int_0^3(e^x-1)dx=e^3-4$ 6a. $\int_0^3\pi(e^x-1)^2dx=\pi\left(0.5e^6-2e^3+4.5\right)$ b. $\int_0^3\pi\left[24^2-(25-e^x)^2\right]dx$

6c. $\int_1^{e^3}\pi\left[3^2-(\ln y)^2\right]dy$

7a. both are the same since the triangle has the same area in both orientations → thus the total and avg inflow over the 10 minutes are the same on both days b. The average amount of water in the tub on Monday is much greater than the average amount on Tuesday since more water came in earlier. The tub is more than half full for much of Monday but not for Tuesday.

8a. area under I(t) from 0 to 4 minus area under O(t) from 0 to 4
b. about 4—fastest rate of inflow (gross)
c. at t=6 or so… outflow rate starts surpassing inflow rate; max water when they are equal
d. when I(t) is most above O(t) so about t=3

9. position function is $D(t)=2t+0.1t^2+40$

a. $\int_{15}^{30}(2+0.2t)dt$ *or* $D(30)-D(15)=97.5m$ b. $D(20)=120$ c. $D(t)=500$ so t=58.56 min

d. $\frac{1}{10}\int_{20}^{30}(2+0.2t)dt=\frac{D(30)-D(20)}{10}=7$ m/s e. $\frac{1}{30}\int_0^{30}(2t+0.1t^2+40)dt=100$ meters east

10a. t=6 b. 28; the increase in the length of the line from t=0 to t=4, which is the number of people in line at t=4 c. decreasing by about 2.5 ppl/min

d. using 2 trapezoids [0,3] & [3,6] and symmetry get about 130 people so ≈10.8 ppl/min e. t ≈13.6

f. 16; area between curves [9,13.6] > area between cures [13.6,16] so more net incr than net decr

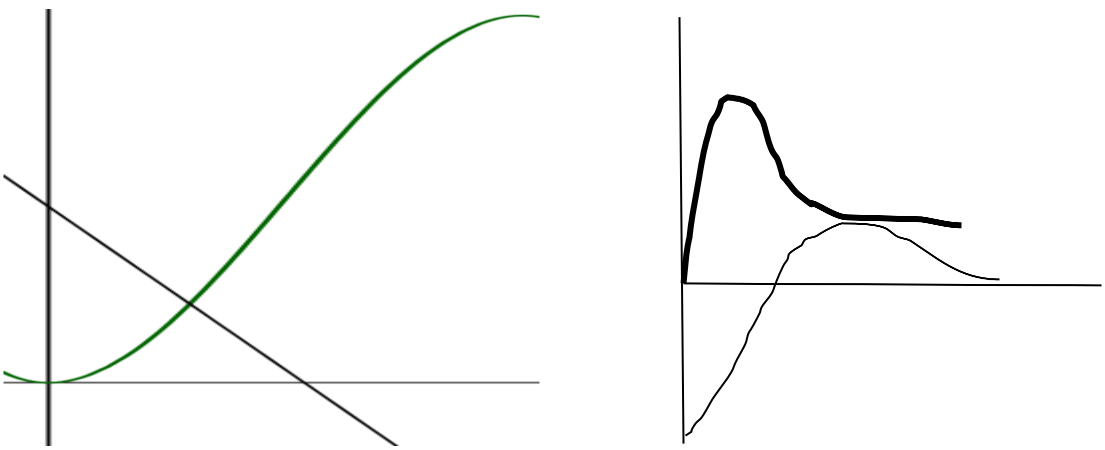
g. an approx.: at t=0 there are 0 ; at t=1 about 5; at t=2 about 12; at t=3 about 20; at t=4 about 28; so about 65/5 or 13

11a. $\int_{10}^{20}(-0.1t^2+6t+210)dt = 2767$ b. total entering is 18933 so avg rate is 236.67

c. $F(t)=\dfrac{-0.1t^3}{3}+3t^2+210t+2500$ d. $F^{-1}(10000)=28$ so at 6:28 pm

e. max $f(t)$ *so* $f'(t)=0=-0.2t+6$ so t=30 or 6:30 pm f. $\dfrac{1}{80}\int_0^{80}F(t)dt = 13033.33$

12.



13. $x-\tan^{-1}x$ from 0 to 1 is 1-π/4 or about 0.215

14. $2\int_0^2 0.5x^2\sqrt{2-x}dx = \int_0^2 x^2\sqrt{2-x}dx$ if u=2-x this is $\int_{x=0}^{x=2}(4-4u+u^2)\sqrt{u}(-du)$

$-\dfrac{2}{7}(2-x)^{7/2}+\dfrac{8}{5}(2-x)^{5/2}-\dfrac{8}{3}(2-x)^{3/2} = \dfrac{2}{7}2^{3.5}-\dfrac{8}{5}2^{2.5}+\dfrac{8}{3}2^{1.5} \approx 1.724$ square meters

Unit 2 Handout #4: More Anti-Derivatives Problems and Applications

1. Evaluate the following definite integrals:

a. $\int_{-2}^{5} (2x-1)dx$ b. $\int_{1}^{4} (\sqrt{x}-\frac{1}{\sqrt{x}})dx$ c. $\int_{0}^{2} x(3x^2-5)^2 dx$ d. $\int_{1}^{e} \frac{1}{x} dx$

2. Find the following:

a. $\int xe^{2x^2-5}dx$ b. $\int \sqrt{\sin x \tan x}dx$ (hint: put in terms of sine/cosine)

c. $\int \frac{4}{1+x^2} dx$ d. $\int \frac{4x}{1+x^2} dx$ e. $\int \frac{4x^2}{1+x^2} dx$

f. $\int \sqrt{1+3\sin x}\cos x dx$

g. $\int x\sqrt{3-x}dx$ (use a *u*-substitution)

h. $\int \frac{3x-7}{x^2+1}dx$ hint: this will have logs and an inverse tangent

i. $\int \frac{x^2-5x+7}{x^2+1}dx$

3. Find the average value of the function $f(x)=2x^2-5$ on the interval [2,5].

4. Answer the following questions about the region bounded by the graphs $y = \frac{4}{x}$ y=0 x=1 x=2

a. What is its area?

b. What is the volume of the solid created when that region is revolved around the x-axis?

c. What is the volume of the solid created when that region is revolved around the line y=-1?

5. A park is empty at 10 am. Then people enter at the rate of $4+3\sqrt{x+4}$ per minute on the interval [0,120], where *x* is the number of minutes after 10 am.

a. How many people entered the park between 10 and 11 am?

b. When did the 3000th person enter? [you may solve the equation graphically instead of algebraically]

c. What was average rate of entry into the park between 10 and 10:30 am?

d. What was the average number of people in the park between 10 and 10:30 am?

6. A particle is moving in a way that its velocity is $v(t) = \dfrac{t}{t^2+1}$ (for $t \geq 0$). Its initial position is zero.

a. How far does it travel in the first two seconds?

b. When is the position of the particle equal to 3?

c. The average velocity from t=1 to t=k is equal to 0.1. What is k? (you can be creative in your calculator usage)

d. What is $\lim_{t\to\infty} v(t)$ and what does it tell us about the particle?

e. Does the particle ever change directions? Explain.

f. When is the particle moving the fastest, and what is its maximum velocity?

7. A tank initially has 200 gallons of water. The rate at which more water is pumped into the tank (in gallons per minute) is $R(t)$, an increasing differentiable function, some values of which are given in the table below:

t	0	4	11	15	20
$R(t)$	20	25	32	37	45

a. Use right Reimann sums with four intervals to approximate $\frac{1}{20}\int_0^{20} R(t)dt$. Is your approximation too high or too low? Explain. What does your answer mean in the context of the problem?

b. What is the value of $\int_0^{20} R'(t)dt$?

c. What is the most and least possible amount of water in the tub at t=2?

d. Water happens to drain from this tank at a rate of $D(t) = 5 + 3t$ (in gallons per minute) over this same interval. What are the largest and smallest possible amounts of water that can be in the tank at t=20?

e. You want to know when the volume of water in the tank is greatest. What is the narrowest interval of time in which the maximum volume must occur?

f. Find the 6-minute period when outflow averages 30 gallons per minute.

8. At noon, there were 100 people standing on a line to buy tickets. Between noon and 3 pm, people joined the line at the rate of $j(t) = 180 - 60t$ people per hour, where t is the hours after noon. People are served and leave the line at the rate of $s(t) = 140 - 30\sqrt{t}$ people per hour from t=0 (noon) until the all people have been served. Answer the following questions. Calculator OK for solving equations.

a. What is the total number of people who bought tickets?

b. When did the last person buy a ticket?

c. At t=1, how many people were in the line? At this time, is the length of the line increasing or decreasing, and at what rate?

d. When was the line the longest, and how many people were in it at that time?

e. When was the length of the line increasing the fastest?

f. Ellie got in line at 1 pm. What time did she get her tickets?

g. Max got his ticket at 3 pm. What time did he get in line?

h. What was the average length of the line between noon and 3 pm?

i. What was the average amount of time that people spent in line? Assume the 100 people there at noon had just arrived.

j. Find a six-minute period where ten people got their tickets.

Answers

1a. 14 b. 8/3 c. 26 d. 1

2a. $0.25e^{2x^2-5}+C$ b. this is $\int \frac{\sin x}{\sqrt{\cos x}}dx=-2\sqrt{\cos x}+C$ c. $4\tan^{-1}(x)+C$ d. $2\ln(1+x^2)+C$

e. divide and get $4x-4\tan^{-1}(x)+C$ f. $\frac{2}{9}(1+3\sin x)^{3/2}+C$ g. $\frac{2}{5}(3-x)^{5/2}-2(3-x)^{3/2}+C$

h: $\int \frac{3xdx}{x^2+1}-\int \frac{7dx}{x^2+1}=1.5\ln(x^2+1)-7\tan^{-1}x+C$

i. $\int \frac{x^2+1}{x^2+1}dx+\int \frac{-5xdx}{x^2+1}+\int \frac{6dx}{x^2+1}=x-2.5\ln(x^2+1)+6\tan^{-1}x+C$

3. 21 4a. $4\ln 2-4\ln 1=4\ln 2=\ln 16$ b. $\int_1^2 \frac{16\pi dx}{x^2}=8\pi$ c. $\int_1^2 \pi[(1+(4/x))^2-(1)^2]dx=8\pi(1+\ln 2)$

5a. 1248 b. $4x+2(x+4)^{3/2}-16=3000$ (need -16 so empty initially) so x=113.9 or about 11:54

c. $\frac{1}{30}\int_0^{30}(4+3\sqrt{x+4})dx$ or 16.7 people/minute d. $\frac{1}{30}\int_0^{30}(4x+2(x+4)^{3/2}-16)dx$ or 222.9 people

6a. $\ln\sqrt{5}=0.805$ units b. $0.5\ln(t^2+1)=3$ *so* $\sqrt{e^6-1}=20.06$ seconds

c. $0.5\ln(k^2+1)-0.5\ln(2)=0.1(k-1)$ so k=32.29 d. 0; particle's velocity eventually nears 0

e. no… velocity is always>0, so only goes right f. $v'=0$: using quotient rule get t=1 and $v(1)=0.5$

7a. 34.85. Too high. The average rate of inflow is approximately 34.85 gal/min

b. 25; cumulate the changes in R gives us the total change in R from 0 to 20

c. 240-250; rate of increase from t=0 to t=2 must be between 20 and 25 gallons per minute.

d. Inflow is between and 568 and 697; outflow = $\int_0^{20} D(t)dt=700$; initial is 200, so between 68 and 197.

e.. sometime between *t*=6.67 and *t*=9 (outflow can't be over 25 until t=6.67 and must exceed 32 at t=9)

f. Outflow in 6 minutes = $\int_k^{k+6} D(t)dt=180$ so $5(k+6)+1.5(k+6)^2-5k-1.5k^2=180$

and $84+18k=180$ so *k*=96/18 → from *t*=5.33 until *t*=11.33

8a. $100+\int_0^3(180-60t)dt=370$ b. $\int_0^x(140-30\sqrt{t})dt=370$ *so* $140x-20x\sqrt{x}=370$ and x=3.63 so 3:38

c. $L(t)=100+40t+20t^{1.5}-30t^2$ and L(1)=130 ppl $L'(t)=180-60t-(140-30\sqrt{t})$ so L'(1)=10 ppl/hr

d. max when L'(t)=0 (from pos to neg) which is $180-60t=(140-30\sqrt{t})$ and t is about 1.22 so 1:13 or so and L(1.22) is about 131 people

e. increase in line = $L'(t)=180-60t-(140-30\sqrt{t})$

maximized when derive of this is zero so -60 + 15/sqrt(t)=0 and t=1/16 or 12:03:45

f. 1 pm : she is person # $100+\int_0^1(180-60t)dt=250$

when did 250th get served? $\int_0^x (140-30\sqrt{t})dt = 250 = 140x - 20x\sqrt{x}$ so x=2.276 or ≈2:16 or 2:17

g. person served $\int_0^3 140-30\sqrt{t}dt = 316$ so $100 + \int_0^x (180-60x)dx = 316$ and x≈1.658 so 1:39 or 1:40.

h. $L(t) = 100 + 40t + 20t^{1.5} - 30t^2$ so we need $\frac{1}{3}\int_0^3 L(t)dt = 111.57$

i. total people hours divided by total people…

the length of the line on the interval [0,3] is $L(t) = 100 + 40t + 20t^{1.5} - 30t^2$

the length of the line on the interval [3,3.63] is $370 - \int_0^t (140-30\sqrt{t})dt$ or $370 - 140t + 20t^{1.5}$

so total people hours is $\int_0^3 (100 + 40t + 20t^{1.5} - 30t^2)dt + \int_3^{3.63} (370 - 140t + 20t^{1.5})dt$ which is 351.56

and divide this by the 370 people to get about 0.95 hours

j. $\int_k^{k+0.1} (140-30\sqrt{x})dx = 10$ so $140x - 20x^{1.5}$ or $14 - 20(k+0.1)^{1.5} + 20k^{1.5} = 10$

and solving graphically we get 1.73 and 1.83 hrs

Unit 2 Handout #5: Practice and More Rates Problems

1. A car is moving at a velocity of 30 meters per second. The brakes are slammed, giving it constant negative acceleration of 4 meters per second squared. How far does it car travel between the time the brakes are slammed and the time it stops? No physics formulas; use differential equations instead; getting velocity and then position functions!

2. Assume that a car is going some unknown velocity when the brakes are slammed, giving it acceleration of -4 meters per second squared. If it takes 150 meters to stop, then how fast was it going?

3. Evaluate each. Remember your special triangles from trigonometry. Note: some thinking may be required! If you are stuck with anti-derivatives, try a graph.

a. $\int_0^3 \frac{x}{x^2+1}dx$

b. $\int_0^{\pi/8} (1-\cos(2x))dx$

c. $\int_0^{0.5} \sin(\pi x)dx$

d. $\int_{-4}^{4} x\cos x dx$

e. $\int_{-2}^{2} \sqrt{4-x^2}dx$

f. $\int_{-3}^{5} \left|x^2-1\right|dx$

g. $\int_1^{\sqrt{3}} \frac{2}{1+x^2}dx$

h. $\int_0^1 \frac{x^3-2x}{x^2+1}dx$

i. $\int \frac{\sqrt{x}-4}{x+1}dx$ let u= $\sqrt{x}$

(tough one!)

4*. Given that $\int_0^4 f(x)dx = 10,\ \ find \int_0^2 f(2x)dx$.

5*. The area under the graph of $f(x) = e^x$ on the interval [0,*a*] is equal to A. The area under the same graph on the interval [0,*b*] is equal to B. If B=3A, what is *b* in terms of *a*?

6*. Given that *h* is a function and $h(1) = -2,\ h'(1) = 2,\ h''(1) = 3,\ h(2) = 6,\ h'(2) = 5,\ and\ \ h''(2) = 13$.

Also suppose that h and its first two derivatives are continuous everywhere. Answer the following:

a. What is the average value of $h'(x)$ on the interval [1,2]?

b. What is the value of $\int_1^2 h''(x)dx$?

c. What is a reasonable approximation of $h(1.3)$?

*These are adapted from / inspired by Stewart's book, *Calculus: Concepts and Contexts*

7*. The temperature (in °F) t hours after 9 am one day is given by the function $f(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$

a. What is the average temperature between 9 am and 9 pm?

b. What is the average temperature between 9 am and noon?

c. Without using your calculator, when was the temperature the highest and what was the highest temperature?

8. Given that $f''(x) = \cos(2x)$ and the equation of the tangent line to $f(x)$ at x=0 is $y = 2x + 5\ldots$.

a. Find $f'(0)$ *and* $f(0)$.

b. What is $f(\pi)$? You may assume we are in radians. Find $f(x)$ first.

9. The rate of inflow of customers (measured in people per hour) into a store t hours after it opens at 10 am is given by the function $r(t)=10t^3-90t+110$ for the domain [0,5]. There were no customers in the store at 10 am.

a. When did the 800th person enter the store?

b. What was the average rate of inflow of people between noon and 3 pm?

c. When was the inflow the slowest, and what was the slowest rate? [Solve analytically]

d. Over the time period from 10 am to 1 pm, what was the average number of customers **in** the store?

10. Find the area bounded by the graphs of $y = x^2$ *and* $y = 18 - x^2$. Then find the volume when that area is revolved around the *x*-axis.

11. Evaluate each of the following:

a. $\int_1^4 \frac{2x^2 + \sqrt{x}}{x} dx$

b. $\int_0^{\pi/6} \sin^2 x \cos x dx$

c. $\int_0^2 \frac{3dx}{3+x}$

d. $\int_0^2 \frac{dx}{4+x^2}$

e. $\int_{-1}^1 \frac{6x^2 - 3x + 1}{2x+3} dx$

f. $\int_0^3 x\sqrt{x+1} dx$

12. A runner is the 1200th person to cross the starting line in a marathon. Other runners pass him at the rate of $\dfrac{300}{x+10}$ per minute, on the domain [0,180]. He passes other runners at the rate of $3+0.05x$ on the same domain. He finishes after 180 minutes.

a. What was the total number of runners who pass him over the interval [0,180]?

b. What place does he finish in? Hint: how many runners does he pass?

c. Write the function that shows the runner's place *x* minutes after the start of the race. What place was he after 60 minutes?

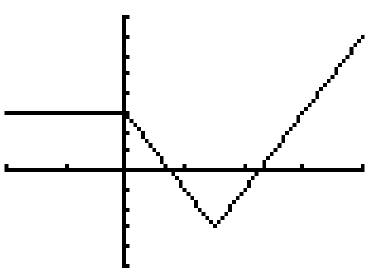
d. Find all times when his place was 1200th.

e. What was the lowest and highest place he was in over the interval [0,180], and at what times did these occur? Find analytically.

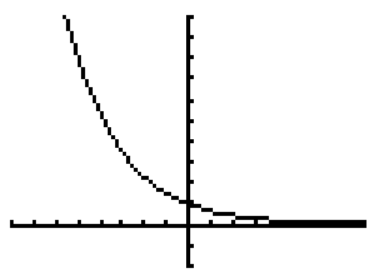
f. On average, what place was he in? (Fnint recommended… Math-9; arguments are function, then *x*, then lower limit, then upper limit)

13. Sketch the anti-derivatives of each function below. For part *d*, draw a function that is continuous everywhere, if it is possible.

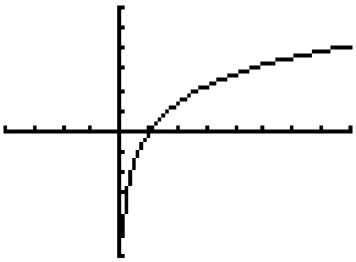
a.



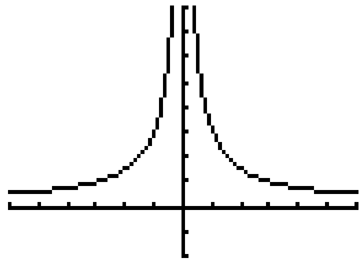
b.



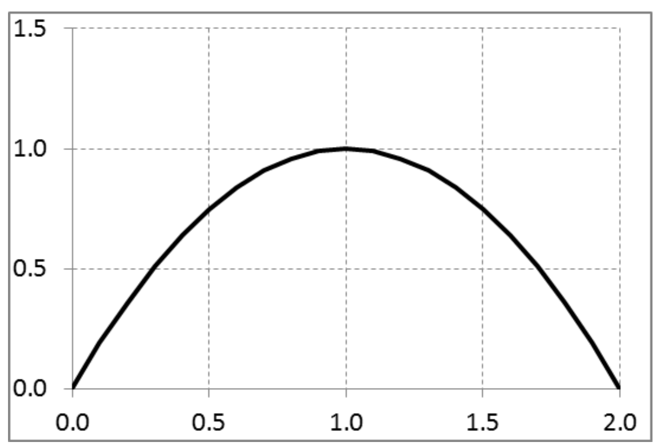
c.



d.



14. **Challenge**: Adapted from Larson's <u>Calculus</u> book: The graph below is that of $y = x(2 - x)$.



a. The line $y = cx$ splits the region bound by the x-axis and the graph of $y = x(2 - x)$ into two regions of equal area. What is the value of c? (calc-intersect OK)

b. The line $y = k$ splits the region bound by the x-axis and the graph of $y = x(2 - x)$ into two regions of equal area. What is the value of k? (calc-intersect OK)

Answers

1. s''=-4 s'=-4t+c and c=30 so s=-2t^2 + 30t +c and c=0
Stops when s'=0 so t=7.5 and s(7.5) = -225/2 + 225 = 112.5 m

2. $s'(t) = -4t + v$ *so* $s(t) = -2t^2 + vt$ $s'(t) = 0$ *when* $t = 0.25v$ so $s(0) = 150 = -\frac{v^2}{8} + \frac{v^2}{4}$ so $v = \sqrt{1200}$

3a. $0.5\ln(x^2+1)$ *so* $0.5\ln 10 - 0.5\ln 1 = 0.5\ln 10$ *or* $\ln\sqrt{10}$ b. $x - \frac{\sin(2x)}{2}$ *so* $(\pi/8 - \sqrt{2}/4)$

c. $-\cos(\pi x)/\pi$ *so* $-0 - -1/\pi = 1/\pi$ d. 0 since function is odd .. f(-x)=-f(x)

e. semi-circle so 2π f. this is x^2-1 from -3 to -1 and 1 to 5 and 1-x^2 elsewhere so 136/3

g. $2\tan^{-1}(x)$ *so* $2\tan^{-1}(\sqrt{3}) - 2\tan^{-1}(1) = 2\pi/3 - 2\pi/4 = \pi/6$

h. divide to get $0.5x^2 - 1.5\ln(x^2+1) = 0.5 - 1.5\ln(2) - (0 - 1.5\ln(1)) = 0.5 - 1.5\ln 2$

i. $\int \frac{2u^2 - 8u}{u^2+1}du = \int 2du - \int \frac{8udu}{u^2+1} - \int \frac{2du}{u^2+1} = 2u - 4\ln|u^2+1| - 2\tan^{-1}u = 2\sqrt{x} - 4\ln|x+1| - 2\tan^{-1}\sqrt{x} + C$

4. 5 5. $e^a - 1 = A$, $e^b - 1 = B$ *so* $e^b - 1 = 3(e^a - 1)$ *and* $e^b = 3e^a - 2$ *so* $b = \ln(3e^a - 2)$

6a. avg value is $\int_1^2 h'(t)dt = h(2) - h(1) = 8$ b. $h'(2) - h'(1) = 3$ c. -2+0.3(2)=-1.4 (too low…)

7a. $\frac{1}{12}\int_0^{12}(50 + 14\sin(\pi t/12))dt$ = 58.91 b. 55.22 c. sin max at sin(pi/2) so t=6 and max=64

8a. f(0)=5 (tangent line hits function at point) and f'(0)=2 (slope of tangent line is slope of function)
b. $f'(x) = 0.5\sin(2x) + C$ *and* $C = 2$ *so* $f'(x) = 2 + 0.5\sin(2x)$
$f(x) = 2x - 0.25\cos(2x) + C$ *and* $f(0) = 5$ *so* $C = 5.25$ *so* $f(\pi) = 2\pi - 0.25 + 5.25 = 2\pi + 5$

9a. ppl in store is $P(t) = 2.5t^4 - 45t^2 + 110t + 0$ so this = 800 when t=4.78 hrs after 10am
b. (P(5)-P(2))/3 = 302.5 people per hour c. $r'(t) = 0 = 30t^2 - 90$ *so* $t = \sqrt{3}$ *and* $r(\sqrt{3}) = 110 - 60\sqrt{3}$

d. $\frac{1}{3}\int_0^3 (2.5t^4 - 45t^2 + 110t)dt = 70.5$ 10a. $\int_{-3}^{3}(18 - 2x^2)dx = 72$ $\int_{-3}^{3}\pi\left[(18 - x^2)^2 - x^4\right]dx = 1296\pi$

11a. 17 b. $(1/3)\sin^3(\pi/6) = 1/24$ c. $3\ln(5) - 3\ln(3)$ *or* $\ln(125/27)$ d. $0.5\tan^{-1}(0.5x) = \pi/8$

e. divide and get $3x - 6 + \frac{19}{2x+3}$ *so* $1.5x^2 - 6x + 9.5\ln(2x+3) = 9.5\ln 5 - 12$

f. substitute u=x+1 and get $\frac{2}{5}(x+1)^{2.5} - \frac{2}{3}(x+1)^{1.5}$ so 64/5-16/3 – (2/5-2/3)=116/15

12a. $\int_0^{180}\frac{300}{x+10}dx = 300\ln(x+10) = 300\ln(190) - 300\ln(10) = 300\ln 19 = 883$

b. he passes $\int_0^{180}(3 + 0.05x)dx = 3x + 0.025x^2 = 1350$ so 1200+883-1250=733rd place

c. $P'(t) = \frac{300}{x+10} - (3 + 0.05x)$ and P(0)=1200 so $P(x) = 1200 + 300\ln(x+10) - 300\ln 10 - 3x - 0.025x^2$
and $P(60) = 1514$ d. using calc-intersect, t=0 or t=127.1

e. max and min of $P(x)$ so $P'(x)=0=\frac{300}{x+10}-3-0.05x$ or $\frac{300}{x+10}=3+0.05x$

(this makes sense as likely at max or min when rate of being passed equals rate of passing)

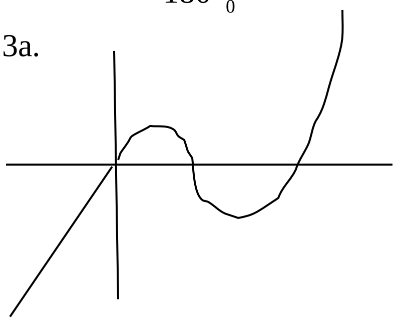
Solving using quadratic formula we get $x=46.4$ (obviously negative x makes no sense)

$P(46.4)=1526$ th place…. For abs max and min also check endpoints $P(0)=1200;\ P(180)=733$

So closest to front of runners at end and furthest from the front after 46.4 minutes

f. use fnInt: $\frac{1}{180}\int_0^{180}(1200+300\ln(x+10)-300\ln 10-3x-0.025x^2)dx=1292.4$

13a.



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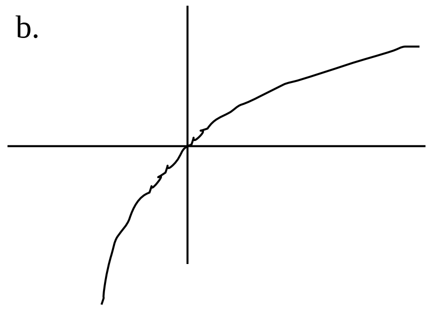
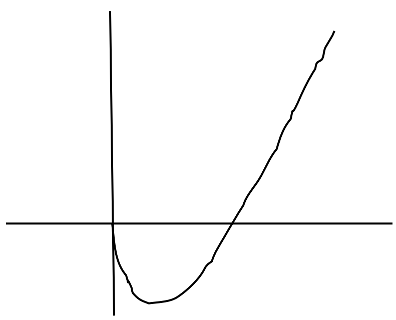
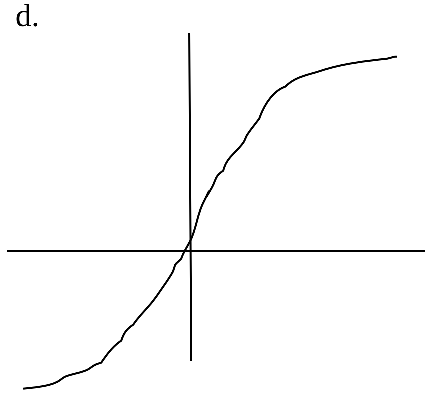


c.



d.



14. area of region is $\int_0^2(2x-x^2)dx=\frac{4}{3}$

a. for given c meet when $(x)(2-x)=cx$ so $x^2+(c-2)x=0$ *and* $x+(c-2)=0$ so x=2-c

so $\int_0^{2-c}(2x-x^2-cx)dx=\frac{2}{3}$ so $x^2-\frac{x^3}{3}-\frac{cx^2}{2}\Big|_0^{2-c}=\frac{2}{3}$ and $(2-c)^2-\frac{(2-c)^3}{3}-\frac{c(2-c)^2}{2}=\frac{2}{3}$

using calc-intersect, c≈0.413

b. easiest to use inverse functions… inverse: area from [1,2] = 2/3 and want to split this in half:

inverse of $y=x(2-x)$: write in vertex form from graph: $y=-(x-1)^2+1$

so $x=1+\sqrt{1-y}$ so $\int_k^1(x-1)dy=\frac{1}{3}$ and $\int_k^1\sqrt{1-y}dy=\frac{1}{3}$ so $\frac{-2}{3}(1-y)^{3/2}\Big|_k^1=\frac{1}{3}$

and $\frac{2}{3}(1-k)^{3/2}=\frac{1}{3}$ so $k=1-(0.5)^{2/3}\approx 0.37$ OR: use the results of a prior question where the area under of a parabolic arch is 2/3 of the rectangle it can inscribed within

Unit 2 Handout #6: Unit Review Questions

Ideas:

1. Definite integrals; use anti-derivatives and maybe some geometry (semi-circle or absolute value function)
2. Area in the plane and volumes of revolution using disks and washers
3. Rates problems → velocity or rate of inflow/outflow
4. Average value
5. Conceptual questions
6. Integration techniques: inverse tangent etc and u-substitution (including trig stuff)
7. Graphing derivatives and anti-derivatives
8. Basic differential equations

1. Evaluate $\int_3^5 \frac{2x+4}{x-1}\,dx$

2. What is the average value of $y=\sin(2x)$ on the interval $[0,\pi/6]$?

3. The region bounded by the curves $y = \frac{1}{x}$, $y = 0$, $x = 1$, *and* $x = 3$ is called R.

a. What is the area of R?

b. What is the volume of the solid created when R is revolved around the *x*-axis?

c. What is the volume of the solid created when R is revolved around the line $y = 2$?

d. What is the volume of the solid created when R is revolved around the line $x = -1$?

4. A bathtub has 20 gallons of water in it. Water is added at the rate (gallons per minute) of $A(t) = \sqrt{t+1}$ on the interval [0,8].

a. How much water is in the tub at t=8?

b. What is the average amount of water in the tub over the 8 minutes?

c. What is the average rate of inflow on the interval [0,8]?

5. Find the following.

a. $\int \frac{x+1}{x^2+1}dx$ b. $\int x \cdot \sqrt[3]{2x+5}dx$ c. $\int x(3-x)^7 dx$

6. A train leaves Boston at noon heading straight for San Francisco. The function $f(x)$ describes its velocity x hours into its trip (in miles per hour).

a. What is the meaning of $\int_1^5 f(x)dx$?

b. Under what scenarios, if any, can $\int_a^b f(x)dx$ be negative?

c. Assume that $f(x)$ is always positive (or zero) and that $\int_3^7 f(x)dx = 320$. What is the train's average velocity between 3 pm and 7pm?

7. Find the average value of the function $f(x) = x^3 + x - 2$ on the interval [3,8].

8. What is the area of the regions bounded by the curves $f(x) = x^3 - 2x^2 - 7x + 3$ and $g(x) = x + 3$?

9. The initial savings account balance is \$5000. They add to the account continuously at a rate of $a(t) = 120\sqrt{t}$ dollars per month. They also take money out of the account at a rate of $s(t) = 100 + 10t.$ Assume the account pays no interest.

a. What is the balance after 12 months?

b. When is the balance the highest, and what is this maximum balance?

c. At the time the balance is the highest, what are the rates of addition and subtraction from the account?

d. When is the value of the account growing most quickly? What is going on in the graph of the balance function (account balance vs. time) at this point?

e. When, if ever, is the account balance equal to \$10,000? (calculator OK)

f. Write an integral showing the average balance in the first 120 months. Evaluate it with fnInt.

10. An object is not moving. It then begins to accelerate at a constant rate. Five seconds later it is 100 meters from where it started.

a. If its acceleration is a, write functions for the object's velocity and position in terms of a. (You may assume that the initial position is 0).

b. Find a.

c. What is the object's velocity when it reached the 100-meter mark?

d. When is the object half-way there (to the 100-meter mark), and what is its velocity at this point?

e. What is the object's average velocity over this five-second interval?

f. What is the object's average distance/position over this five-second interval?

11. If $f''(x) = \dfrac{2}{\sqrt{x+3}}$ and the tangent line to $f(x)$ at x=1 is $y = 3x + 2$ then find $f(x)$. No calculator.

12. What is the volume created when the graph of $f(x) = \cos x$ on the interval [0,π/2] is revolved around the *x*-axis? Remember (?).. $\cos(2x) = 2\cos^2(x) - 1$.

13. Given that $\int_0^k |x-2|dx = 12$, find *k*.

14. If the carbon monoxide in the air is currently 1.8 parts per million (ppm) and is changing at the rate $f(t)$ ppm per year at time t in years, then $\int_0^5 f(t)dt$ means:

a. the level will be 5 ppm at some time t

b. the rate will be 5 ppm/year at some time t

c. the level at the end of 5 years

d. the rate at the end of 5 years

e. the increase in the level during the next 5 years

Answers

1. divide and get $\int_3^5 (2+\frac{6}{x-1})dx = 2x + 6\ln(x-1) = 4 + 6\ln 4 - 6\ln 2 = 4 + 6\ln 2$ (last step is rules of logs)

2. $\frac{1}{\pi/6}\int_0^{\pi/6} \sin(2x)dx = \frac{6}{\pi}\cdot\frac{1}{2}(-\cos 2x) = \frac{3}{\pi}[-0.5 - -1] = 1.5/\pi$ 3a. $\int_1^3 \frac{1}{x}dx = \ln 3$

b. $\int_1^3 \pi\left(\frac{1}{x}\right)^2 dx = \pi\cdot\frac{-1}{x} = \pi\cdot\frac{2}{3}$ c. $\int_1^3 \pi\left[2^2 - (2-1/x)^2\right]dx = \pi\int_1^3\left(\frac{4}{x} - \frac{1}{x^2}\right)dx = \pi\left[4\ln x + \frac{1}{x}\right] = \pi[4\ln 3 - 2/3]$

d. $\int_0^{1/3} \pi\left[4^2 - 2^2\right]dy + \int_{1/3}^1 \pi\left[\left(\frac{1}{y}+1\right)^2 - 2^2\right]dy = 4\pi + 2\pi\ln 3$

4a. $20+\int_0^8 \sqrt{t+1}dt = 20+\frac{2}{3}(t+1)^{3/2} = 20+\frac{2}{3}(27)-\frac{2}{3}(1) = 112/3$

b. $w(t) = 20+\frac{2}{3}(t+1)^{3/2}-\frac{2}{3} = \frac{58}{3}+\frac{2}{3}(t+1)^{3/2}$ so avg = $\frac{1}{8}\int_0^8\left(\frac{58}{3}+\frac{2}{3}(t+1)^{3/2}\right)dt$

$= \frac{1}{8}\left[\frac{58t}{3}+\frac{4}{15}(t+1)^{5/2}\right] = \frac{1}{8}\left[\left(\frac{464}{3}+\frac{4}{15}(243)\right)-\left(\frac{4}{15}\right)\right] = 27.4$ gallons

c. $\frac{1}{8}\int_0^8 \sqrt{t+1}dt = \frac{13}{6}$ gal/min

5a. $\int \frac{x}{x^2+1}dx+\int \frac{1}{x^2+1}dx = 0.5\ln(x^2+1)+\tan^{-1}x+C$

5b. $u = 2x+5$ *so* $x = (u-5)/2$ *and* $du = 2dx$ *so* $\int\left(\frac{u-5}{2}\right)u^{1/3}\frac{du}{2} = \frac{1}{4}\int(u-5)u^{1/3}du$

which equals $\frac{1}{4}\int(u^{4/3}-5u^{1/3})du = \frac{1}{4}\left[\frac{3}{7}u^{7/3}-\frac{15}{4}u^{4/3}\right]$ so $\frac{3}{28}(2x+5)^{7/3}-\frac{15}{16}(2x+5)^{4/3}+C$

c. $u = 3-x$ *so* $x = 3-u$ *and* $dx = -du$ so $\int(3-u)u^7(-du) = \int(u-3)u^7du = \int\left(u^8-3u^7\right)du$

which is $\frac{u^9}{9}-\frac{3u^8}{8}+C = \frac{(3-x)^9}{9}-\frac{3(3-x)^8}{8}+C$

6a. distance traveled between 1 and 5 pm b. if it goes backwards c. 80 miles per hour

7. $\frac{1}{5}\int_3^8(x^3+x-2)dx = 204.25$ 8. $\int_{-2}^0(x^3-2x^2-8x)dx+\int_0^4(-x^3+2x^2+8x)dx$ = 20/3 + 128/3 = 148/3

9a. balance function is $b(t) = 5000+80t^{1.5}-(100t+5t^2)$ so $b(12) = 6405.5$

b. $b'(t) = 0$ *when* $a(t) = s(t)$ t=123.19 and $b(t) = \$26186$; c. $a'(123.19) = \$1332 = -s'(t)$

d. maximize $a(t)-s(t)$ *so* $(a-s)'(t) = 0$ *or* t=36; max slope of b' means an inflection point

e. $b(t) = 5000+80t^{1.5}-(100t+5t^2) = 10000$ months 27.5 and 203.1

f. $\frac{1}{120}\int_0^{120}(5000+80t^{1.5}-(100t+5t^2))dt = \$17{,}065$

10a. $v(t) = at$ $p(t) = 0.5at^2$ since both are initially 0 b. $0.5a(5)^2 = 100$ so a=8 meter/sec sq

c. 40 m/s d. t=3.54 and velocity is 28.28 e. 100 meters in 5 seconds so 20 m/s f. 33.33

11. $f'(x) = 4\sqrt{x+3}+C$ and $f'(1) = 3$ so $f'(x) = 4\sqrt{x+3}-5$

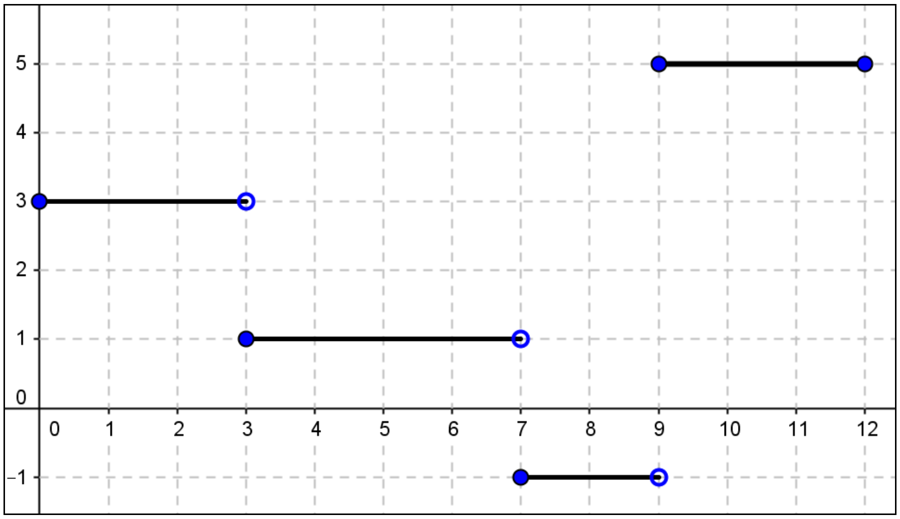
so $f(x) = (8/3)(x+3)^{3/2}-5x+C$ and $f(1) = 5$ so $f(x) = (8/3)(x+3)^{3/2}-5x-34/3$

12. $\int_0^{0.5\pi}\pi\cos^2 x dx = \pi\int_0^{0.5\pi}\frac{(1+\cos(2x))}{2}dx = \pi\left[\frac{x}{2}+\frac{\sin(2x)}{4}\right] = \pi\left[\frac{\pi}{4}\right] = 0.25\pi^2$

13. $2+2\sqrt{5}$ 14. e

Unit 3 Handout #1: Functions Defined by Integrals

1. A graph of the function $f(t)$ is shown below.



The function $F(x)$ is defined as $F(x)=\int_0^x f(t)dt$. Find the following:

a. $F(1)$

b. $F(2)$

c. $F(4)$

d. $F(12)$

e. Where is $F(x)$ increasing? Where is it decreasing?

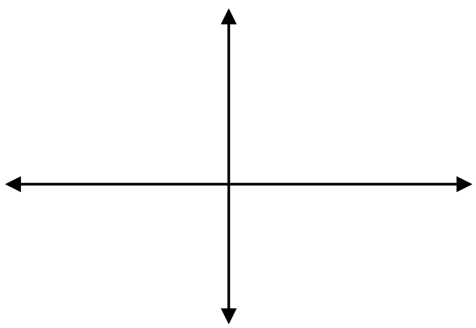
f. Are there any local maximums or minimums of $F(x)$ on the interval [0,12]? Explain?

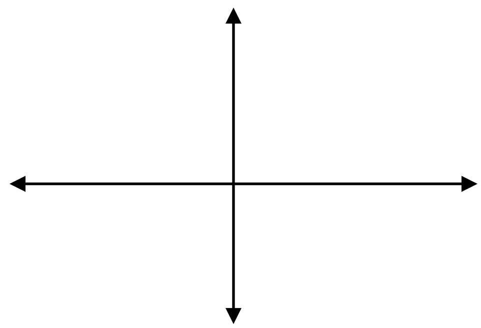
g. Sketch a graph of $F(x)$.

2. The function $F(x)$ is defined as $F(x)=\int_0^x e^{-t^2}\,dt$. We cannot write $F(x)$ without the integral since there is no chain rule satisfaction (after Cal C you'll be able to handle it with series!). But we can learn something about $F(x)$:

a. Sketch a graph of $f(x)=e^{-x^2}$ (reason it out: domain, range, end behavior, etc.—it should look familiar from statistics). Then show where on the graph of $f(x)=e^{-x^2}$ you can see the value of $F(1)$. Use your calculator to find the value of $F(1)$.

b. Sketch a graph of $F(x)$. Hint: do it for $x \geq 0$ first. And what is $F(0)$?





3. The function $b(t)$ gives the rate of people entering a concert t minutes after 7 pm one evening (in people per minute). Let the function $P(x)=50+\int_{10}^{x} b(t)\,dt$.

a. What does the function $P(x)$ represent? What are the significance of the 10 and the 50?

b. If the concert hall was empty at 7 pm, what is the value of $\int_0^{10} b(t)\,dt$?

c. Assume 70 people enter the concert hall between 7:10 and 7:20. If $P(x)$ can also be defined as $P(x)=k+\int_{20}^{x} b(t)\,dt$, then what is the value of k?

d. What is the meaning of $P'(30)$? How does it relate to $b(t)$?

4. If $f'(x)=\sin(x^2)$ and $f(5)=1$, what is $f(6.5)$? (calculator required)

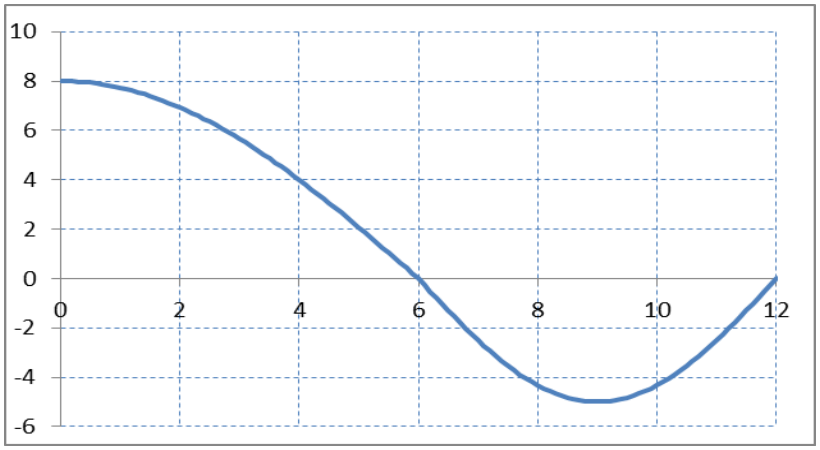
Many of the following problems (5-10) are from <u>Calculus</u>, by Finney, Demana, and Waits.

5. Find K such that $\int_{-1}^{x} f(t)dt + K = \int_{2}^{x} f(t)dt$, given that $f(x)=x^2-3x+1$. It may be helpful to think in terms of area.

6. Find K such that $\int_{0}^{x} f(t)dt + K = \int_{2}^{x} f(t)dt$, given that $f(x)=\sin^2 x$. You will probably want to use fnInt.

7. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y=\sin(kx)$ is always $2/k$.

8. Let $H(x) = \int_0^x f(t)dt$, where $f(t)$ is the function graphed below. Its domain is $0 \le t \le 12$.



a. Find $H(0)$.

b. On what interval is $H(x)$ increasing? Explain.

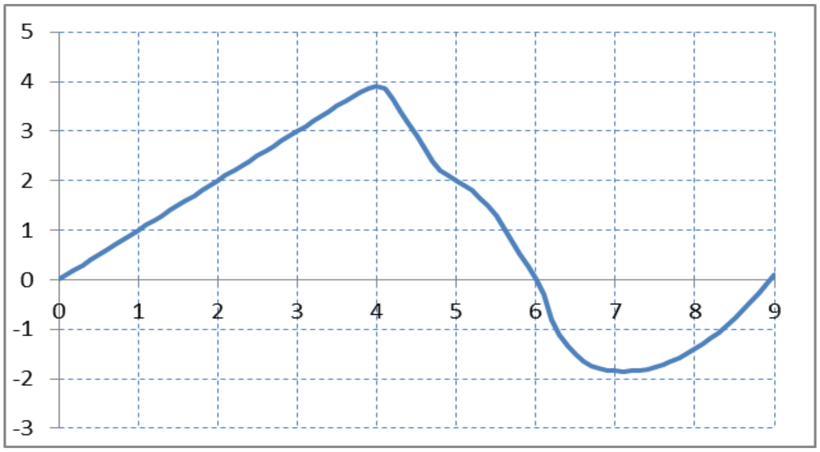
c. On what interval is the graph of $H(x)$ concave up? Explain.

d. Is $H(12)$ positive or negative? Explain.

e. At what x value is $H(x)$ maximized? Explain.

f. At what x value is $H(x)$ minimized? Explain.

9. The graph of differentiable function $f(x)$ is shown below. A particle moves along the x-axis; its position at time t (measured in seconds) is given by the function $s(t) = \int_0^t f(x)dx$.



a. What is the velocity of the particle at t=5?

b. Is the acceleration at t=5 positive or negative?

c. What is the position at t=3?

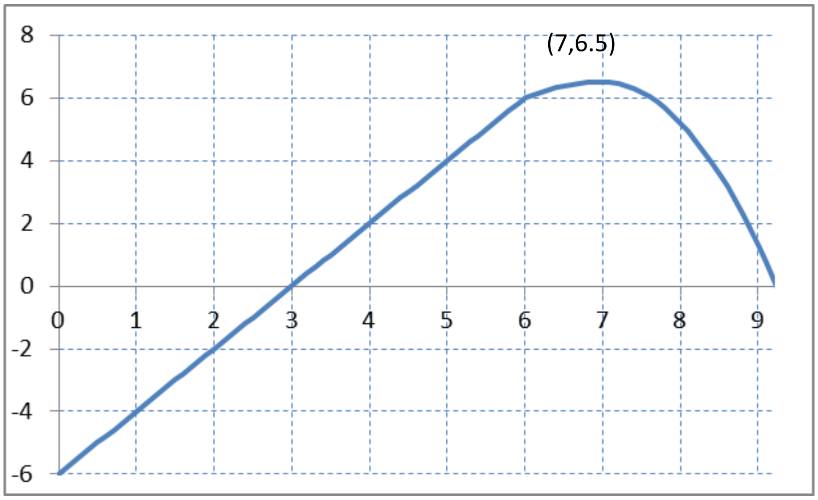
d. At what time in the first 9 seconds is $s(t)$ maximized?

e. Approximately when is the acceleration zero?

f. At what times is the particle moving toward the origin? Away from the origin?

g. On which side of the origin is the particle at t=9?

10. The graph of differentiable function $f(x)$ is shown below. A particle moves along the x-axis; its position at time t (measured in seconds) is given by the function $s(t) = \int_0^t f(x)dx$.



a. What is the particle's velocity at t=3?

b. Is the particle's acceleration positive or negative at t=3?

c. What is the particle's position at t=3?

d. When does the particle pass through the origin?

e. Approximately when is the acceleration zero?

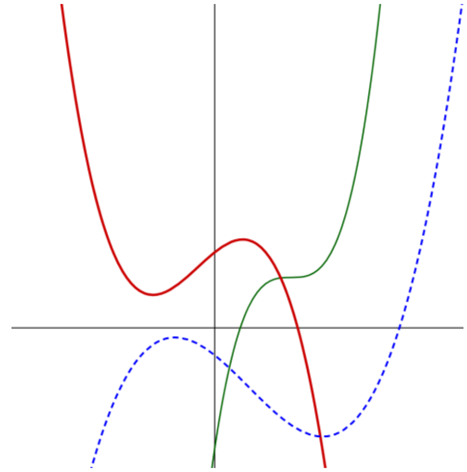
f. When is the particle moving towards the origin? Away from the origin?

g. On which side of the origin is the particle at t=9?

h. When is the particle's speed decreasing? (note: speed is always positive)

i. Sketch a graph of $s(t)$.

11. Some third-degree polynomial functions ("cubic functions") are graphed on the right. It turns out that all cubic functions are symmetric about their inflection points. Explain why. Hint: write a generic cubic as a function defined by an integral.



12. Assuming $f(x)$ has a domain of all reals and is differentiable everywhere, which of the following must be equal to $f(x)$? Explain.

a. $f(x)=\int_0^x f'(t)dt$ b. $f(x)=f(3)+\int_3^x f'(t)dt$ c. $f(x)=f(c)+\int_c^x f'(t)dt$

Answers

1a. 3 b. 6 c. 10 d. 26 e. increasing from 0 to 7 and from 9 to 12; decreasing 7 to 9
f. mins at 0 and 9 ; maxes at 7 and 12
3a. People: 50 plus the number that have entered between 7:10 and time x.
b. 50 c. 120 d. rate of change in people in hall at 7:30 → $b(30)$
4. 1.11 5. -1.5 6. -1.19
7. since period of sin(kx) is 2π/k one arch occurs halfway which is at x=π/k
Therefore area is $\int_0^{\pi/k}\sin(kx)dx$ which is $\frac{-\cos(kx)}{k}$ evaluated pi/k to zero
So $\frac{-\cos(k(\pi/k))}{k}-\frac{-\cos(k0)}{k}=\frac{1}{k}-\frac{-1}{k}=\frac{2}{k}$
8a. 0 b. (0,6) since accumulating positive values
c. concave up means 2nd deriv is positive means slope of f(x)>0 so 9<x<12
(can also think of it as decreasing at decreasing rate → and it is never increasing at incr rate)
d. looks positive since positive area from 0 to 6 is larger then negative area from 6 to 12
e. at x=6 since it stops cumulating positive values and starts including negative values (or since the first derivative of H – which is f – is zero going from positive to negative)
f. lowest at x=0 but has a relative min at x=12
9a. 2 b. negative since velocity is decreasing c. 4.5 since area of triangle with base=height=3
d. 6 e. when velocity is not changing so approximately x=4 and x=7
f. between 6 and 9 ; between 0 and 6 g. right/positive side since area (0,6)>area (6.9)
10a. zero b. positive c. -9 d. t=6 e. t=7 f. towards 3<t<6 ; away at other times
g. right side—by area under curve between 6 and 9
h. (0,3) when it is going left more slowly and (7.5,9.2) when it goes right more slowly
11. a cubic's derivative is a quadratic, so $f(x)=c+\int_h^x\left[a(x-h)^2+k\right]dx$
Or: let the cubic be $f(x)$ with inflection point at x=a. Then its derivative $f'(x)$ is a parabola with vertex at x=a. The symmetry of the parabola means that $\int_a^{a+h}f'(x)dx=\int_{a-h}^{a}f'(x)dx=-\int_a^{a-h}f'(x)dx$.
Therefore, $f(a+h)-f(a)=-1\cdot\left(f(a-h)-f(a)\right)$, describing the cubic's symmetry.
12. b and c are; *a* is only true if f(0)=0

Unit 3 Handout #2: Functions Defined by Integrals and Their Derivatives

1. The function $r(t)$ measures the rate of inflow of water into a bathtub t minutes after the tap is opened. Its units are gallons per minute, and assume that the person in the bath adjusts the inflow in some undetermined manner. Also assume that no water drains out of the tub.

a. If $f(x)=\int_0^x r(t)dt$ for $x \geq 0$ then what are the meaning and units of $f(x)$?

b. If $g(x)=\int_5^x r(t)dt$ for $x \geq 5$ then what are the meaning and units of $g(x)$?

c. True or false: the graphs of $f(x)$ and $g(x)$ are parallel. Explain.

d. If $h(x)=\int_{x-5}^x r(t)dt$ for $x \geq 5$ then what are the meaning and units of $h(x)$?

e. Which functions, if any, can be negative: $f(x),\ g(x),\ h(x), and\ r(t)$? Explain.

f. Which functions, if any, can have negative first derivatives? Explain.

g. What is $f'(12)$ and what does it mean? Answer in terms of $r(t)$.

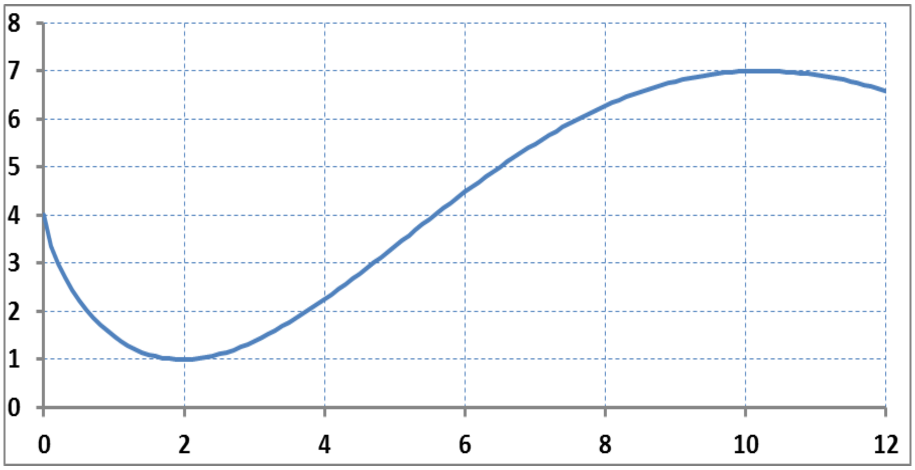
h. What is $g'(12)$ and what does it mean? Answer in terms of $r(t)$.

i. What is $h'(12)$ and what does it mean? Answer in terms of $r(t)$.

j. Is it true that $f'(x) = g'(x)$ for all $x \geq 5$. Why or why not?

k. If $k(x) = \int_x^{20} r(t)dt$ for $0 \leq x \leq 20$ then what does it represent? How does $k'(x)$ relate to $f'(x)$?

2. The graph of $f(x)$ below shows the rate of people entering a park (in people per minute) that is empty at time x=0. Let $F(x)=\int_0^x f(t)dt$. Our goal is to understand something about $F'(x)$.



a. Where on the graph can you see $F(6)$? What meaning does it have?

b. Let h be a small number (maybe ~0.5). Where on the graph can you see $F(6+h)-F(6)$? What does it represent? How about $\dfrac{F(6+h)-F(6)}{h}$?

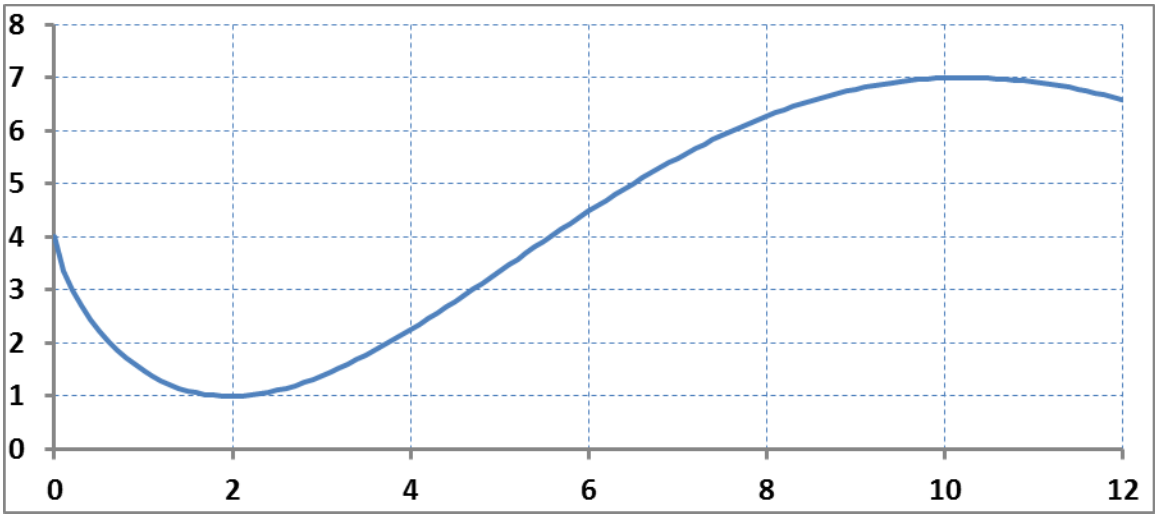
c. What is the meaning of $\lim\limits_{h\to 0}\dfrac{F(6+h)-F(6)}{h}$? What is its value? What does this tell us about $F'(6)$?

d. For $0\le x\le 12$, is $F'(x)$ ever negative? Explain.

e. Write $F'(x)$ in terms of $f(x)$.

f. Let $G(x)=\int_3^x f(t)dt$. Do similar steps to what you did above to determine $G'(6)$.

3. The graph of $f(x)$ below shows the rate of people entering a park (in people per minute) that is empty at time x=0. Let $F(x)$ be defined as $F(x)=\int_{x-4}^{x} f(t)dt$ for $x \geq 4$.



a. Where on the graph can you see $F(8)$? What does it represent?

b. Where on the graph do you see $F(8+h)-F(8)$?

c. What is $F'(8)=\lim_{h\to 0}\frac{F(8+h)-F(8)}{h}$? What does it represent?

d. Write $F'(x)$ in terms of $f(x)$.

e. Can $F'(x)$ ever be negative? Explain.

f. Let $G(x)=\int_{0}^{2x} f(t)dt$. Find $G'(4)$ and $G'(x)$ using the limit definition of the derivative (as in *c* above).

Derivatives of Functions Defined by Integrals

4. Find the derivative of each function below: You may define the function without the integral if you like (and if it is possible), but you should not ever need to do so.

a. $f(x) = 4 + \int_0^x t\cos t\,dt$

b. $f(x) = \int_0^x (t^2+5)^3\,dt$

c. $f(x) = 2x + 5 + \int_8^x \ln t\,dt$

d. $f(x) = \int_x^{11} \sin(t^2)\,dt$

e. $f(x) = 6 - \int_{-3}^5 t^3\,dt$

f. $f(x) = 7 + \int_x^0 5\sqrt{t}\,dt$

g. $f(x) = 3 + \int_0^{2x} 3t^2\,dt$

h. $f(x) = \int_{0.5}^1 \sin(\pi t)\,dt$

i. $f(x) = \int_5^{x^2} \ln(t+1)\,dt$

j. $f(x) = 11 + \int_{x^2}^9 (2t+1)^5\,dt$

k. $f(x) = \int_{\sin x}^{\cos x} t^2\,dt$

l. $f(x) = \int_{x^2}^{x^3} e^t\,dt$

5. Find the equation of the tangent line of each function below at the specified point.

a. $f(x) = 6 + \int_2^x (2t+1)dt$ at x=2

b. $f(x) = \int_0^x \sin(\pi t)dt$ at x=1

c. $f(x) = 3 + \int_x^{x^2} e^t dt$ at x=1

6. Find the second derivative of each function below:

a. $f(x) = \int_0^x t\cos t dt$

b. $f(x) = \int_x^5 (2t+1)^4 dt$

c. $f(x) = 8 + \int_x^{2x} \ln t dt$

7. The function $g(t)$ is the rate at which Juno gains weight at time *t*, in ounces per week. It is always positive. Let $W(t) = 20 + \int_0^t g(x)dx$. Answer the following questions. Give units when applicable.

a. What is the significance of the 20 in $W(t)$?

b. What is the meaning $W(30) = 800$?

c. What is the meaning of $\frac{1}{20}\int_0^{20} g(t)dt$?

d. True or false: $\frac{1}{20}\int_0^{20} g(t)dt = \frac{W(20)}{20} - 1$?

e. What is the meaning of $\frac{1}{20}\int_0^{20} W(t)dt$?

f. If $h(t) = \int_t^{100} g(x)dx$, then what does the equation $h(60) = 160$ indicate?

g. What must be true of the sign of $h'(t)$?

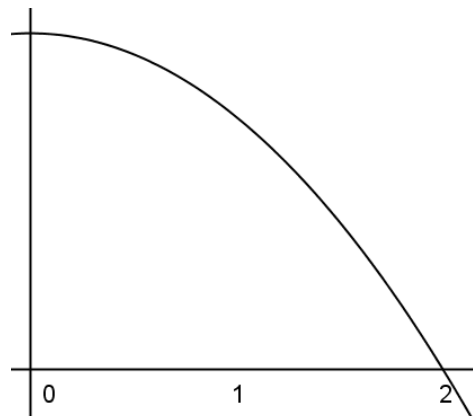
h. Let $k(t) = \int_{t-10}^{t} g(x)dx$. What does $k(100) = 12$ tell us?

i. Is it true that $k'(t) > 0$ for all *t*>10? Explain briefly.

8. If $f(x) = 100 + \int_x^{10} w(t)dt$ and $w(t)$ is always positive, then we know that $f'(x)$ is: (choose one and explain)

a. always positive
b. always negative
c. always concave up
d. always concave down
e. none of the above

9. Let $g(x) = \int_0^x f(t)dt$ for the graph of $f(x)$ as shown



a. On the interval (0,2), is the graph of $g(x)$ increasing or decreasing, or can it not be determined?

b. On the interval (0,2), is the graph of $g(x)$ concave up or concave down, or can it not be determined?

10. Given that $f(x)$ is always defined and positive and $\lim_{x\to\infty} f(x) = 0$, what can we say about $\lim_{x\to\infty} \int_1^x f(t)dt$? Must it be finite?

Answers

1a. # of gallons in the tub at time x b.# of gallons flowing into tub since 5 min after the water went on
c. true; they differ by the amount that flowed in during the first 5 minutes, which is a constant
d. number of gallons flowing into the tub in the last 5 minutes
e. none—no outflow and can't have negative water in the tub

f. only h and r; number of gallons flowed in since a given time can't be negative, but h can be because there may have been faster inflow earlier

g. r(12); rate of change in water in tub at time 12 is equal to inflow rate at time 12

h. r(12); rate of change in total water flowed in between t=5 and given time is increasing at time 12 by the inflow rate at that time

i. r(12)-r(7); rate of change in amount flowing in during last 5 minutes equals inflow at time 12 minus inflow at time 7 (since we are about to stop including that time period)

j. yes—rate of change in water in tub does not depend on when you started counting, only on flows

k. total amount of water flowing in between time x and 20 minutes; $k'(x) = -f'(x) = -r(x)$ since as time increases, there's less time of inflow you are counting…or $k(x) = -\int_{20}^{x} r(t)dt$ k'=–r(x).

2a. area under curve [0,6]; number of people in the park at 6.

b. Area between [6,6+h]; number of people entering h minutes after 6; Average rate of entry between [6,6+h] in people per minute. Average height of area [6,6+h]

c. Instantaneous rate of change of people in the park at 6; equal to $f(6) \approx 4.5$ ppl/min

d. No; number of people in park is never falling. e. $f(x)$; incr in # of ppl in park is equal to entry rate.

f. also 4.5 ppl/min since change in number of people who entered park since 3 is equal to inflow at 6

3a. people entering park from 4 to 8 b. area under curve from [8,8+h] minus area from [4,4+h]

c. $f(8) - f(4) \approx 4$; change in people who came in the last 4 minutes is rate at 8 minus rate at 4.

d. $f(x) - f(x-4)$ e. Yes; if rate of inflow now is below where it was 4 minutes ago

f. $G'(4) \approx 12.5$ and $G'(x) = 2f(2x)$ since rectangle is 2h wide and you are dividing by h

4a. $x\cos x$ b. $(x^2+5)^3$ c. $2+\ln x$ d. $-\sin(x^2)$ e. 0 f. $-5\sqrt{x}$ g. $24x^2$ h.0

i. $2x\ln(x^2+1)$ j. $-2x(2x^2+1)^5$ k. $-\sin x\cos^2 x - \cos x \sin^2 x$ l. $3x^2e^{x^3} - 2xe^{x^2}$

5a. $y-6 = 5(x-2)$ b. $y - \frac{2}{\pi} = 0$ c. $y - 3 = e(x-1)$

6a. $f'(x) = x\cos x$ *so* $f''(x) = \cos x - x\sin x$ b. $f'(x) = -(2x+1)^4$ *so* $f''(x) = -8(2x+1)^3$

c. $f'(x) = 2\ln(2x) - \ln x = \ln(4x)$ *so* $f''(x) = \frac{4}{4x} = \frac{1}{x}$

7a. The puppy's initial weight is 20 ounces. b. At t=30 weeks, the puppy weighs 800 oz (50 pounds).

c. The average rate of increase in weight over the first 20 weeks of its life (in oz/wk) d. true

e. The average weight of the puppy (in oz) over the first 20 weeks of its life.

f. From week 60 to week 100 the puppy gained 160 oz.

g. Always negative since as window gets shorter (t going from 60 towards 100) less weight is gained.

h. The puppy gained 12 ounces between weeks 90 and 100.

i. No, $k'(t)$ can be pos or neg depending on how $g(t)$ looks. If puppy gains weight quickly then slowly then quickly, then k will be a large positive, then small positive, then large positive, so k' can be <0

8. b; accumulating positives, and as x gets larger, the accumulation window gets smaller so less is accumulated

9a. increasing b. concave down (increasing at a decreasing rate)

10. It may or may not be finite: depends how quickly f approaches zero

Unit 3 Handout #3: Functions Defined by Integrals (and rates too!)

1. Use your calculator to determine $f(7)$ if $f(5)=3$ and $f(x)=x^2+\int_a^x \left(1+\sin(x^2)\right)^2 dx$

2. What is the slope of the function $f(x)=6+\int_0^{x^2} \sin(t^2)dt$ where $x=\sqrt[4]{\dfrac{\pi}{3}}$?

3. An object at rest accelerates at a constant rate. Six seconds later it is 1000 meters from where it began.
   a. What is its velocity at *t*=6?
   b. What is its average speed in these 6 seconds?
   c. When is the object 500 meters from its starting place?
   d. On average, how far is it from its starting place over this six-second interval?

4. For $t \geq 0$, a particle's velocity is given by the equation $v(t) = 3te^{-t^2/4}$. Its initial position is 7.

a. What is the particle's position at t=2?

b. What is the average velocity on the interval [0,4]?

c. What is the acceleration at t=1?

d. What is the maximum velocity and when does it occur?

e. Are there any direction changes?

f. When is the position equal to 12?

g. If $P(t)$ is the position function for the particle, then what is $\lim_{t \to \infty} P(t)$ and what does it mean?

5. What, if any, relative maximums does the function $f(x)=\int_{-2.5}^{x} t\ln(t+3)\,dt$ have on the interval [-2,10]? (Just give the *x*-values). No calculator allowed, not even for graphing!

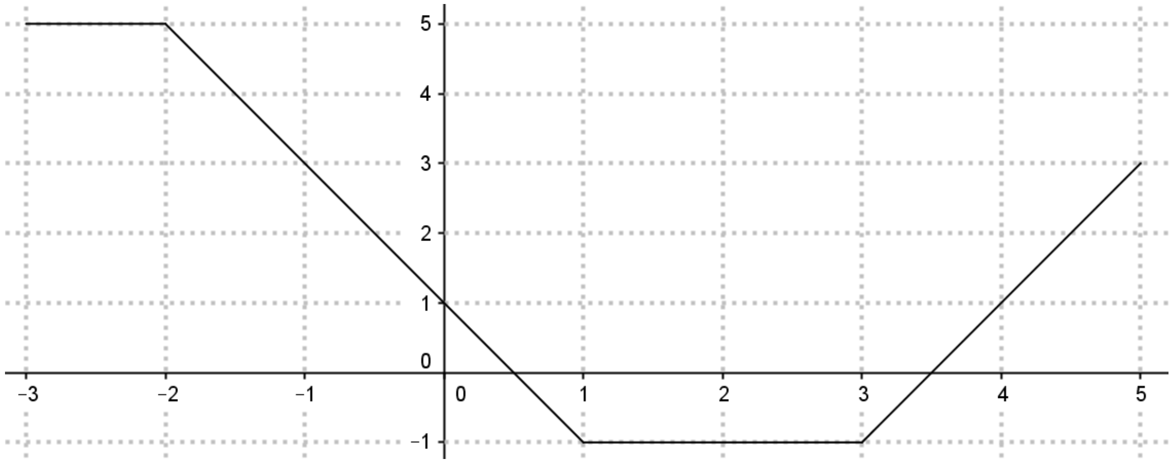
6. The function $f(x)=8+\int_{-6}^{x} te^{3t}\,dt$ has one inflection point and one relative max or min. At what *x* values do these occur, and is it a relative max or min? Again, no calculator allowed.

7. Write the equation of tangent line to each function at the given point:

a. $f(x)=x+\int_{e}^{2x} \ln t\,dt \quad at\ x=e/2$

b. $f(x)=6+\int_{x}^{3} \sqrt{t^2+7}\,dt \quad at\ x=3$

8. The graph of $g(x)$ is below. Define G and H as $G(x) = -8 + \int_{-3}^{x} g(t)\,dt$ and $H(x) = \int_{x-3}^{x} g(t)\,dt$.



a. What is $G(-1)$?

b. What is the y-intercept of $G(x)$?

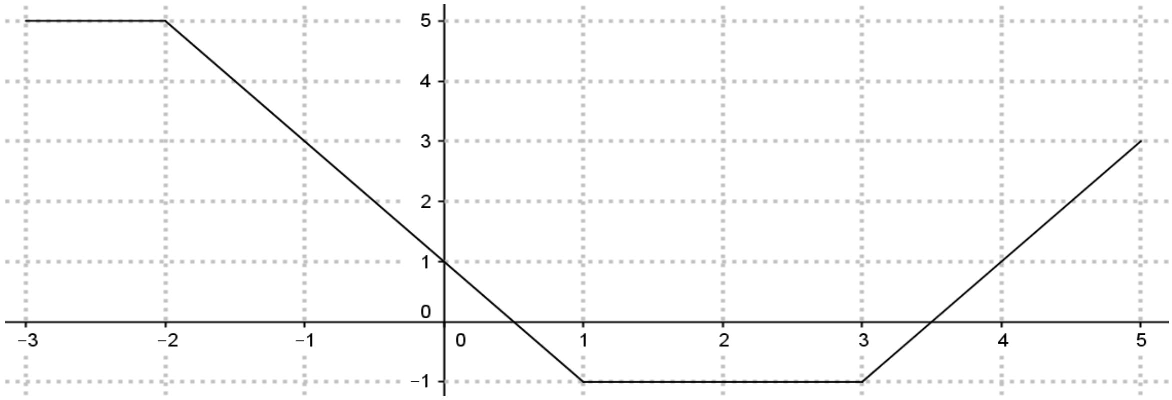
c. What is/are the zero(s) of $G(x)$?

d. Without doing any calculations, which is larger, $G(3)$ *or* $G(4)$? Why?

e. On what intervals is $G(x)$ increasing?

f. What are the coordinates of the maximum value of $G(t)$ on the interval [-3,4]?

8 (cont). The graph of $g(x)$ is below: $G(x) = -8 + \int_{-3}^{x} g(t)dt$ and $H(x) = \int_{x-3}^{x} g(t)dt$.



g. What is $H(1)$?

h. What is $H'(2)$?

i. On what intervals is $H'(x) > 0$?

j. Is $H'(x)$ ever equal to zero on this interval? If so, where?

k. Find the smallest positive zero of $H(x)$ on this interval, if there are any. Is this the only zero?

l. What are $G''(2)$ *and* $H''(2)$?

m. What are the equations of the tangent lines to the graphs of $G(x)$ *and* $H(x)$ *at* $x = 0$?

9. The rate of oil extraction from a given oil well is given by the equation $r(t) = 10te^{-0.2t}$ on the domain [0,30]. At t=30, the well runs dry. Assume t is in months and $r(t)$ is in thousands of barrels.

a. Write a function $C(x)$ describing the **total amount** of oil (cumulative) extracted from the well at time x.

b. What is the value and meaning of $C(0)$? $C(4)$? $C(20)$?

c. What is the total amount of oil that ever gets extracted? [Calculator required- until Cal C!]

d. At some time w, the following is true. What does this mean (in terms of oil)? $\int_0^w r(t)dt = 0.25\int_0^{30} r(t)dt$

e. What is the average rate of oil extraction between months 3 and 7? Your answer should be in barrels per month. This means from the start of month three through the end of month seven.

f. At what time is the oil be extracted the fastest, and what is the rate of extraction at that time? No calculator on this one!

10. The function $w(x)$ is a periodic function with period k. It is differentiable everywhere and has a domain of all real numbers. Also, the average value of $w(x)$ over one period is equal to a. Answer the following questions. Let $F(x)=\int_0^x w(t)dt$.

a. What is the value of $\int_0^k w(x)dx$?

b. What is the value of $\frac{1}{k}\int_c^{c+k} w(x)dx$? How about $\frac{1}{k}\int_0^{5k} w(x)dx$?

c. Must $F(x)$ be a periodic function? If not, under what circumstances will it be periodic?

d. What is $F(x+3k)-F(x)$?

e. How does the graph of $F(x)$ compare to that of $G(x)=\int_2^x w(t)dt$?

f. What does the graph of $H(x)=\int_{x-k}^x w(t)dt$ look like?

g. Let $K(x)=\int_{x-3}^x w(t)dt$. Must $K(x)$ be periodic?

11. For $f(x) = 6 - \int_0^x e^{-t^2}\,dt$, find the *x*-values on the interval [0,10] that maximize and minimize $f(x)$. Justify your answer.

Answers

1. 3+ $\sqrt{7^2 - 5^2 + \int_5^7 \left(1+\sin(x^2)\right)^2 dx}$ which is about 30.16 2. $\sqrt[4]{3\pi}$ *or* $\sqrt[4]{\frac{\pi}{3}} \cdot \sqrt{3}$

3a. $accel = k$ *so* $velocity = kt + C$; since initial velocity is zero, $v = kt$

Position is then $P = \frac{kt^2}{2} + C$; since initial position is zero, $P = \frac{kt^2}{2}$

Since $P = 1000$ *when* $t = 6$ $k = \frac{1000}{18} = \frac{500}{9}$ So $v = kt = \frac{500t}{9} = \frac{1000}{3}$ meters per second

b. average speed = 1000 meters in 6 seconds, so 166.67 meters per second

c. $P = \frac{kt^2}{2} = \frac{500t^2}{18}$ so P=500 when $t = \sqrt{18} = 3\sqrt{2} = 4.243$ sec d. $avg\ P = \frac{1}{6}\int_0^6 \frac{500t^2}{18}\,dt = 333.33$ m

4a. $P(t) = 13 - 6e^{-0.25t^2}$ *so* $P(2) = 13 - \frac{6}{e} = 10.79$

b. $\frac{P(4) - P(0)}{4} = 1.47$ c. $v'(t) = 3e^{-0.25t^2} - 1.5t^2 e^{-0.25t^2}$ *so* $v'(1) = 1.17$

d. max v when v'=0 so $0=3e^{-0.25t^2}-1.5t^2e^{-0.25t^2}=e^{-0.25t^2}(3-1.5t^2)$ so critical number is $t=\sqrt{2}$ since v'>0 to left of this and v'<0 to right, it must be a max. The velocity is $3\sqrt{2}e^{-0.5}=3\sqrt{\frac{2}{e}}=2.57$

e. direction changes occur when velocity changes sign. On this interval velocity is always non-negative, so there are no direction changes.

f. $P(t)=13-6e^{-0.25t^2}=12$ *so* $e^{-0.25t^2}=1/6$ *and* $-025t^2=-\ln 6$ *and* $t=\sqrt{4\ln 6}$ *or* 2.68

g. This is the eventual position of the particle, which is 13. The position asymptotically approaches it.

5. need $f'(x)=0$ where it was positive before and negative afterwards… $f'(x)=x\ln(x+3)$, which is zero when x=-2 or x=0. This first derivative is positive for x<-2, negative -2<x<0, and positive x>0. So the relative maximum occurs at x=-2.

6. $f'(x)=xe^{3x}$ *and* $f''(x)=e^{3x}(1+3x)$ so the inflection point is at x=-1/3 and when x=0 we have a critical number, which (using sign testing) must be a relative minimum

7a. y-0.5e = 3(x-0.5e) b. y-6 = -4(x-3)

8a. 1 b. 3 c. must be between -2 and -1 and some algebra yields -1.30 (exactly $0.5-\sqrt{3.25}$)

d. they are equal since $\int_3^4 g(t)dt=0$, due to symmetry e. [-3,0.5) or (3.5,5]

f. (0.5,3.25) – slightly higher than (5,2.5) g. 6 h. -4 i. when g(x)>g(x-3) so (3.5,5]

j. whenever g(x)=g(x-3) so t=3.5 (it is just a coincidence that the g's here are 0—they could be anything and H'=0 as long as the outputs are the same…

k. $4-\sqrt{3}$ using geometry: a trapezoid below curve from [w,0] equals rectangle from [1,w+3]

You can also say $\int_{x-3}^{x} g(t)dt=0$ and if x is between 2 and 3 (where one zero must be) then

$$\int_{x-3}^{x} g(t)dt=0=\int_{x-3}^{1} g(t)dt+\int_{1}^{x} g(t)dt \text{ so } \int_{x-3}^{1}(1-2t)dt+\int_{1}^{x}(-1)dt \text{ so } \left(1-1^2\right)-\left((x-3)-(x-3)^2\right)+(-x--1)=0.$$

There is another zero between 4 and 5, at $x=3+\sqrt{3}$

l. 0 and 2, respectively

m. G(x): G(0)=3 and G'(0)=1 so y=x+3 H(0)=11 and H'(0)=-4 so y=-4x+11

9a. $C(x)=\int_0^x 10te^{-0.2t}dt$ b. total oil extracted at diff times: fnInt says $C(0)=0; C(4)=47.8; C(20)=227.1$

c. C(30)=245.66 d. A quarter of the total oil has been extracted by time w

e. 2.00 to 6.9999 or 2 to 7 → 17.33 f. when $r'(t)=0$ so t=5 and $r(5)=18.39$

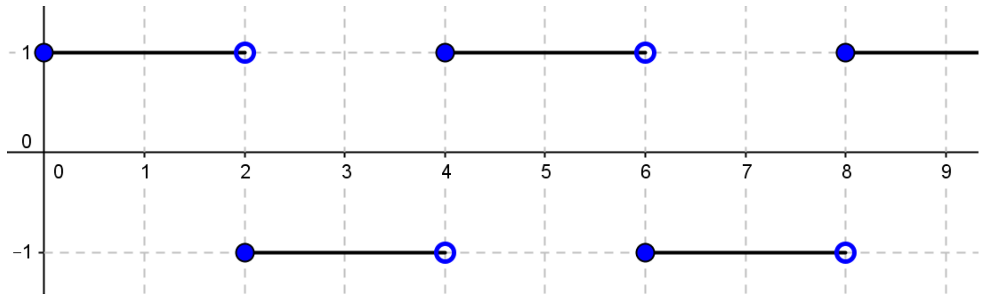
10a. *ak* b. *a* and 5*a* c. only be periodic if *a* is equal to zero d. 3ak e. parallel—shift by $\int_2^x w(t)dt$

f. the horizontal line y=ak g. yes; at any point in the cycle of w, K cumulates the last 3 units of w which will be the same.

11. Since e^{-x^2} is always positive, the integral is always positive and growing. But it is being subtracted, so the maximum value if f occurs when x=0 and the minimum when x=10.

Unit 3 Handout #4: Functions Defined by Integrals (with some rates…)

1. The graph of part of the function $f(x)$ is shown on the right. Its domain is $x \geq 0$ and it continues infinitely to the right, with a period of 4.



a. If $H(x) = \int_0^x f(t)dt$ *for* $x \geq 0$ then sketch a graph of $H(x)$.

b. If $K(x) = \int_2^x f(t)dt$ *for* $x \geq 0$ then how does the graph of $K(x)$ compare to that of $H(x)$?

c. If $G(x) = \int_{x-3}^x f(t)dt$ *for* $x \geq 3$ then find the smallest positive two zeros of $G(x)$ as well as $G(4.8)$.

d. Sketch a graph of $W(x) = \int_{x-2}^x f(t)dt$ *for* $x \geq 2$.

2. For the function $F(x) = \int_0^x \frac{t-3}{t^2+7}\,dt$ answer the following questions without using your calculator. (this is question 61on page 408 of the Anton textbook).

a. Where does $F(x)$ attain its minimum value?

b. Over what intervals is $F(x)$ increasing? Decreasing?

c. Over what intervals is $F(x)$ concave up? Concave down?

3a. If $f'(x) = 5\cos^2(x^2)$ and $f(3) = 7$ then find $f(9)$

b. For some constant a, $f(x) = x^2 + \int_a^x \ln(t^2)\,dt$. If $f(4) = 20$ find $f(2)$

4. A jet takes off from an airport by accelerating at a constant rate from a stopped position. Given that its acceleration is 1.6 meters per second squared and it can take off when its velocity is 90 m/s, how long must the runway be? (No physics formulas!)

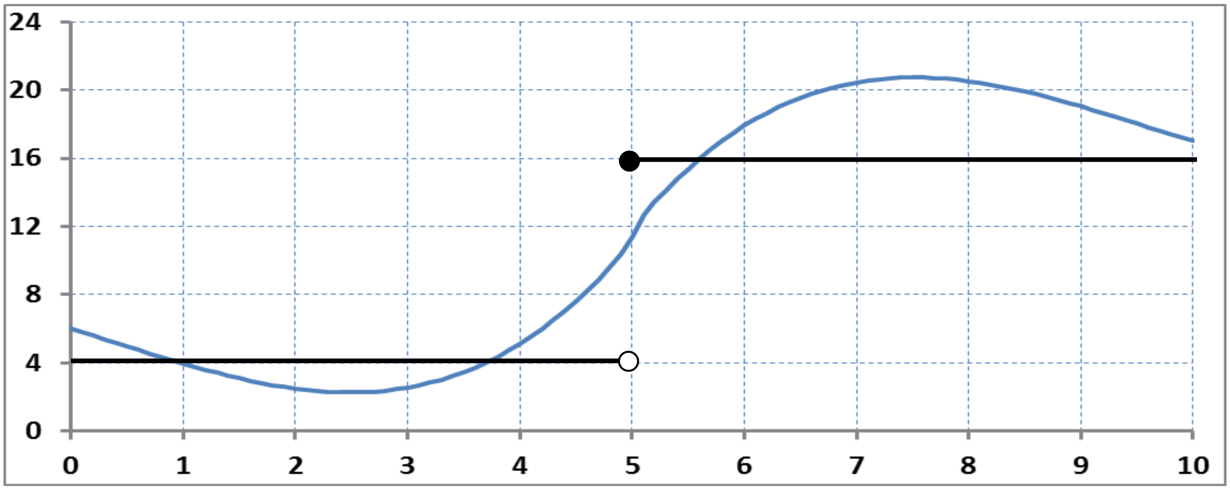
5. A sprinter racing 100 meters explodes out of the starting block with an acceleration of 4 meters per second squared for the first two seconds. She then runs the remainder of the race at a constant velocity.

a. What is her time for the race?

b. How much faster did she run the second half of the race (compared to the first half?)

c. Would you believe that the average speed of an Olympic sprinter for a 200 meter sprint is faster than the same sprinter's average speed for a 100-meter sprint?

6. In the graph below, $f(x)$ is the piece-wise function and $g(x)$ is the continuous function and $h(x) = 4 + \int_0^x g(t)dt - \int_0^x f(t)dt$. Use the graph to answer the following questions. Think bathtub!



a. Is $h(1) > 6$? Explain.

b. Find all x values (approximate) where $h'(x) = 4$.

c. At approximately what x values on the interval [0,10] does $h(x)$ have local maximums? Minimums?

d. At what x value on the interval [0,10] is $h(x)$ the largest? Smallest?

e. Where on the interval [0,10] is $h'(x)$ minimized?

f. What is the approximate slope of the secant line to $h(x)$ on interval [6,7]?

g. Where are the inflection points of $h(x)$?

h. At approximately what values on [0,10] is $h(x)$ concave up?

7. The instantaneous rate of people entering an amusement park t minutes after noon one Saturday is given by the function $E(t) = -0.0018t^2 + 0.32t + 18$ where $0 \le t \le 220$. The function $E(t)$ is in people per minute. (fnInt is OK)

a. What was the total number of people that entered the park between 1pm and 2pm that day?

b. What was the average rate of people entering the park over this 220-minute interval?

c. Assume that no one had entered the park before noon and no one leaves. Write a function describing the total number of people in the park t minutes after noon.

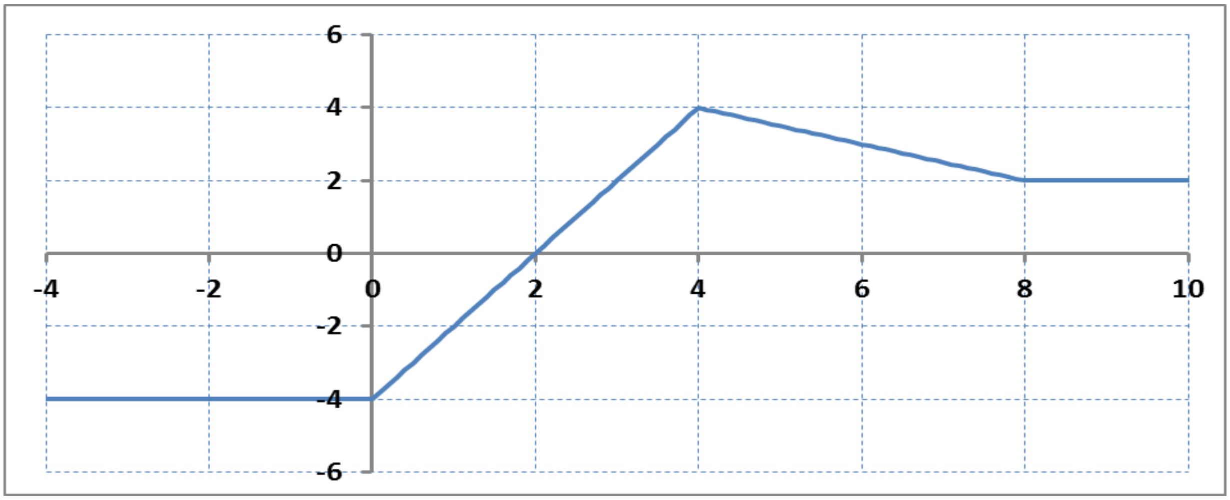
d. Exactly when did the 3000$^{\text{th}}$ person enter the park that day?

e. At what time in the interval [0,220] were people entering the park most rapidly, and what was the rate of entry at this time? You must use calculus to answer this question.

f. Assuming nobody leaves the amusement park before 3:40 pm, what was the average number of people in the park?

g. Challenge: Assume everybody left at 3:40. How long, on average, did people spend in the park? There are a few ways to approach this… one hint is to follow the units!

8. The functions $f(x)$ *and* $g(x)$ are defined as $f(x) = x^2 + \int_{-2}^{x} s(t)dt$ and $g(x) = \int_{x-3}^{x} s(t)dt$ where $s(t)$ is graphed below. The domain for $f(x)$ is $-4 \le x \le 10$ and for $g(x)$ is $-1 \le x \le 10$.



a. Find $f(-2)$ and $f(0)$. Why does there have to be a zero of $f(x)$ between *x*=-2 and *x*=0?

b. What is $f'(-1)$? What is $f''(-1)$?

c. On what interval is $f(x)$ increasing?

d. Find the maximum and minimum of $f(x)$ on the interval [-2,10].

e. What is $g'(1)$?

f. Find all zeros of $g(x)$ on the interval [-1,10].

g. What is the equation of the tangent line to $g(x)$ where *x*=3?

h. Tough: find the *x* values of the maximum and minimum of $g(x)$ on the interval [-1,10].

i. The function $h(x)$ is defined as $h(x) = ax + b + \int_{0}^{x} s(t)dt$ where *a* and *b* are constants. The coordinates of a local extreme of $h(x)$ are (7,20). Find *a* and *b* and determine if this is a max or a min.

The next two problems are adapted from Anton's Calculus book

9. A particle moves along the x-axis. At t=0, it is at the origin and has a velocity of 25 cm per second. It has 0 acceleration for the first 4 seconds, then is acted on by a force that produces a constant acceleration of -10 cm per second squared.

a. Sketch the graph of acceleration versus time on the interval $0 \le t \le 12$.

b. Sketch the graph of velocity versus time on the interval $0 \le t \le 12$.

c. Find the location of the particle at t=8 and t=12.

d. When is the particle's location the same as it is at t=4.19?

e. What is the furthest right the particle gets on the interval $0 \le t \le 12$ and when does this occur?

10. A car stopped at a toll booth accelerates at a rate of 4 feet per second squared. When it starts, it is 2500 feet behind a truck traveling at a constant speed of 50 feet per second. How long will it take for the car to catch the truck, and how much distance will it have covered since the toll booth by then?

11. A particle is moving back and forth along a linear track. Its initial position is 40 cm right of the left endpoint. Its velocity and any point in time is given by the equation $v(t) = 3t^2 - 32t + 60$. You may use your calculators however you like on these questions.

a. Write functions for the particle's position $d(t)$ (cm right of the left endpoint) and acceleration $a(t)$.

b. Where is the particle at *t*=4 and which direction is it traveling?

c. Is the particle's speed increasing or decreasing at *t*=4? Why?

d. What is the particle's average velocity between *t*=1 and *t*=3?

e. What is the particle's average velocity between *t*=0 and *t*=10? Why?

f. What is the distance that the particle traveled between *t*=4 & *t*=10? What was its average speed?

g. On the interval [0,4], on average how far right of the left endpoint is the particle?

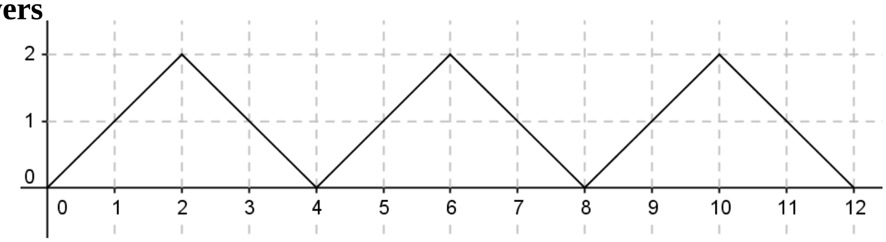
h. When is the particle exactly 30 cm right of the left endpoint?

i. When is the particle's velocity minimized and what is the velocity at that time? What is happening on the graph of $d(t)$ at this point?

j. On the interval [0,10], what is the farthest the particle gets from the left endpoint and when does it get there? What is happening on the graph of $v(t)$ at this time?

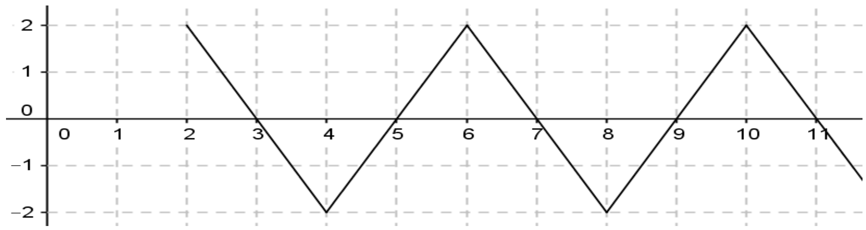
Answers

1a.



b. Parallel to H and always 2 units lower vertically c. 3.5 and 5.5 G(4.8)=-1

d.



2a. x=3 since what is being accumulated goes from negative to positive there

b. x>3 it is increasing, x<3 decreasing

c. concave up from -1 to 7; concave down elsewhere (but not at -1 or 7 exactly)

3a. $f(9) = f(3) + \int_3^9 f'(x)dx = 22.092$ b. 3.68 since $f(4) - f(2) = 16 - 4 + \int_2^4 \ln(t^2)dt = 16.32$

4. a=1.6 so v=1.6t+c=1.6t so d=0.8t^2+c=0.8t^2 so v=90 when t=56.25 and d=2531 meters

5a. $v = \begin{cases} 4t & t < 2 \\ 8 & t \geq 2 \end{cases}$ in first two seconds she runs 8 meters then runs the other 92 in 92/8 so 13.5

b. 1st half was 2 + 42/8 or 7.25 seconds; 2nd half is 50/8=6.25 so one second faster c. it is true!

6a. No, since $h(0) = 4$ and the cumulative increase from 0 to 1 is a traingular-ish region with area<2

b. $h' = 4$ when $g' - f' = 4$ which occurs at approximately 4.5, 6.7, and 8.5

c. local max at x=1 and 5; local min at x=3.7 and 5.5 d. largest at x=10; smallest around x=3.7

e. at x=5 f. about 3.5

g. h''= g'-f''=g' which is 0 at x=2.5 or 7.5 and sign of g' changes here so sign of h'' does as well

h. h''=g', which is >0 from 2.5 to 7.5 except at x=5 where h'' is undefined

7a. 1901 b. 24.16 people per minute c. $P(t) = -0.0006t^3 + 0.16t^2 + 18t$ d. 106.3 so about 1:46

e. t=88.9 and E(t)=32.22 people per minute f. 2964 people

g. $\int_0^{220} (220-t)(-0.0018t^2 + 0.32t + 18)dt = 652{,}109/60 = 10868 \; ppl \;\; hours$ divided by 5315 ppl=2.04

OR total ppl-hrs is $\int_0^{220} (-0.0006t^3 + 0.16t^2 + 18t)dt = 652109/60 = 10868 \; ppl\, hrs$ then divide by ppl…

8a. $f(-2)=4$ and $f(0)=-8$. The IVT says that continuous functions on a closed interval must have all outputs between the outputs of the two endpoints at least once on that interval and 0 is between -4 and 8.
b. $f'(-1)=-6;\ f''(-1)=2.$ c. $f'(x)=2x+s(x)$ and this is 0 when $s(x)=-2x$ which occurs at x=1 (sketch y=-2x). For x>1, $f'(x)>0$ so f is increasing (1,10].
d. The critical number is x=1 and the endpoints are x=-2 and 10. $f(1)=-10$, $f(-2)=4$ and $f(10)=108$ so (1,-10) is the min and (10,108) is the max.
e. 2 f. 3.5 is the only one g. $g(3)=$-3 and $g'(3)=6$ so $y+3=6(x-3)$
h. $g'=s(x)-s(x-3)$ where x= 6.4 or x=-1; 6.4 is max and min is anywhere from -1 to 0 inclusive
i. $h'(x)=0=a+s(x)$ so $s(x)=-a$ when x=7 so a=-2.5; $h(7)=20=-17.5+b+9.75$ so b=27.75.
this must be a max since to the left, the integral part is increasing more than the linear part is decreasing and to the right the opposite occurs (ie: $h'(x)=-2.5+s(x)$ which is >0 when x<7 and <0 when x>7 so first deriv test tells us it is a max; or 2nd deriv is $s'(7)=-0.5$ so concave down and a max)

9c. 120 and -20 d. at 8.81 seconds (distance right from 4.19 to 6.5 is same as left from 6.5 to 8.81)
e. at t=6.5 stops going right so 131.25 cm

10. $a=4$ *so* $v=4t$ *and* $d=2t^2$; truck's distance is 50t so $2t^2=50t+2500$ when t=50 and d=5000

11a. $d(t)=40+\int_0^t(3x^2-32x+60)dx=t^3-16t^2+60t+40$ $a(t)=v'(t)=6t-32$
b. $d(4)=88$ and $d'(4)=v(4)=-20<0$ so left c. $a(4)=-8$ velocity getting more negative means speed is increasing d. $\left(d(3)-d(1)\right)/2=9$ e. $0.1*(d(10)-d(0))=0;$ ends at same place as it starts
f. calculator shortcut is to sum up speeds is $\int_4^{10}\left|3t^2-32t+60\right|dt$ = 112.97
or find when it turns… when $v(t)=0$ which is about t=8.24 so travels
$|d(10)-d(8.24)|+|d(8.24)-d(4)|=32.49+80.49\approx 112.97$, so avg speed is about 18.83
g. $\frac{1}{4}\int_0^4\left(t^3-16t^2+60t+40\right)dt=90.67$ h. $t^3-16t^2+60t+40=30$ so using calc-intersect t=6.44, 9.72
i. min of $v(t)$ is when $v'(t)=0$ and $v''(t)>0$ so t=16/3 & $v(t)=$-76/3; inflection point b/c $d''(t)=0$
j. max of $d(t)=105.67$ when t is about 2.43; at this time v(t)=0 and is going from positive to negative

Unit 3 Handout #6: A Few Practice Questions on Unit 3

1. Find $f'(x)$ for each function below

a. $f(x)=\int_5^x \sin(2t)dt$

b. $f(x)=\int_{2x}^{x^2} \sqrt[3]{t^2-1}\, dt$

2. For $f(x)=\int_{-7}^{x}(t-3)e^{-t}dt$ without using a calculator, find all of the *x*-values where

a. $f(x)$ has a relative max or min (Identify which!)

b. $f(x)$ has an inflection point

3. Given that $f(x)=2x+\int_a^x \ln(t^2+1)dt$ and $f(3)=12$, find the equation of the tangent line to $f(x)$ at x=5. Note: *a* is a constant. Calculator OK.

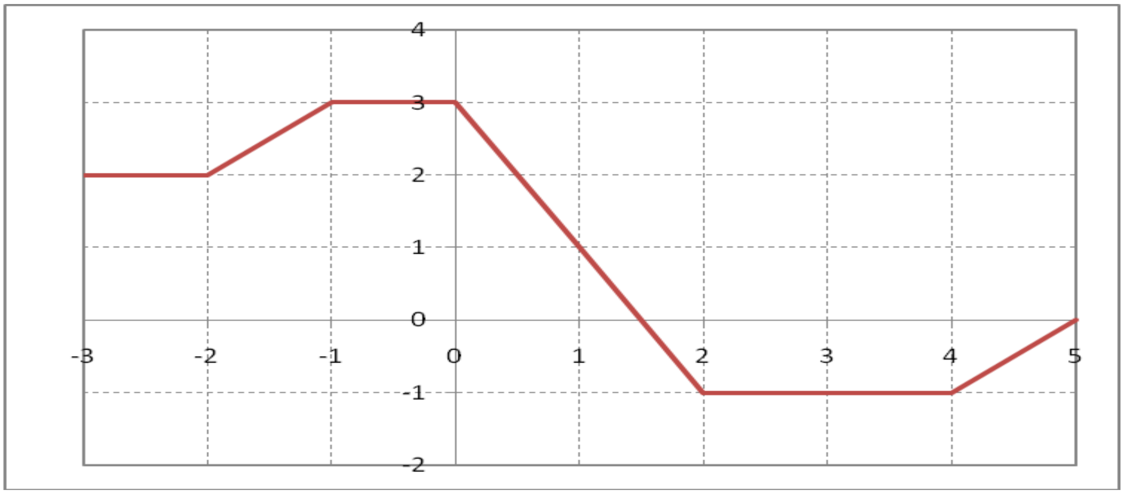
4. A car travelling at 31 meters per second (about70 mph) slams on the brakes, giving it a constant (negative) acceleration. It comes to a stop 5 seconds later. How far did it travel after hitting the brakes? No physics formulas, unless you first derive them with calculus.

5. Let $f(x)$ be defined for all $x\geq 0$ such that $f(x)>0,\ f'(x)<0,$ *and* $f''(x)>0$ for all *x* in the domain of $f(x)$. Let $g(x)=\int_a^x f(t)dt$ *and* $h(x)=\int_x^{x+3} f(t)dt$ for all $x\geq 0$.

a. What, if anything, do we know about the sign of $g(x)$? How about $g'(x)$ and $g''(x)$?

b. What, if anything, do we know about the sign of $h(x)$? How about $h'(x)$ and $h''(x)$?

6. The graph of $r(x)$ is shown below. The function $G(x)$ is defined as $\int_{-1}^{x} r(t)dt$ on the interval [-3,5] and the function $H(x)$ is defined as $\int_{x-2}^{x} r(t)dt$ on the interval [-1,5].



a. Find the equation of the line tangent to the graph of $G(x)$ at x=0.5.

b. Find $G''(1)$.

c. What is/are the coordinate<u>s</u> of the local maximum(s) of $G(x)$ on the interval [-3,5]?

d. On what interval, if any, is $G(x)$<0?

e. The maximum value of $H(x)$ occurs when x is between what two consecutive integers?

f. What is $H'(3)$?

Answers

1a. sin(2x) b. $2x \cdot \sqrt[3]{x^4 - 1} - 2 \cdot \sqrt[3]{4x^2 - 1}$

2a. at x=3 and it is a min (first derivative test) b. at x=4

3. $f(5) = f(3) + 2(5-3) + \int_3^5 \ln(t^2+1)dt = 21.63$ $f'(5) = 2 + \ln 26 \approx 5.26$ So $y - 21.63 = 5.26(x-5)$

4. accel = -6.2 m/s^2 so traveled 77.5m after hitting the brakes

5a. don’t know about the sign of g (b/c a could be something non-zero); g’>0 and g’’<0

b. h>0; h’<0, and h’’>0

6a. y-4.25 = 2(x-0.5) b. -2 c. at (1.5, 5.25) d. $-3 \le x < -1$ e. 0 and 1 f. -2

Unit 4 Handouts #1: Volumes with Disks and Washers

1. A region in the first quadrant is bounded by the graphs of x=0, y=9, and $y=(x-2)^3+8$.
   a. Write an integral to find the area when you slice it vertically. Evaluate it.
   b. Write an integral to find the area when you slice it horizontally. Use fnInt to evaluate it. (note: your integral will have y's in it, but use x's with fnInt).
   c. Find the volume when you revolve this area around the line y=9. Write the integral and evaluate it with your calculator.
   d. Find the volume when you revolve this area around the line y=-3. Calculator OK.
   e. Find the volume when you revolve this area around the y-axis. Calculator OK.
   f. Find the volume when you revolve this area around the line x=-2. Calculator OK.

2. Answer the following about the region in the first quadrant below $y = 10 - x^2$ and above $y = 3x$.

a. Find the volume when it is revolved around the x-axis (no calculator)

b. Find the volume when it is revolved around the y-axis (calculator OK). You will need two integrals.

c. Find the volume when it is revolved around the line x=5. (calculator OK)

d. Find the volume when it is revolved around the line y=11. (calculator OK)

3. Answer the questions about the region in the fourth quadrant bounded by the graphs of $y = x^2 - 4,\ x = 0,\ and\ y = 0.$

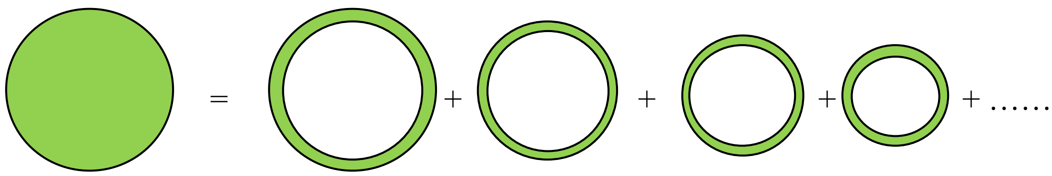
a. What is the volume of the solid created when this region is revolved around the line y=5? (calc OK)

b. What is the volume of the solid created when this region is revolved around the line x=3? (calc OK)

4. The region in the first quadrant bounded by $y = x^2$ *and* $y = \sqrt{x}$ is revolved around the line y=3. Find the volume of the resulting solid. (evaluate by hand).

5. A sphere with radius 1 is cut by a plane 0.8 units from center. Find the volume of smaller piece. Hint: put it somewhere on the coordinate plane and think of this region as some area being revolved around some line. Evaluate without your calculator.

6. When finding area in the plane, we typically generate it by adding up infinitesimal rectangles—the height is some (usually vertical) distance and the width is usually dx. But there are other ways! A disk (a filled-in circle) has radius 3 and is centered at the origin. We can think of its area as the sum of a bunch of super-thin circles (the technical definition of circle—not filled in) with the same center as the disk. (See diagram below). Given that the area of each of these circles can be approximated by the product of the circumference and an infinitesimal width, write an integral giving us the area of the disk and evaluate it.



Answers

1a. $\int_0^3 (1-(x-2)^3)dx = 6.75$ b. $\int_0^9 x\,dy = \int_0^9 (\sqrt[3]{y-8}+2)dy = 6.75$ c. $\int_0^3 \pi\left[1-(x-2)^3\right]^2 dx \approx 90.882$

d. $\int_0^3 \pi\left[(12^2)-((x-2)^3+11)^2\right]dx \approx 418.056$ e. $\int_0^9 \pi x^2 dy = \int_0^9 \pi(\sqrt[3]{y-8}+2)^2 dy \approx 33.929$

f. $\int_0^9 \pi\left[(x+2)^2-2^2\right]dy = \int_0^9 \pi\left[(\sqrt[3]{y-8})+4)^2-2^2\right]dy \approx 118.752$

2a. $\int_0^2 \pi\left[(10-x^2)^2-(3x)^2\right]dx = 1936\pi/15$ b. $\int_0^6 \pi\cdot\left(\frac{y}{3}\right)^2 dy + \int_6^{10} \pi\cdot\left(\sqrt{10-y}\right)^2 dy \approx 50.265$

c. $\int_0^6 \pi\cdot\left[5^2-\left(5-\frac{y}{3}\right)^2\right]dy + \int_6^{10} \pi\left[5^2-\left(5-\sqrt{10-y}\right)^2\right]dy \approx 305.782$ d. $\int_0^2 \pi\left[(11-3x)^2-(1+x^2)^2\right]dx \approx 377.829$

3a. $\int_0^2 \pi\left[(9-x^2)^2-(5)^2\right]dx \approx 221.168$ b. $\int_{-4}^0 \pi\left[3^2-(3-\sqrt{y+4})^2\right]dy \approx 75.398$

4. $\int_0^1 \pi\left[(3-x^2)^2-(3-\sqrt{x})^2\right]dx = \pi\int_0^1 (x^4-6x^2+6\sqrt{x}-x)dx = \frac{17\pi}{10}$

5. I put a circle centered at the origin and sliced it with the line x=0.8. The region to the right (above the x-axis) is then revolved around the x-axis, giving a volume of:

$$\int_{0.8}^1 \pi y^2 dx = \int_{0.8}^1 \pi(1-x^2)dx = \pi\left(x-\frac{x^3}{3}\right) = \pi(2/3)-\pi(0.629) \approx 0.117$$

6. $\int_0^3 2\pi r\,dx = \int_0^3 2\pi x\,dx = 9\pi$ … The area of a solid circle πr^2 is the sum of the circumferences of an infinite number of smaller circles that fit inside it: $\int_0^r 2\pi x\,dx$. It works with spheres s well… $\int_0^r 4\pi x^2 dx = \frac{4}{3}\pi r^3$

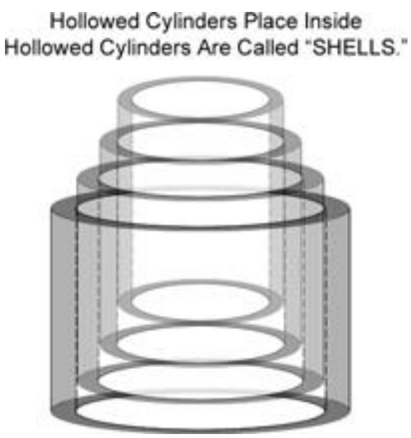
Unit 4 Handout #2: Volumes by Cylindrical Shells

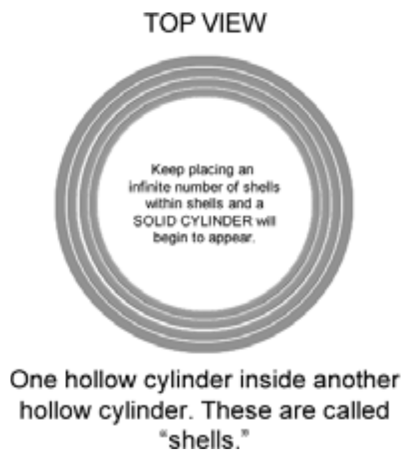
<u>Class Problems</u>

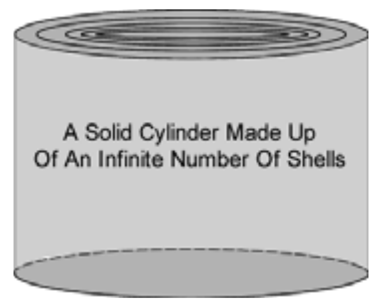
1. The region bounded by the graph of $y = e^x - 1$, the x-axis, and the line x=2 is revolved around the y-axis. Find the volume of the resulting solid.

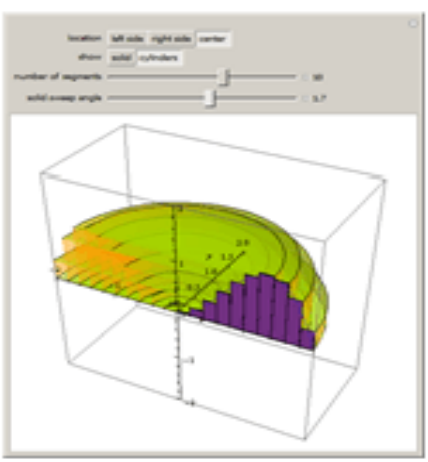
   a. Write an integral using washers and slicing perpendicular to the y-axis. You will need the inverse function.

   b. Write an integral for the volume using shells.



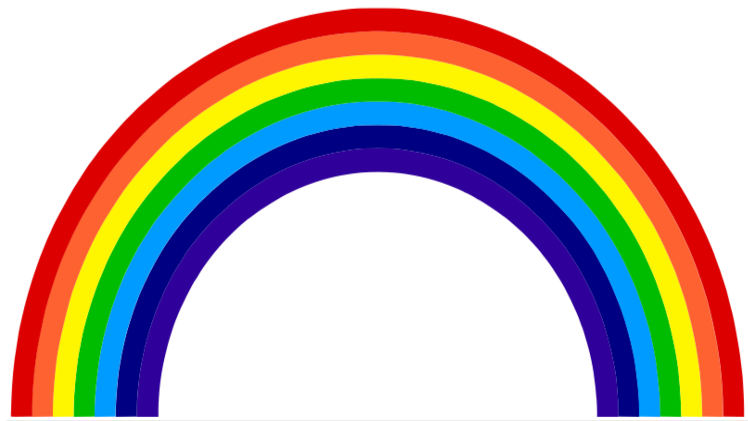






2. In making the movie of "The Wizard of Oz" the set designers needed to paint an enormous rainbow (this is totally made up, if you are curious). Each stripe was 5 meters across at the ends and the radius of the semi-circular inner "hole" was 60 meters. The cost per square meter of each paint color is given below:

Color	cents/sq meter
Red	6
Orange	8
Yellow	10
Green	8
Blue	4
Indigo	8
Violet	6



a. Write an expression that shows the amount spent on paint for the rainbow. [Note: the red is on the outside and the violet on the inside!]

b. Find the average cost per square meter of paint (weighted by the amount of paint used). You may use your lists in your calculator to compute it if you want.

3. A penguin cluster is shaped like a circle with a radius of 200 meters. The penguins tend to congregate on the edges. In fact, the density of penguins (in # per square meter) as a function of the distance from the center is given by the function $d(r) = 0.1 + 0.001r$.

a. What is the total population of the cluster?

b. How many penguins live within 100 meters of the center of the cluster?

c. What is the average density of the cluster? (penguins per square meter)? Do a quick check for whether this answer is reasonable.

Problems. You may use fnInt to evaluate integrals, but do a few by hand for practice.

4. The region in the first quadrant below the graph of $f(x) = -2(x-1)(x-5)$ is revolved around the *y*-axis. Find the volume. This can be difficult to do with washers because this is not a particularly easy function to find the inverse of (in fact, it fails the horizontal line test-- but there are ways to get around this). Use cylindrical shells.

5. The region R in the first quadrant is bounded by the graphs of $y = x^2 - 4x + 7,\ x = 1,\ and\ x = 4$.

a. Find the volume of the solid that results when R is revolved around the *y*-axis.

b. Find the volume of the solid that results when R is revolved around the line *x*=-1.

6. The region in the first quadrant bounded by $y = \sqrt{x}$ *and* $x = 9$ is revolved around the *y*-axis.

a. Write an integral showing the volume of the solid created. Use washers. Evaluate it.

b. Write an integral showing the volume of the solid created. Use shells. Evaluate it.

7. The region R is in the first quadrant, bound by the graphs of $y = 0.25x$ *and* $y = \sqrt[3]{x}$. Note: they meet at the origin and at (8,2). **For each, write an integral using washers and one using shells**.

a. What is the volume created when R is revolved around the *y*-axis?

b. What is the volume created when R is revolved around the line $x = 8$?

8. The area in the first quadrant below the graph of $y = x(x-1)^2$ and up to the line x=1 is revolved around the y-axis. Use shells to find the resulting volume. Can you do this with washers instead?

9. You want to revolve the region in the first quadrant bound by the y-axis and the graphs of $y = e^{-x^2}$, $x = 0$ *and* $x = 1$ around the y-axis. The exponential function is shaped like the normal distribution.

a. Write an integral for the volume using disks.

b. Can we integrate this with the analytical techniques we have covered thus far?

c. Write an integral for the volume using shells and evaluate it analytically.

10. The area in the first quadrant under the graph of $y = -(x-2)(x-6)$ is revolved around the y-axis.

a. Write the volume using shells.

b. Assume instead that a rectangle fitting tightly around this part of the parabola was revolved around the y-axis instead. Find its volume without integrals—hint: cylinders. Note: 3 sides of the rectangle are x=2, x=6, and y=0.

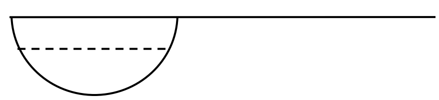
c. What portion of the rectangle's **area** does the parabola's **area** represent?

d. What portion of the revolved rectangle's **volume** does the revolved parabola's **volume** represent?

e. Do you think this will always be the case, or is there something special going on here?

11. A bead (for stringing into necklaces) is made by drilling a cylindrical hole through the center of a sphere. A sphere has a radius of 5mm has a hole with radius 1mm drilled through it. What is the volume of the bead? Hint: think of it as some shape revolved around some line.

12. A tablespoon measuring spoon is a hemisphere with radius 1 unit. When half a tablespoon of liquid is in the measuring spoon, what percentage of the way from the bottom to the top is the surface of the liquid? Calc-intersect is OK.



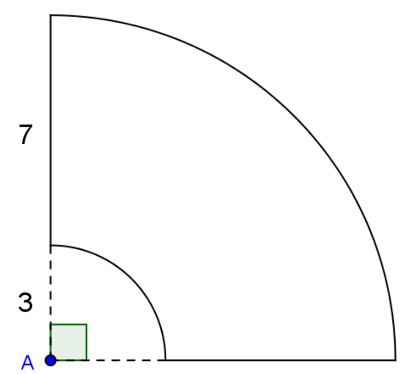
13. The population density of a circular town depends on the distance from the center, according to the function $d(x) = 8000e^{-x^2}$ people per square kilometer. The radius of the city is 4 km.

a. How many people live in the town? No fnInt.

b. What is the average population density of the town? (In other words, the population density of the town as a whole.)

c. Half the people live within what distance of town center?

14. In a person's yard, leaves falling from a tree at point A land in a region shaped like a quarter of a washer (see diagram, where distances are measured in meters). The density of leaves (in leaves per square meter) at any point is a function of the distance of that point from the tree, given by $L(d) = \dfrac{3000}{d^2+1}$. What is the total number of leaves in the yard? Evaluate the integral by hand.



15. A vase is shaped like area in the first quadrant above the graph of $y = 3\ln x$ between $y = 0$ and $y = 4$ revolved around the y-axis.

a. What volume of water does it hold? No fnInt. (think… disks or shells?)

b. When the vase is 40% full, what is the height of water from the bottom?

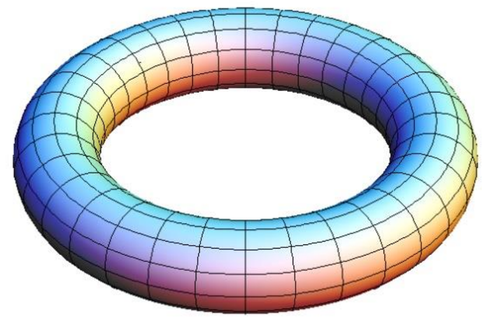
16. Hard one: A torus (in addition to being a model of a Ford car) is a shape like an inner tube. You get it by revolving a circle around a line that does not intersect the circle.

a. Sketch the circle $x^2 + y^2 = 1$.

b. Revolve it around the line $x = 4$. Write an integral showing the resulting volume. Use shells.

c. Evaluate it analytically by splitting it into two integrals (splitting using the 4-*x*)… one can be evaluated with the rules and the other geometrically.

d. A torus is created by revolving a circle with radius *r* around a line, such that the distance from the center of the torus to the center of the revolved circle (in the case above this was 4) is *R*. What is the volume? [note: you can also do this utilizing symmetry as in problem 10e]



Answers

1a. $\int_0^{e^2-1} \pi\left[2^2-(\ln(y+1))^2\right]dy$ b. $\int_0^2 2\pi rh\,dx = \int_0^2 2\pi x(e^x-1)dx = 40.14$

2a. sum of area times $/unit area:

$$\frac{\pi\left(65^2-60^2\right)}{2}\cdot 6+\frac{\pi\left(70^2-65^2\right)}{2}\cdot 8+\frac{\pi\left(75^2-70^2\right)}{2}\cdot 4+\frac{\pi\left(80^2-75^2\right)}{2}\cdot 8+\frac{\pi\left(85^2-80^2\right)}{2}\cdot 10+\frac{\pi\left(90^2-85^2\right)}{2}\cdot 8+\frac{\pi\left(95^2-90^2\right)}{2}\cdot 6$$

which turns out to be: $613.40

b. total cost over total area $\frac{613.4}{0.5\pi(95^2-60^2)}=0.072$ so 7.2 cents per square meter

3a. $\int (area)\cdot(density) = \int_0^{200} 2\pi r(0.1+0.001r)dr = 29{,}322$ b. $\int_0^{100} 2\pi r\cdot(0.1+0.001r)dr = 5{,}236$

c. penguins/area = $\frac{29322}{40000\pi} = 0.233$. This is reasonable since the density ranges from 0.1 to 0.3 and more area has higher density

4. Volume = $\int 2\pi rh\,dx = \int_1^5 2\pi x(-2)(x-1)(x-5)dx \approx 402.124$

5a. shells $V = \int_1^4 2\pi rh\,dx = \int_1^4 2\pi x(x^2-4x+7)dx \approx 202.633$

b. shells… $V = \int_1^4 2\pi rh\,dx = \int_1^4 2\pi(x+1)(x^2-4x+7)dx \approx 278.031$

6a. $\int_0^3 \pi[(9)^2-(y^2)^2]dy$=610.73 b. $\int_0^9 2\pi x(\sqrt{x})dx$=610.73

7a. Using washers: $\int_0^2 \pi[(4y)^2-(y^3)^2]dy$ and using shells: $\int_0^8 2\pi x(\sqrt[3]{x}-0.25x)dx$

b. Using washers: $\int_0^2 \pi[(8-y^3)^2-(8-4y)^2]dy$ and using shells: $\int_0^8 2\pi(8-x)(\sqrt[3]{x}-0.25x)dx$

8. $\int_0^1 2\pi x\cdot x(x-1)^2\,dx \approx 0.209$ → can't do it with washers because can't really invert the function.

9a. $\int_0^{1/e} \pi(1)^2\,dy + \int_{1/e}^1 \pi(-\ln y)dy$ b. can't really integrate lny yet… c. $\int_0^1 2\pi xe^{-x^2}dx = -\pi e^{-x^2} = \pi(1-1/e)$

10a. $\int_2^6 2\pi x\cdot(-1)(x-2)(x-6)dx = 268.08$

b. the 4th side is y=4 (y-value of parabola's vertex) and the rectangle is a square
its volume is big cylinder minus little cylinder: $\pi(6^2)(4)-\pi(2^2)(4)=128\pi$

c. $\int_2^6 (-1)(x-2)(x-6)dx = 10.67/16 \; or \; 2/3$ d. also 2/3 ; cool!

e. it works b/c the region is symmetrical around its horizontal center…center of mass or something…

11. Using shells: $\int_1^5 2\pi x(2\sqrt{25-x^2})dx = 64\pi\sqrt{6}$ Using washers $\int_{-\sqrt{24}}^{\sqrt{24}} \pi\left[(25-y^2)-1\right]dy = 64\pi\sqrt{6}$

12. Volume of hemisphere is $\frac{2}{3}\pi$. Making the center of the top the origin, one cross-section's equation is y= $-\sqrt{1-x^2}$ so $\int_{-1}^{k} \pi x^2 dy = \frac{1}{3}\pi$ → $\int_{-1}^{k} \pi(1-y^2)dy = \frac{1}{3}\pi$

So $y - \frac{y^3}{3}\Big|_{-1}^{k} = \frac{1}{3}$ so $k - \frac{k^3}{3} = \frac{-1}{3}$ and with calc-intersect k≈-0.3473 so it is ≈65.3% from the bottom

13a. sum up area times population density so $\int_0^4 2\pi rh dx = \int_0^4 2\pi x(8000e^{-x^2})dx = -8000\pi e^{-x^2} \approx 25133$

b. 25133 people/ 16π sq km ≈ 500

c. $\int_0^w 2\pi x(8000e^{-x^2})dx = 12566.5$ so $-8000\pi e^{-w^2} + 8000\pi(1) = 12566.5$ so w ≈0.833 km

14. $\int_3^{10}\left(\left(\frac{2\pi x dx}{4}\right)\cdot\frac{3000}{x^2+1}\right) = 750\pi\ln\left(\frac{101}{10}\right) \approx 5449$

15a. I'd use disks: $V = \int_0^4 \pi x^2 dy = \int_0^4 \pi(e^{y/3})^2 dy = \pi\int_0^4 e^{2y/3}dy = 1.5\pi e^{2y/3} \approx 63.108$ units

(shells would require two integrals—or a cylinder and an integral—and it would be a hard integral)

b. $\pi\int_0^k e^{2y/3}dy = (0.4)63.108$ so k is about 2.774

16b. $\int_{-1}^{1} 2\pi(4-x)(2)(\sqrt{1-x^2})dx$ note: the 2nd 2 is because the height is twice the y-value

c. $16\pi\int_{-1}^{1}\sqrt{1-x^2}dx + 2\pi\int_{-1}^{1}\sqrt{1-x^2}(-2xdx)$ … the first integral is the area of a semi-circle (π/2) and the second one magically becomes zero!

$= 16\pi(0.5\pi) + 2\pi(2/3)(1-x^2)^{3/2} = 8\pi^2$

d. $2Rr^2\pi^2$ … using the symmetry deal from 10e it is $\frac{\pi}{4}[\pi(R+r)^2(2r) - \pi(R-r)^2(2r)]$, which turns out to be the same thing!

Unit 4 Handout #3: Volumes of Solids with Known Cross Sections
fnInt OK unless otherwise specified.

1. The region in the first quadrant bound by the graphs of $y = x^2$ and $y = x$ is the base of a solid. Cross-sections perpendicular to the *x*-axis are squares. (See model #1) What is the volume of this solid?

2. The area in the first and second quadrants bound by the x-axis and the graph $y = 4 - x^2$ is the base of a solid. Cross-sections perpendicular to the *x*-axis are semi-circles. (See model #2) What is the volume of this solid?

3. The area in the first and second quadrants bound by the *x*-axis and the graph $y = 4 - x^2$ is the base of a solid. Cross-sections perpendicular to the ***y*-axis** are semi-circles. (See model #3) What is the volume of this solid? Since your integral will have a *dy* in it, you will need to solve for *x* in terms of *y*.

4. The surface of a pond is the region in the first quadrant bound by the coordinate axes and the line $y = 6 - \frac{2}{3}x$. The depth of the pond is equal to one-half of the *x*-coordinate. So it is 4.5 feet deep at the right and beach at the left (ie, the *y*-axis). See model #4 for the (upside-down) pond.

a. How much water is in the pond? Note: you are adding up rectangular prisms whose width is dx, height is *y*, and depth is 0.5*x*.

b. Someone says the average depth is $\frac{1}{9}\int_0^9 0.5x\,dx$. Do you agree? Explain.

c. What is the average depth of the pond?

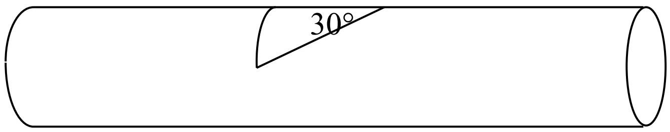
5. A pond's surface is the region on the first quadrant bound by the coordinate axes and the graph of $y = 8 - 0.5x^2$. The pond is shallowest on the west (left) side—by the y-axis-- and gets deeper as you move further away from the y-axis. The depth as a function of the x-coordinate is given by the equation $d(x) = 1 + \sqrt{x}$. See model #5. Use fnInt for this question.

a. How much water is in the pond?

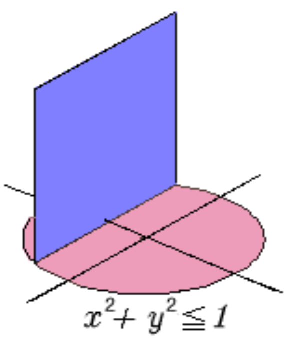
b. What is the average depth of the pond?

6. The area above the x-axis and below the graph of $y = 8 - 2x^2$ is the base of a solid. The solid's height is equal to the distance from the x-axis, or $h(y) = y$. Find the volume of the solid. See model #6. Try to evaluate this integral using u-substitution if you want practice; otherwise use fnInt.

7. A wedge is cut from a log. One cut is perpendicular to the log, going exactly half-way through. The other cut is at a 30° angle. The radius of the log is 10 inches. What is the volume of the wedge? See model #7; note that all cross-sections perpendicular to the semi-circular base are 30/60/90 triangles. It may be helpful to write the area of a 30/60/90 triangle in terms of its shortest side. Also, put the semi-circle on the coordinate axes, using the equation $y = \sqrt{100 - x^2}$.



8. A solid's base is a circle with a radius of 1. Each cross-section perpendicular to the base is a square (whose side is the length of a chord of the circle). What is the volume of this solid? Use the unit circle $x^2 + y^2 = 1$ and slice it perpendicular to the x-axis. Evaluate this integral by hand.

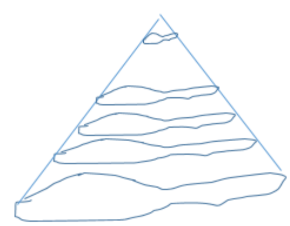


9. Answer the following questions about the region in the first quadrant bound by the y-axis and the lines $y = x$ and $y = 12 - 2x$.

a. Assume that a three-dimensional solid is created where this region is the base. As you slice it perpendicular to the x-axis, the cross-section of each slice is a square. Find the volume of this solid.

b. Instead, assume that a solid is created where each cross-section perpendicular to the y-axis is a square. What is its volume?

10. Cross sections of a solid are all similar to the base and their length decreases linearly to zero (see diagram at right).



a. Does this describe a hemisphere? cone? Pyramid?

b. Let the base have width w and area B. Find the width and area of the cross-section half way to the top. How about the cross-section 1/4th of the way from the top to the bottom?

c. Let the height (measured from the apex perpendicular to the base) be h. Thus the point on the base directly below the apex has a y-coordinate of zero and the apex has a y-coordinate of h. For some generic y, $0 \le y \le h$, what is the area of the cross-section (in terms of B,h and y)?

d. Write an integral showing the volume of the solid and evaluate it to show that the volume of the solid is $V = \frac{1}{3}Bh$.

Answers

1. $\int_0^1 s^2 dx = \int_0^1 (x-x^2)^2 dx = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} = \frac{1}{30}$

2. $\int_{-2}^{2} 0.5\pi r^2 dx = \int_{-2}^{2} 0.5\pi \left(\frac{4-x^2}{2}\right)^2 dx = \frac{\pi}{8}\int_{-2}^{2}(4-x^2)^2 dx = \frac{64\pi}{15}$

3. $\int_0^4 0.5\pi r^2 dy = \int_0^4 0.5\pi x^2 dy = \int_0^4 0.5\pi\left(\sqrt{4-y}\right)^2 dy = 4\pi$

4a. $\int_0^9 y \cdot depth \cdot dx = \int_0^9 \left(6 - \frac{2}{3}x\right)(0.5x)dx$=40.5 4b. No; more area in some places than in others!

c. avg depth is $\frac{water\,volume}{surface area} = \frac{40.5}{27} = 1.5\ldots.$ You can also think of this as taking a weighted average of the depth by the amount of surface area with that depth: so $\frac{\int (depth)(ydx)}{\int ydx}$

5a. $\int_0^4 y \cdot d(x) \cdot dx = \int_0^4 (8-0.5x^2)\cdot(1+\sqrt{x})\cdot dx$=45.71 b. $\frac{45.71}{\int_0^4 (8-0.5x^2)dx} = \frac{45.71}{21.33} = 2.14$

6. $\int_0^8 (b)(h)dy = \int_0^8 (2x)(y)dy = \int_0^8 \left(2\cdot\sqrt{4-0.5y}\right)(y)dy$ =68.27

Using u-substitution: let $u = 4-0.5y \;\; so \;\; y = 8-2u \;\; and \;\; du = -0.5dy$

So $\int \left(2\cdot\sqrt{4-0.5y}\right)(y)dy = \int 2\sqrt{u}(8-2u)(-2du) = -4\int(8u^{1/2} - 2u^{3/2})du = -4\left(\frac{16}{3}u^{3/2} - \frac{4}{5}u^{5/2}\right)\ldots.$

7. area of 30/60/90 in terms of its shortest side is $\frac{1}{2}s\cdot s\sqrt{3} = \frac{s^2\sqrt{3}}{2}$. Volume is:

$$\int_{-10}^{10} \frac{s^2\sqrt{3}}{2}dx = \int_{-10}^{10}\frac{y^2\sqrt{3}}{2}dx = \int_{-10}^{10}\frac{(100-x^2)\sqrt{3}}{2}dx = \frac{\sqrt{3}}{2}\int_{-10}^{10}(100-x^2)dx$$

by symmetry this is $\sqrt{3}\int_0^{10}(100-x^2)dx = \frac{2000\sqrt{3}}{3}$ 8. $\int_{-1}^{1}(2y)^2 dx = 4\int_{-1}^{1}(1-x^2)dx = 8\int_0^1(1-x^2)dx = \frac{16}{3}$

9a. $\int_0^4 (12-3x)^2 dx = \frac{-1}{9}(12-3x)^3 = 192$ b. $\int_0^4 x^2 dy + \int_4^{12} x^2 dy = \int_0^4 y^2 dy + \int_4^{12}(6-0.5y)^2 dy$=64

10a. no; yes; yes b. 0.5w, 0.25B; 0.25w, B/16 c. $\left(1-\frac{y}{h}\right)^2 \cdot B$ d. $\int_0^h \left(1-\frac{y}{h}\right)^2 \cdot B \cdot dy$

$\frac{-h}{3}\left(1-\frac{y}{h}\right)^3 B$ from h to 0 is $\left(\frac{-h}{3}\left(1-\frac{h}{h}\right)^3 B\right) - \left(\frac{-h}{3}\left(1-\frac{h}{h}\right)^3 B\right) = \frac{1}{3}Bh$

Unit 4 Handout #4: Volumes Practice

fnInt OK unless otherwise specified.

1. Region R is in the first quadrant; it is bound by the graphs of $y = x \text{ and } y = x^3$.

   a. Find the volume of the solid whose base is R and cross-sections perpendicular to the x-axis are squares. Also find the area of the largest square cross-section.

   b. Find the volume of the solid whose base is R and cross-sections perpendicular to the ***y*-axis** are semi-circles.

   c. Write two integrals showing the volume of the solid created by revolving R around the line x=3. One should use shells and the other washers.

   d. Find the volume of the solid whose base is R and cross sections perpendicular to the x-axis are equilateral triangles. (Find the area of an equilateral triangle with side s).

2. The region in the first quadrant bound by the graphs of $y = \dfrac{12}{x+1}$, $x = 0,$ *and* $y = 3$ is called R.

   a. Find the area of R.

   b. Find the volume of the solid created by revolving R around the x-axis.

   c. A solid is created whose base is R and whose cross-sections when sliced perpendicular to the x-axis are squares. Find its volume.

   d. A solid is created whose base is R and whose cross-sections when sliced perpendicular to the y-axis are squares. Find its volume.

3. The region in the first quadrant bounded by the x-axis and the graphs of $y = 2\sqrt{x}$ and $y = 8 - x$ is called R.

a. Sketch R and find the coordinates of the point where the graphs intersect algebraically.

b. Find the area of R.

c. Find the volume of the solid created by revolving R around the line y=6.

d. Find the volume of the solid created by revolving R around the y-axis.

e. Assume that R is the base of a three-dimensional solid and the cross sections slicing perpendicular to the x-axis are all squares. What is the volume of this solid?

4. A solid's base is the parabola $y = x^2$ from y=0 to y=4. Its cross-sections perpendicular to the y-axis are semi-circles. Find its volume.

5. A city is built around a circular lake with radius of 1 mile. The population density is $f(r)$ people, where r is the distance from the center of the lake in miles. Write an integral indicating the number of people who live within 1 mile of the lake. [no one lives in the lake!]

6. The triangle with vertices (0,0), (0,2), and (1,0) is the base of a solid.
   a. If cross-sections perpendicular to the *x*-axis are squares, then what is the area of this strange solid?
   b. What if the cross-sections are semi-circles instead of squares?
   c. Given your answer to part *a*, could you answer part *b* without any integrals? How so?

7. The base of a solid is an equilateral triangle with side 2. When the solid is sliced by planes parallel to one side of the triangle, each cross-section is a rectangle whose height is one half of its base (so the bases of these rectangles lie in the triangle, parallel to one of the sides.) What is the volume of this strange solid? Hint: put one vertex at the origin, another on the *y*-axis, and find where the third must be.

8. The surface of a lake happens to be shaped exactly like the area between the graphs of $y = \sqrt{x}$ *and* $y = 0.5x$ on the interval [0,4], where the units correspond to miles. Its depth is given by $d(x) = 0.1 + 0.1x$. So it is shallow in the west and deeper in the east.

a. What is the total volume of this lake?

b. What is the average depth of this lake?

9. The base of an object is the region in the first quadrant bounded by the graphs of $y = e^{0.5x}$, $x = 2$, *and* $x = 4$.

a. If cross-sections perpendicular to the *x*-axis are squares, then find the volume.

b. If cross-sections perpendicular to the *y*-axis are squares, then find the volume.

10. Define the region R as the area in the first quadrant bounded by the graphs of $f(x) = x + 1$ and $g(x) = x^2 - 4x + 5$. Write integrals that express each of these:

a. The area of R.

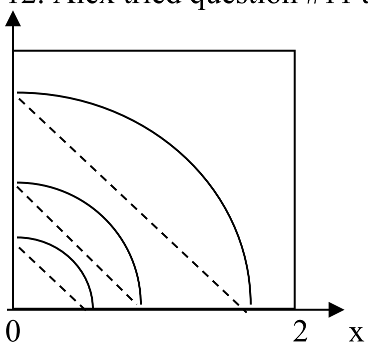
b. The volume of the solid whose base is R and whose cross-sections perpendicular to the *x*-axis are semi-circles.

c. The volume of the solid created by revolving R around the line *y*=-2.

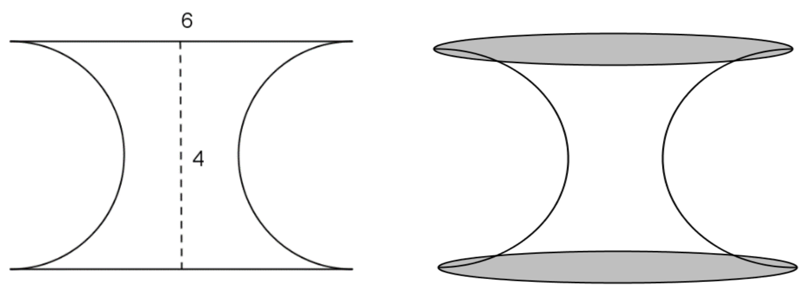
d. The volume of the solid created by revolving R around the line *x*=-2.

11. A paperweight's base is a square with side 2. It has semi-circular cross sections when it is sliced at 45 degrees to the sides. In other words, the largest semi-circular cross section has a diameter that is the diagonal of the square. What is the volume of this paperweight?

12. Alex tried question #11 above using the diagram below. At any x-value, the diameter of the circle is $x\sqrt{2}$. If he adds them up for zero to two he gets one-half the volume of the paperweight. So the volume is $2\int_0^2 \frac{1}{2}\pi\left(\frac{x\sqrt{2}}{2}\right)^2 dx$. This equals $4\pi/3$, which is not correct. What did Alex do wrong?



13. The shape on the left is a rectangle with two semi-circles cut out of it. It is revolved around its center (the dashed line) to create the spool on the right. Find the volume of the spool.



ANSWERS

1a. $\int_0^1 \left(x-x^3\right)^2 dx \approx 0.0762$ lgest sq is $\frac{4}{27}$ if $x = 1/\sqrt{3}$ b.diam is $\sqrt[3]{y}-y$ so $\int_0^1 0.5\pi\left(\frac{\sqrt[3]{y}-y}{2}\right)^2 dy \approx 0.0299$

c. shells: $\int_0^1 2\pi rh dx = \int_0^1 2\pi(3-x)(x-x^3)dx$ washers: $\int_0^1 \pi(R^2-r^2)dy = \int_0^1 \pi[(3-y)^2-(3-\sqrt[3]{y})^2]dy$

d. area of equilateral triangle is $\frac{s^2\sqrt{3}}{4}$ so $\frac{\sqrt{3}}{4}\int_0^1 \left(x-x^3\right)^2 dx \approx 0.0330$

2a. $\int_0^3 \left(\frac{12}{x+1}-3\right)dx \approx 7.64$ b. $\int_0^3 \pi\left[\left(\frac{12}{x+1}\right)^2-3^2\right]dx = 81\pi$ c. $\int s^2 dx = \int_0^3 \left(\frac{12}{x+1}-3\right)^2 dx = 35.19$

d. $\int s^2 dy = \int_3^{12}\left(\frac{12}{y}-1\right)^2 dy \approx 11.73$ 3a. meet at (4,4) b. $\int y dx \ so \int_0^4 2\sqrt{x}dx + \int_4^8 (8-x)dx = 56/3$

3c. $\int \pi[R^2-r^2]dx$ so $\int_0^4 \pi\left[6^2-(6-2\sqrt{x})^2\right]dx + \int_4^8 \pi\left[6^2-(-2+x)^2\right]dx = 96\pi + 224\pi/3 = 512\pi/3$

d. Washers: $\int \pi[R^2-r^2]dy$ so $\int_0^4 \pi\left[(8-y)^2-(0.25y^2)^2\right]dy \approx 136.53\pi$

Shells: $\int 2\pi rh dx = \int_0^4 2\pi rx(2\sqrt{x})dx + \int_4^8 2\pi rx(8-x)dx$ e. $\int y^2 dx = \int_0^4 \left(2\sqrt{x}\right)^2 dx + \int_4^8 (8-x)^2 dx = 160/3$

4. $\int_0^4 0.5\pi r^2 dy = \int_0^4 0.5\pi(\sqrt{y})^2 dy = 4\pi$

5. $2\pi\int_1^2 r\cdot f(r)dr$

6a. $\int_0^1 (-2x+2)^2 dx = 1.3333$ b. $\int_0^1 \frac{\pi}{2}(-x+1)^2 dx = 0.5236$

c. a semi-circle takes up π/8 of the square so 1.333* π/8=0.5236

7. vertices are $(0,0),(0,2),\ and\ (\sqrt{3},1)$. The equations of the sides are $y = \frac{1}{\sqrt{3}}x\ and\ y = 2-\frac{1}{\sqrt{3}}x$

$V = \int bh = \int 0.5b^2\ \ so\ V = \int_0^{\sqrt{3}} 0.5\left(\left(2-\frac{x}{\sqrt{3}}\right)-\left(\frac{x}{\sqrt{3}}\right)\right)^2 dx = \int_0^{\sqrt{3}} 0.5\left(\left(2-\frac{2x}{\sqrt{3}}\right)\right)^2 dx = \frac{2\sqrt{3}}{3}$

8a. volume is sum of very thin rectangular prisms: base is vertical segment between graphs, width is dx, and height is the depth of water (0.1+0.1x) so volume = $\int_0^4 (\sqrt{x}-0.5x)(0.1+0.1x)dx = 0.347$ mi^3

b. $0.347/\int_0^4 (\sqrt{x}-0.5x)dx =\ 0.347/(4/3) = 0.26$ mi

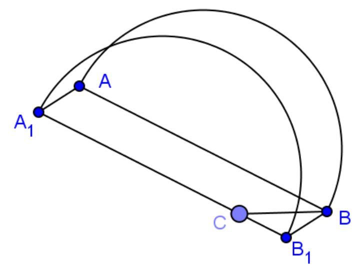
9a. $\int_2^4 (e^{0.5x})^2\,dx$ b. x=2lny so $\int_0^e 2^2\,dy + \int_e^{e^2} (4-2\ln y)^2\,dy$

10a. $\int_1^4 (x+1-(x^2-4x+5))dx$ b. $\frac{\pi}{2}\int_1^4 \left(\frac{-x^2+5x-4}{2}\right)^2 dx$ c. $\int_1^4 \pi\left[(x+3)^2-(x^2-4x+7)^2\right]dx$

d. $\int_1^4 2\pi(x+2)(-x^2+5x-4)dx$

11. rotate the square 45°in the plane; its corners are $(\sqrt{2},0), (0,\sqrt{2}), (-\sqrt{2},0), (0,-\sqrt{2})$. The sides of the square are on lines with slope 1; in the 1st quad the line is $y=-x+\sqrt{2}$; this is the radius of some of the semi-circles when sliced vertically. $V=\int_0^{\sqrt{2}} 0.5\pi(-x+\sqrt{2})^2\,dx + \int_{-\sqrt{2}}^0 0.5\pi(x+\sqrt{2})^2\,dx$. B/c of symmetry this is $V=\int_0^{\sqrt{2}} \pi(-x+\sqrt{2})^2\,dx = \frac{2\pi\sqrt{2}}{3} \approx 2.962$

12. The volume of a semi-circular disk is the area of the semi-circle times the width. But the width has to be measured along a perpendicular to the disk's diameter, and Alex did not do this. He measured along BC and not BB_1. Since BC is $\sqrt{2}$ times BB_1 Alex's area is $\sqrt{2}$ times the correct answer.



13. Put center of left semi-circle at (0,0) and revolve around x=3. Slice horizontally:

Volume = $\int_{-2}^{2} \pi(3-x)^2\,dy$ & $x^2+y^2=4$

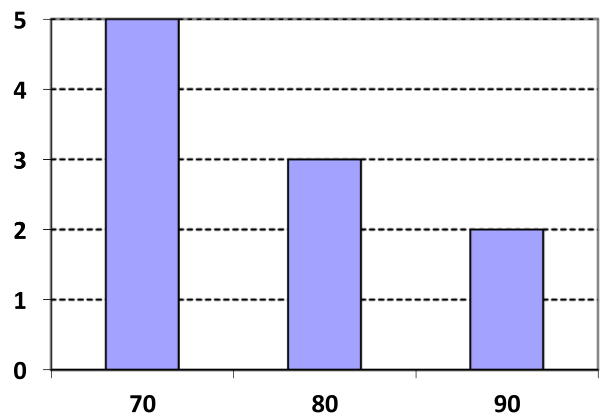
Volume = $\pi\int_{-2}^{2} 9dy - 6\pi\int_{-2}^{2}\sqrt{4-y^2}\,dy + \pi\int_{-2}^{2}(4-y^2)dy = 36\pi - 6\pi(2\pi) + \pi\left(\frac{32}{3}\right) = \frac{140\pi}{3} - 12\pi^2$

(middle integral is best evaluated geometrically!)

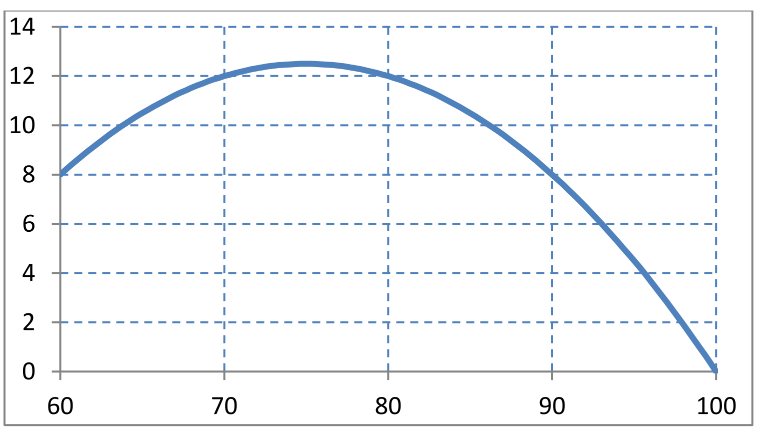
Unit 4: Handout #5: Weighted Average, Density, and Volumes

1. On a test, five students earned scores of 90, seven earned 80, and three earned 70. What is the average score?

2. The chart below shows the frequency of people's scores on a test. Find the average score:



3. Scores on a test ranged between 60 and 100. Any score in this range was possible, not just integers.. The density of people getting each score (in people per point) is approximated by the function $f(x) = -0.02x^2 + 3x - 100$.



a. How many people took the test?

b. What was the average score?

4. On the interval [0,60] minutes after noon, the rate of inflow of people into a (previously-empty) stadium is given by the function $r(t) = 20\sqrt{t}$.

a. What was the total number of people entering the stadium?

b. When did the 3000$^{\text{th}}$ person arrive?

c. What was the average number of people in the stadium?

d. What was the average time of arrival for the people (how many minutes after noon?)

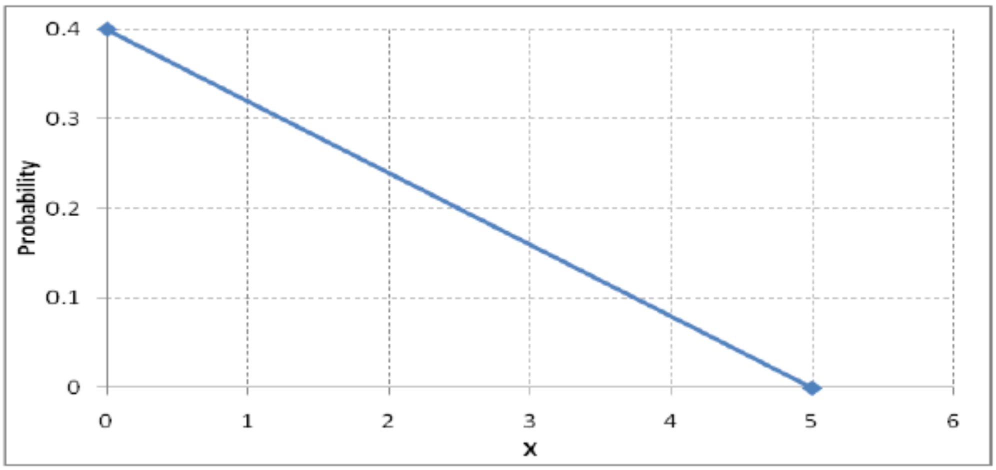
e. The game starts at t=60 (1 pm). On average, how long did people wait (ie, time between getting in the stadium and the game starting)?

5. The gates of a football stadium open at 10am, and the game begins at noon. The rate of people entering t minutes after 10 am is given by the function $r(t) = 0.1t^2 - 8t + 500$ on the domain [0,120].On average, how long did people wait? (in other words, how long before game time did they enter?) One way to think of this is like #4 above, but we are adding up people-minutes then dividing by people and people minutes is a very thin rectangle under the graph (people) times a height that represents minutes.

<u>Questions 6 and 7 involve density curves</u>

In statistics, a density curve (such as the normal distribution) is like a histogram in that it shows what values a variable may take and how frequently they take those values. A density curve is used for variables that have an infinite number of possible values (like height, as it can be any decimal). The *x*-axis represents the different possible values, and the *y*-axis represents the relative frequency of occurrence. The area under a density curve must equal 1, as in 100%.

6. The density curve below is for a random variable. It shows that the variable takes values anywhere from 0 to 5, with lower values happening more frequently than larger values.



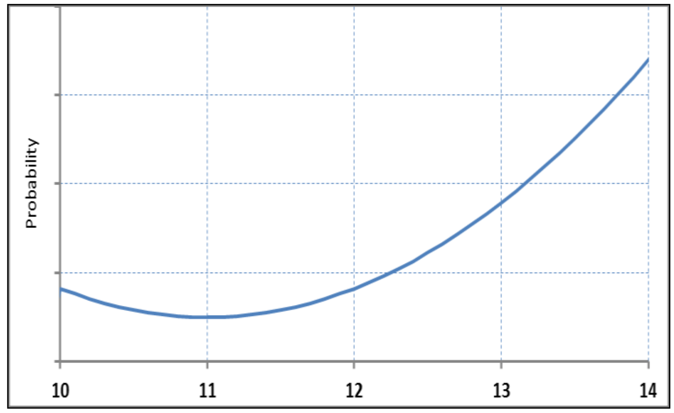
a. Use geometry or calculus to verify that the area beneath the curve is 1.

b. Use algebra or calculus to find the median of the distribution.

c. This variable is skewed right (the right extremes are further from the center/median than the left extremes are). How should the mean compare to the median?

d. Find the exact value of the variable's mean using calculus (for random variables, the mean is often called the "expected value").

7. A variable takes values between 10 and 14. The density curve showing the probabilities of different values is given by $d(x)=\frac{9}{140}(x-11)^2+0.1$.



a. Verify that the area beneath the curve is 1.

b. Find the median. Calc-intersect OK.

c. Find the expected value.

d. Find the standard deviation. It is the square root of the variance and one convenient formula for variance is the expected value of x^2 minus the expected value of *x*, squared. Or $\sigma^2(x)=E(x^2)-[E(x)]^2$

Answers

1. $\frac{5(90)+7(80)+3(70)}{15}=81.33$ b. $\frac{5(70)+3(80)+2(90)}{10}=77$

3a. $\int_{60}^{100}(-0.02x^2+3x-100)dx=373.33$ b. $\frac{\int Score\cdot freq}{\int freq}=\frac{\int_{60}^{100}x(-0.02x^2+3x-100)dx}{373.33}$=77.14

4a. $\int_0^{60}20\sqrt{t}dt=6197$ b. number of people in stadium at time t is $P(t)=\int_0^t 20\sqrt{x}dx=\frac{40}{3}t^{3/2}$ which equals 3000 when t=37 minutes c. $\frac{1}{60}\int_0^{60}P(t)dt=\frac{1}{60}\int_0^{60}\frac{40}{3}t^{1.5}dt=2478.71$

d. weighted average of time is $\dfrac{\int (time * ppl)dt}{\int ppl * dt} = \dfrac{\int_0^{60} t \cdot 20\sqrt{t}dt}{\int_0^{60} 20\sqrt{t}dt} = 36$ minutes after noon

another way: 2478.71 people on avg so total # of ppl minutes in stradium = 2478.71*60=148,723. Divide by 6197 ppl to get 24 mins on avg in the stadium, so got there 24 min before 1, which is 12:36.

e. weighted average of time in stadium = $\dfrac{\int (time * ppl)dt}{\int ppl * dt} = \dfrac{\int_0^{60} (60-t) \cdot 20\sqrt{t}dt}{\int_0^{60} 20\sqrt{t}dt} = 24$

or just subtract your answer to part d from 60…

5. two approaches:

1. average of wait time weighted by rate of arrivals

$$\frac{\int_0^{120} (120-t)r(t)dt}{\int_0^{120} r(t)dt} = \frac{\int_0^{120} (120-t)(0.1t^2 - 8t + 500)dt}{\int_0^{120} r(t)dt} = 50.4 \text{ minutes}$$

2. number of people at time t is given by $\dfrac{0.1t^3}{3} - 4t^2 + 500t$ so total people-minutes in the stadium is given by $\int_0^{120} \left(\dfrac{0.1t^3}{3} - 4t^2 + 500t \right) dt = 3024000$; now divide by total people (60,000) and get 50.4 minutes. Note that this numerator is a different cubic than the one in the first approach, but on the interval [0,120], they have the same value! Cool!

6a. ½*b*h=1 b. if w is median and y=0.4-0.08x then (0.5)(5-w)(0.4-0.08w)=0.5 and (5-w)^2=12.5 so $5 - \sqrt{12.5} = 5 - \dfrac{5}{\sqrt{2}} \approx 1.464$ OR $\int_0^{w} (0.4 - 0.08x)dx = 0.5$

c. mean will be above the median since extreme values pull mean up

d. $\int_0^5 x \cdot P(X = x) = \int_0^5 x(0.4 - 0.08x)dx = 5/3$

7a. $\int_{10}^{14} \left(\dfrac{9}{140}(x-11)^2 + 0.1 \right) dx = \dfrac{3}{140}(x-11)^3 + 0.1x = 84/140 + 0.4 = 1$

b. $\int_{10}^{w} \left(\dfrac{9}{140}(x-11)^2 + 0.1 \right) dx = 0.5$ so $\dfrac{3}{140}(w-11)^3 + 0.1w + \dfrac{3}{140} - 1 = 0.5$ and w=13.02

c. $\int_{10}^{14} x \left(\dfrac{9}{140}(x-11)^2 + 0.1 \right) dx =$ 12.69

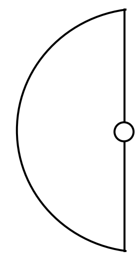
Unit 4 Handout #6: Challenging Questions with Volume and Density
Use fnInt. These are as hard as it gets this semester!

1. A solid's base is the region in the first quadrant between $f(x) = \sin x$ and the x-axis on the interval [0,π]. Its cross-sections, when sliced perpendicular to the x-axis, are rectangles whose height is always $6 - x$. What is its volume?

2. A pond's surface area is defined as the area in the first quadrant bounded by the y-axis and the graphs of $y = x^2$ and $y = 18 - x^2$. Its depth increases with distance from the y-axis and is given by the function $d(x) = 0.2 + 0.1x$. Assume units are km.

   a. What is the area of the surface?
   b. What is the volume of water in the pond?
   c. What is the average depth?

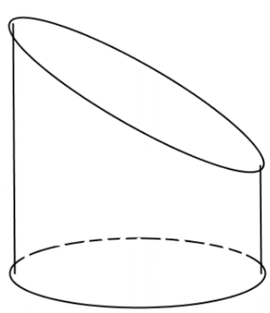
3. The city of Chicago is shaped like a semi-circle, with its center on the western edge of Lake Michigan. Its population density (in thousands of people per square kilometer) x kilometers from city center is given by the equation $d(x) = 20e^{-0.03x^2}$. Assume the radius of the city proper is 6 km (I have no idea how accurate this is).



a. What is the total population of the city proper?

b. What is the population density of the city as a whole? (you can think of this as the average density)

4. A cylinder with radius 6 is sliced at an angle, such that the resulting piece is 10 units high on one end and 4 units high on the other (see diagram). What is its volume? Hint: slice it vertically. Or do it without calculus!



5. A cool problem! Over the course of a day, if no air-conditioning is used, the temperature (as a function of hours since midnight) in an apartment is given by the equation $d(t) = 75 + 20\sin\left(\frac{\pi}{12}(t-9)\right)$.

a. What is the average temperature over the course of a day?

b. What is the average temperature between noon and 6 pm?

(continued…) The apartment's occupant has the air conditioner on and set to 80°. If the "naturally-occurring" temperature is below 80°, the air conditioning remains off. Whenever the temperature would be above 80°, it cools it to 80°.

c. For how many hours a day is the air conditioning on?

d. With the air conditioning as described, what is the average temperature in the apartment over the course of the day?

e. If the cost of air conditioning is about 20 cents per degree-hour (cooling the apartment by one degree for one hour costs 20 cents), then how much is spent each day on air conditioning?

f. Instead, assume the cost of air conditioning is not linear, but gets more expensive the more it is needed. The total cost per hour (in cents) of cooling the apartment by w degrees is given by the function $C(w) = 17 \cdot w^{1.2}$, so if the outside temperature is 90° and the air conditioning cools it to 80°, the hourly cost is $C(10) = 17 \cdot 10^{1.2}$. Now what is the daily cost of air conditioning?

6. A pond's surface is a circle of radius 70 meters.

a. If the depth of the pond is a function of the distance from the (nearest) edge given by $h(d) = 5\sqrt{d}$ then find the average depth of the pond. (hint: shells)

b. Instead, assume the depth of the pond is a function of the how far north a given point is from the southernmost point of the pond. This function is $h(d) = \sqrt{0.1 + 2d}$. Now find the average depth.

7. Answer the questions below about the area under the graph of the parabola $f(x) = ax - x^2$ (where a is a positive constant) and above the x-axis (all in the first quadrant).

a. When the region is revolved around the x-axis, find the volume (in terms of a)

b. When the region is revolved around the y-axis, find the volume (in terms of a)

c. For what value of a are the volumes the same?

8. (From Finney et.al. Calculus). A bowl has a shape that can be generated by revolving the parabola $y = 0.5x^2$ between y=0 and y=5 about the y-axis.

a. What is the volume of this bowl?

b. If it is filled so that its depth is h, then how much water is in it?

c. Water is being added at a constant rate of 3 cubic units per second. How quickly is the water level rising when the water is 4 units deep? [Remember related rates?]

9. **"Two three-four-fives":** The base of a solid is a 3-4-5 triangle. Each cross section cut perpendicular to the hypotenuse is a 3n-4n-5n triangle, whose hypotenuse lies on the base of the solid. What is the volume of this strange solid (perhaps 234.5?)

10. People begin to enter a movie theater at noon. The movie begins at 1 pm. The rate of arrivals (in people per minute t minutes after noon) is given by $P(t) = 15 - 15\cos\left(\frac{\pi t}{60}\right)$. The theater fills at 1.

a. What is the total number of people in the theater at 1?

b. What is the average rate of arrivals between noon and 1?

c. When is the theater 2/3rd full?

d. Without doing any calculations, should the average number of people in the theater between noon and 1 pm be more or less than 450? Explain briefly.

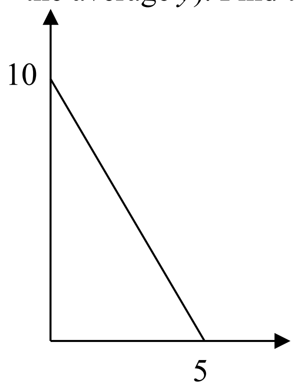
e. On average, how many people are in the theater?

f. On average, how long do people wait (ie, what is the average time people are in the theater before the movie begins)?

11. A sphere of radius r is sliced by a plane; the smaller piece is called a segment. When the flat part of the segment is facing down, the height of the segment is h. (In other words, the plane is (r-h) units from the center of the sphere.) Show that the volume of the segment can be written as $V = \frac{\pi}{3}h^2(3r-h)$.

12. A density curve for a random variable is defined by the function $d(x) = \frac{(x-5)^2}{9}$ on the interval [5,8]. Find the median, mean, and standard deviation of this variable. Hint: two formulas for standard deviation are $\sigma = \sqrt{E(x-E(x))^2}$ *or* $\sigma = \sqrt{E(x^2)-[E(x)]^2}$, where $E(w) = \overline{w}$, the mean (expected value) of w. I think the second formula is easier!

13. The center of mass of a two –dimensional object is the point whose coordinates are (the average *x*, the average *y*). Find the center of mass of the triangle below. No calculator.



13b. Show that the center of mass is the average of the coordinates of the vertices of the triangle.

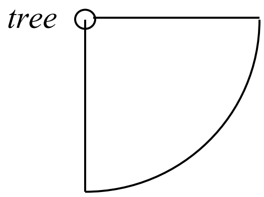
14. Find the center of mass of the region bound by the parabola $y = x^2$ and the line $y = 4$.

15. A lawn is a rectangle with sides running 50 meters north/south and 30 meters east-west. There are many leaves on the lawn. Due to a strange configuration of trees, the concentration of leaves (in leaves per square meter) is a function of the distance north of the southern edge of the lawn, given by the function $L(h) = 110 - 80\cos\left(\frac{\pi}{20}(h-35)\right)$.

a. How many leaves are in the lawn?

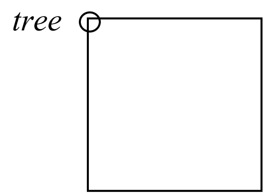
b. Paul rakes all the leaves to the southern edge of the lawn. The work involved in this raking is measured in units of “leaf-meters”—the amount of work required to move one leaf one meter. How many leaf-meters of work is involved in Paul’s raking?

16. A lawn is one quarter of a circle with radius 15 meters. There's an oak tree at the circle's center and leaves have fallen on the lawn. The density of leaves (in leaves per square meter) is a function of the distance from the base of the tree given by the equation $d(m) = \dfrac{100m}{0.25m^2 + 1}$.



a. How far from the tree is the density of leaves the greatest? (no calculator)
b. What is the total number of leaves on the lawn? (fnint ok- the integral is ugly but doable)
c. What is the average distance of the leaves from the base of the tree? (fnint ok)
d. How many leaf-meters of work is involved in raking all the leaves to the base of the tree? (fnint ok)

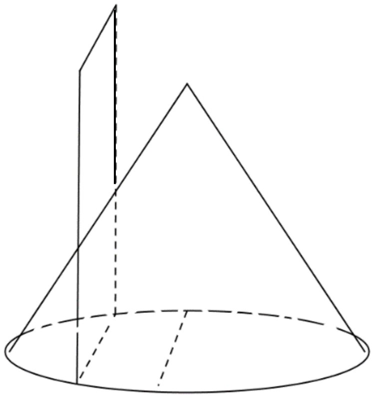
17. Some may note that it was a nice coincidence in the last problem that the lawn was a quarter circle, given that the leaf density is a function of the distance from the circle's center. The problem is still doable if the lawn is a different shape. Now assume the lawn is a square wide side 15 meters; the tree is at a corner and the density of leaves (in leaves per square meter) as a distance from the tree is still given by the function $d(m) = \dfrac{100m}{0.25m^2 + 1}$. We want to find the total number of leaves on the lawn.



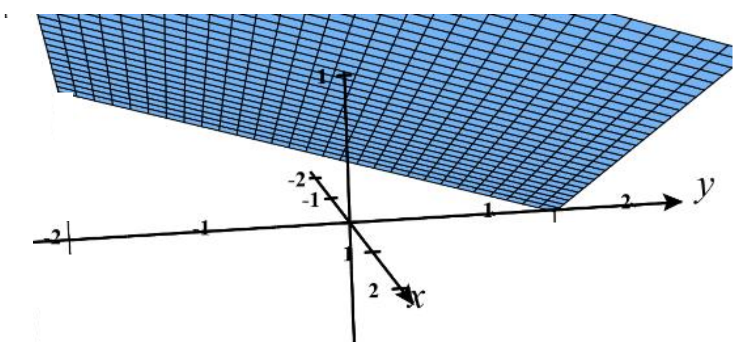
a. Find the number of leaves in the quarter circle of radius 15 meters.
b. To find the number of leaves in the part of the square lawn outside the quarter circle, you need to find the length of the arc *r* meters away from the tree, where *r*>15. When *r* is 16, how long is the arc?
c. Show that, for *r*>15, the length of the arc *r* meters away is $\left(\dfrac{\pi}{2} - 2\cos^{-1}\left(\dfrac{15}{r}\right)\right)\cdot r$

d. Now find the total number of leaves on the lawn.

18. A cone with a base radius of 2 and a height of 3 is sliced by plane perpendicular to its base, one unit from the edge. Find the volume of the smaller piece. You can use disks (circular segments) or shells (using arc length!).

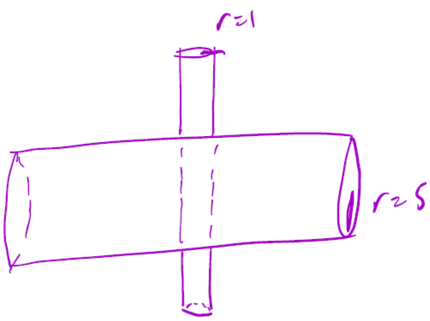


19. The plane graphed is $z = 0.4x - 0.3y + 1$. Find the volume of the region between the plane and the *xy*-plane (ie, *z*=0) whose base is the square where $0 \le x \le 1$ *and* $0 \le y \le 1$

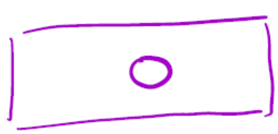


20. A cylinder with a radius of 1 intersects a cylinder of radius 5 in a way that the lines through the center of the two cylinders meet at a right angle. Find the volume common to both cylinders.

Side view:



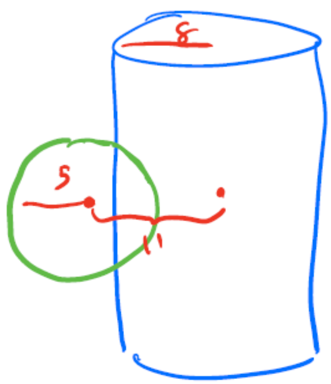
Top view:



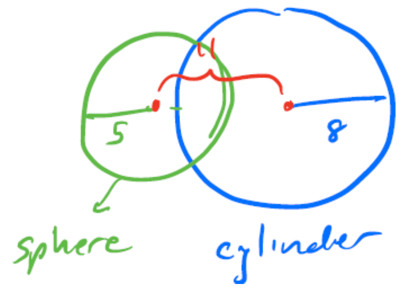
End view:

21. A cylinder of radius of 8 and a sphere of radius 5 intersect where the entire sphere is between the top and bottom of the cylinder. At this point, the center of the sphere is 11 units from the line in the center of the cylinder. Find the volume of the region of overlap.

Side view:



Top view:



ANSWERS

1. add up rectangles so $\int_0^{\pi} bh\,dx = \int_0^{\pi} (\sin x)(6-x)\,dx = 8.86$

2a. $\int_0^3 (18-2x^2)\,dx = 36$ b. solid with this base and height $d(x)$ so add up rectangles

$$\int_0^3 bh\,dx = \int_0^3 (18-2x^2)(0.2+0.1x)\,dx = 11.25$$

c. total volume/area of surface=0.3125 (reasonable since depth goes from 0.2 to 0.5 but more of pond is shallow than deep)

3a. Using shells (actually half-shells since we only get half of a circle) I get

Total population= $\int_0^6 0.5(2\pi rh)\,dx = \int_0^6 \pi x \cdot 20e^{-0.03x^2}\,dx = 691.6$ thousand people

b. The area is 18*pi=56.55 square km so the average density is 12.23 thousand people per sq km. Reasonable? 20 thousand in middle and 6.8 thousand on edges… more on edges but population density only decreases slightly near the center—passes the smell test, I guess.

4. I made the circular base $x^2 + y^2 = 36$ and sliced it from the top perpendicular to the x-axis. Each cross-section is a rectangle. The base is twice the y, so $2\sqrt{36 - x^2}$ and the height descends linearly from 10 when x=-6 to 4 when x=6, so it is given by the equation $h = 7 - 0.5x$. Thus the total volume is $\int_{-6}^{6} bh\,dx = \int_{-6}^{6} 2\sqrt{36 - x^2}(7 - 0.5x)dx = 791.68$. Note: no calculus was needed because this is the same as a cylinder with height 7 and radius 6 (the notch removed from one side is added to the other)

5a. 75 degrees (no calculus necessary!) b. 93.01 degrees (hot hot hot!)

c. 10.07 hours d. (1/24) * (10.07*80 + $\int_{0}^{9.965} d(t)dt + \int_{20.035}^{24} d(t)dt$) = 70.93 degrees

e. $\int_{9.965}^{20.035} [d(t) - 80] * (20)dt = 1951.77$ cents or about \$19.52 f. $\int_{9.965}^{20.035} [(17 \cdot (d(t) - 80)^{1.2})dt = \$26.95$

6a. $\dfrac{\int_{0}^{70} 2\pi(70 - x)(5\sqrt{x})dx}{4900\pi} = 22.31$ or $\dfrac{\int_{0}^{70} 2\pi(x)(5\sqrt{70 - x})dx}{4900\pi} = 22.31$

b. $\dfrac{\int_{-70}^{70} 2\sqrt{4900 - y^2}\sqrt{0.1 + 2(y + 70)}dy}{4900\pi} = 11.37$

7a. $\int_{0}^{a} \pi(ax - x^2)^2 dx = \dfrac{\pi a^5}{30}$ b. shells: $\int_{0}^{a} 2\pi x(ax - x^2)dx = \dfrac{\pi a^4}{6}$ c. 5

8a. $\int_{0}^{5} \pi x^2 dy = \int_{0}^{5} \pi 2y\,dy = 25\pi$ b. $V = \int_{0}^{h} \pi 2y\,dy = h^2\pi$

c. since $V = h^2\pi$ $\dfrac{dV}{dt} = 2h\pi\dfrac{dh}{dt}$. When $h = 4$ *and* $\dfrac{dV}{dt} = 3$, $\dfrac{dh}{dt} = \dfrac{3}{8\pi}$ units per second

or get fancy: $\dfrac{dh}{dt} = \dfrac{dh}{dV} \cdot \dfrac{dV}{dt}$ since $\dfrac{dV}{dt} = 3$, $\dfrac{dh}{dt} = \dfrac{dh}{dV} \cdot 3 = \dfrac{3}{\frac{dV}{dh}}$

In part b, $V = \int_{0}^{h} \pi 2y\,dy$ so $\dfrac{dV}{dh} = 2\pi h$ so $\dfrac{dh}{dt} = 3/(2\pi h) = \dfrac{3}{8\pi}$

9. The vertices of the triangle I am using are (0,12/5), (16/5,0), and (-9/5,0); the hypotenuse lies on the x-axis. For any cross section, if the hypotenuse has length h, the area is $0.24h^2$. So the volume is:

$\int_{-9/5}^{0} 0.24\left(\frac{4}{3}x + \frac{12}{5}\right)^2 dx + \int_{0}^{16/5} 0.24\left(\frac{-3}{4}x + \frac{12}{5}\right)^2 dx$ which is 2.304. (cool! got 2, 3, and 4, but—alas—no 5)

10a. $\int_0^{60} P(t)dt = 900$ b. 15 (900 ppl/60 minutes)

c. $\int_0^{x} P(t)dt = 600$ so $15x - \frac{900}{\pi}\sin\left(\frac{\pi x}{60}\right) = 600$ occurs when x=49.76 minutes after noon

d. less since the arrival rate is increasing, more people arrive closer to 1 pm than to noon

e. $\frac{1}{60}\int_0^{60}\left[15x - \frac{900}{\pi}\sin\left(\frac{\pi x}{60}\right)\right]dx = 267.62$

f. either sum of ppl minutes divided by people, which is 267.62*60/900=17.84

or avg wait time (where wait time is 60-x): $\frac{\int_0^{60}(P(t)*(60-t))dt}{\int_0^{60} P(t)dt} = 17.84$

11. $\int_{r-h}^{r} \pi\left[\sqrt{r^2 - x^2}\right]^2 dx = \pi\left(r^2 x - \frac{x^3}{3}\right) = \pi\left[\frac{2r^3}{3} + \frac{(r-h)^3}{3} - r^2(r-h)\right] =$

$\frac{\pi}{3}\left[2r^3 + r^3 - 3r^2 h + 3rh^2 - h^3 - 3r^3 + 3r^2 h\right] = \frac{\pi}{3}\left[3rh^2 - h^3\right] = \frac{\pi}{3}h^2(3r - h)$

12. median $\int_5^{m}\frac{(x-5)^2}{9}dx = 0.5$ so $(m-5)^3 = 13.5$ and m=7.38 Mean : $\int_5^{8} x\cdot\frac{(x-5)^2}{9}dx = 7.25$

Std dev: square root of [avg of x^2 minus (avg of x)^2] = Sqrt of $\int_5^{8} x^2\cdot\frac{(x-5)^2}{9}dx - 7.25^2$ = 0.581

13. Avg x coord: $\frac{\sum wx}{\sum w} = \frac{\int_0^{5} x(10-2x)dx}{\int_0^{5}(10-2x)dx}$ = 5/3 avg y-coord= $\frac{\int_0^{10} y(5-0.5y)dy}{\int_0^{10}(5-0.5y)dx}$ = 10/3 so $\left(\frac{5}{3},\frac{10}{3}\right)$

14. the weighted average x-value. $\frac{\int_{-2}^{2} x(4-x^2)dx}{\int_{-2}^{2}(4-x^2)dx} = \frac{2x^2 - 0.25x^4}{4x - x^3/3}$ =0 (not surprisingly)

The weighted average y-value; $\frac{\int_0^{4} y(2\sqrt{y})dy}{\int_0^{4}(2\sqrt{y})dy} = \frac{0.8y^{5/2}}{(4/3)y^{3/2}} = \frac{0.8*32}{32/3} = 2.4$

15a. $\int_0^{50} 30\left[110-80\cos\left(\frac{\pi}{20}(h-35)\right)\right]dy = 165{,}000$

b. Slice east-west in very thin slices (dy). Each slice's area is $30dy$ square meters and thus has $30dy \cdot L(y)$ leaves. These need to be raked a total of y meters so the total work done is

$\int_0^{50} 30y\left[110-80\cos\left(\frac{\pi}{20}(h-35)\right)\right]dy = 3.722$ million leaf-meters.

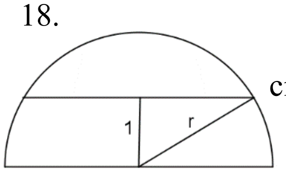
Reasonable? With 165,000 leaves; each is raked an average of approximately 25 meters. This would be 4.125 million leaf-meters, so it passes the "reality check".

16a. 2 m b. $\int_0^{15} \frac{50\pi x^2}{0.25x^2+1}dx = 7{,}617$ c. $\frac{1}{7{,}617}\int_0^{15} \frac{50\pi x^3}{0.25x^2+1}dx = 8.612$ d. $\int_0^{15} \frac{50\pi x^3}{0.25x^2+1}dx = 65{,}600$

17a. $\int_0^{15} 0.5\pi x \cdot \frac{100x}{0.25x^2+1}dx = 7{,}617$ b. 12.90

d. $\int_0^{15} 0.5\pi x \cdot \frac{100x}{0.25x^2+1}dx + \int_{15}^{15\sqrt{2}} \left(\frac{\pi}{2} - 2\cos^{-1}\left(\frac{15}{x}\right)\right)\cdot x \cdot \frac{100x}{0.25x^2+1}dx = 8{,}752$

18.



circular segments $\int_{r=1}^{r=2}\left(\frac{2\cos^{-1}(1/r)}{2\pi}\cdot \pi r^2 - \sqrt{r^2-1}\right)dy$ where $dy = 1.5dr$ so 1.383

shells $\int_{r=1}^{r=2}\left(\frac{2\cos^{-1}(1/r)}{2\pi}\cdot 2\pi r\cdot(3-1.5r)\right)dr = 1.383$

19. $\int_0^1 (0.85+0.4x)dx = 1.05$

20. The overlap is a cylinder plus two "buttons"—one on top and one on the bottom. Each "button" has a circular base of radius one and cross sections parallel to the length of the cylinder are rectangles whose heights h units form the center are $\sqrt{25-h^2}-\sqrt{24}$ and whose bases are $2\sqrt{1-h^2}$

So volume = $2\sqrt{24}\pi + 2\int_{-1}^{1}\left(2\sqrt{1-x^2}\cdot\left(\sqrt{25-x^2}-\sqrt{24}\right)\right)dx$

Unit 4: Extra Practice with Weighted Averages

In unit 2, the average value of $f(x)$ on [a,b] was defined as $\frac{1}{b-a}\int_a^b f(x)dx$

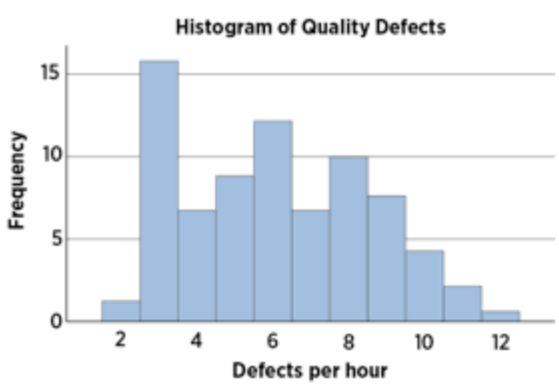
In unit 4, we are using weighted averages. The average value of $f(x)$ with weights $w(x)$ on the interval [a,b] is $\frac{\int_a^b f(x)\cdot w(x)dx}{\int_a^b w(x)dx}$

When do I use what?

For weighted averages, we need two variables: the variable we are trying to take the average of and the variable that shows how much we want to weight each observation. Think carefully about how they are represented! Does the y-variable represent the variable you are averaging? The weights? Neither?

Example #1

In the diagram below, let x be the defects per hour (on the x-axis) and y be the frequency.

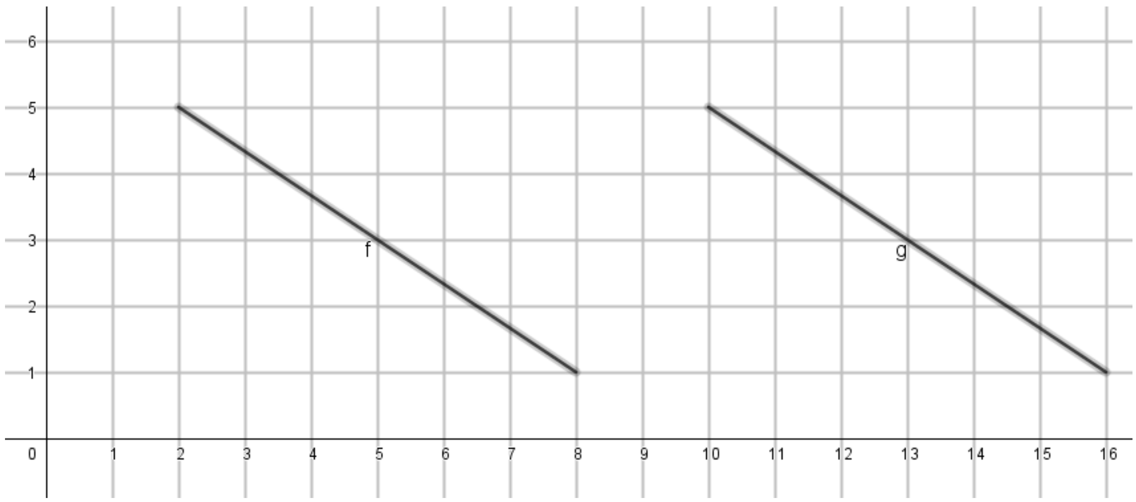


a. The expression $\frac{1}{11}\sum_{i=2}^{12} y$ represents an average value. What is it the average of?

b. Write an expression for the average number of defects per hour.

Example #2

On the chart below, the x value represents the arrival times (in minutes after noon) and y value is frequency (people per minute). The graph below shows the frequency of arrival times for two different groups of people, group f and group g.



a. Write an expression for the average value of $f(x)$ on the interval [2,8]. What does it represent?

b. True or false: average value of $f(x)$ on [2,8] equals average value of $g(x)$ on [10,16]?

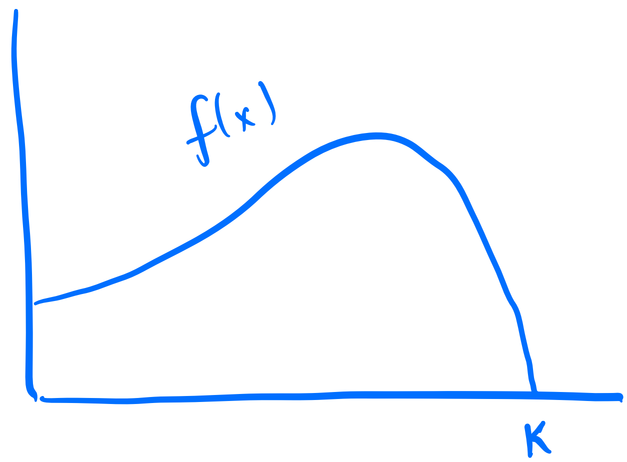
c. Is the average arrival time for group f the same as the average arrival time for group g? Explain.

d. Write an expression for the average arrival time for people in group f.

e. Imagine nobody leaves before minute 30. Write an expression for the average amount of time people in group g spent in the place.

Example #3

The surface of pond is shaped like the region bound by the x-axis and the graph of f below on the interval [0,k]. The depth is a function of the x-coordinate given by $h(x)$.



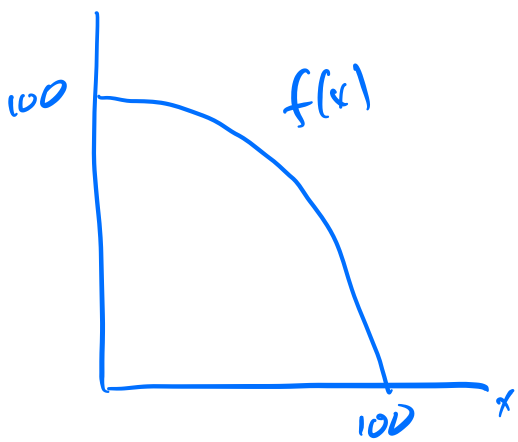
a. What is the meaning of $\frac{1}{k}\int_0^k f(x)dx$?

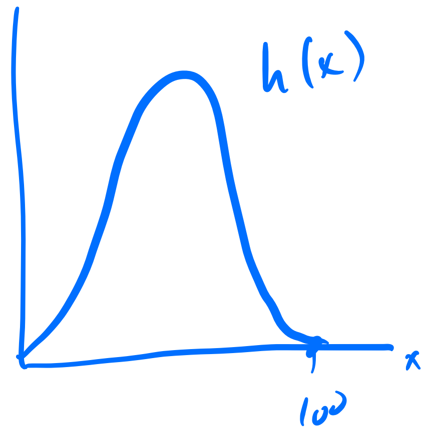
b. Write an expression showing the average depth of the pond.

c. True or false: the expression $\frac{1}{k}\int_0^k h(x)dx$ represents the average depth of the southern edge of the pond (the edge that lies along the x-axis)? Explain.

Example #4

The surface of pond is a quarter circle with radius 100 feet. The depth of the pond is a function of the distance from the center of the circle, given by $h(x)$.





a. What is the average value of $f(x)$ on the interval [0,100] and what does it mean?

b. What is the average depth of the pond.

c. Instead, if the depth was equal to the x coordinate, then what would the average depth be?

d. Instead, if the depth was equal to a function of the y-coordinate, given by $q(k) = 0.2e^{0.5k}$ then what is the volume of water in the pond?

Answers

1a. the average height of the bars. This has nothing to do with the number of defects per hour. b. $\dfrac{\sum_{i=2}^{12} xy}{\sum_{i=2}^{12} y}$

2a. $\frac{1}{6}\int_2^8 f(x)dx$. The average height of line segment f. It has nothing to do with arrival times.

b. true—both have an average y-value of 3. c. No: group f arrived much earlier on average (the last arrival from group f showed up before the first arrival from group g).

d. $\dfrac{\int_2^8 x \cdot f(x)dx}{\int_2^8 f(x)dx}$ (averaging x with weights f) e. $\dfrac{\int_{10}^{16} (30-x)\cdot g(x)dx}{\int_{10}^{16} g(x)dx}$ avg of (30-x) with weights g

3a. The average width of the pond b. $\dfrac{\int_0^k h(x)\cdot f(x)dx}{\int_0^k f(x)dx}$

c. True. Since it represents the southern edge of the pond, each piece of that edge is equally-weighted: the width of the pond does not factor into this calculation.

4a. the area is 2500π so 25π. It is the average width of the pond along the north/south axis.

b. Average h with weights that are arcs (with a tiny width) so

$$\frac{\int_0^{100} h(x)\cdot 0.25(2\pi x)dx}{\int_0^{100} 0.25(2\pi x)dx} = \frac{\int_0^{100} h(x)\cdot 0.25(2\pi x)dx}{2500\pi}$$

c. average x with weights that are thin rectangles cut vertically (since areas of equal depth are these thin rectangles)

$$\frac{\int_0^{100} h(x)\cdot f(x)dx}{\int_0^{100} f(x)dx} = \frac{\int_0^{100} h(x)\cdot \sqrt{10000-x^2}\,dx}{2500\pi}$$

d. equal depth regions are horizontal segments with tiny heights. So $\int_0^{100} 0.2e^{0.5y}\cdot\sqrt{10000-y^2}\,dy$

Where the $\sqrt{10000-y^2}$ is x in terms of y.

Unit 4 Handout #7: Practice Questions

Part I: Basic Volumes

1. Answer the following questions about the area bounded in the first quadrant by the graphs $x = 0,\ y = x^2 + 2,\ and\ \ y = x + 4$. You may use fnInt to evaluate the integrals once you set them up.

   a. What is the area of the region?

   b. What is the volume created by rotating the region around the *x*-axis?

   c. What is the volume created by rotating the region about the line *y*=-3?

   d. What is the volume created by rotating the region about the line *y*=6?

   e. What is the volume created by rotating the region around the *y*-axis? Hint: you may need to write two separate integrals. Or you can just use shells.

2. Find the area bounded by the curves $y = -x^2 + 3x \quad and \quad y = 2x^3 - x^2 - 5x$. Remember: area is by definition non-negative!

3. Answer the following questions about the region in the first quadrant bounded by the curves $y = 0 \quad y = x - 2 \ \ and \ \ y = \sqrt{x}$.

   a. What is its area?

   b. Find the volume when the region is revolved around the *x*-axis.

   c. Find the volume of the solid that results from the region being revolved around the line $y = -3$.

   d. Find the volume of the solid that results from the region being revolved around the line $x = -3$.

4. Answer the following questions about the region in the first quadrant bound by the *y*-axis and the lines $y = x \ \ and \ \ y = 12 - 2x$.

   a. Assume that a three-dimensional solid is created where this region is the base. As you slice it perpendicular to the *x*-axis, the cross-section of each slice is a square. Find the volume of such a solid.

   b. Instead, assume that a solid is created where each cross-section is a semi-circle whose diameter lies along the base. What is its volume?

5. Region R is in the first quadrant; it is bound by the graphs of $y = x \ \ and \ \ y = x^2$.

   a. Find the volume when R is revolved around the *x*-axis.

   b. Find the volume when R is revolved around the *y*-axis.

   c. Find the volume of the solid where parallel cross-sections perpendicular the *x*-axis are squares.

   d. Find the volume of the solid where parallel cross-sections perpendicular to the *x*-axis are equilateral triangles. (Find the area of an equilateral triangle with side *s*).

Part II: More Challenging Questions

6. A movie begins at 1 pm; the theater doors open at noon. The rate of entry into the theater t minutes after noon is given by the function $f(t) = 0.02(t-20)^2 + 5$ for $0 \le t \le 60$. The theater was empty at noon and no one leaves before the movie starts.

   a. How many people were in the theater at 1 pm?

   b. On average how many people were in the theater?

   c. What was the average rate of entry into the theater (in people per minute)?

   d. On average, how long did people wait for the movie to begin (ie, the time from when they entered until the movie began)?

7. The surface of a pond is the region in the first quadrant above the graph of $y = x + 6$ and below the graph of $y = 20 - 5\sqrt{x}$.

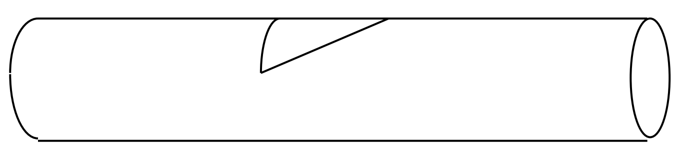
   a. Assume the pond gets deeper the further you are from the y-axis and its depth x units from the y-axis is given by the equation $d(x) = 2x + 5$. Find the average depth of the pond.

   b. Assume instead that the pond's depth is a function of the distance from the x-axis and this function is $d(y) = 0.2y$. Now calculate the average depth of the pond.

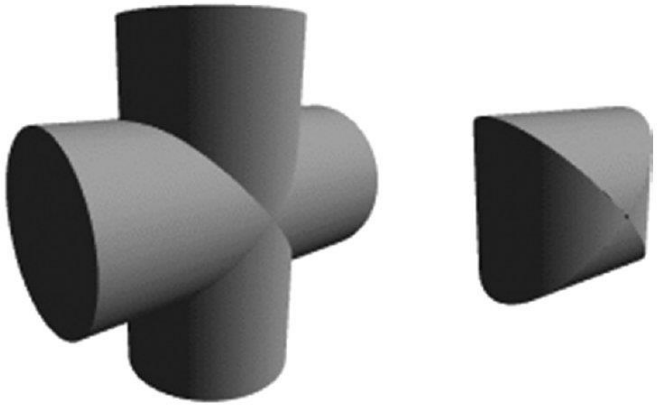
8. A cylindrical piece of metal has a radius of 10 cm and a height of 2 cm. The density of the metal depends upon the distance from the center of the cylinder (the center of each circular cross-section, to be more precise) according to the function $d(x) = 3 + 0.1x^2$. How much does the piece of metal weigh?

9. A bead is created by drilling a cylindrical hold with radius 2 cm through a sphere with radius 8 cm. Find the volume of the bead.

10. A notch is cut out of a log. The log has a radius of 20 cm. One cut is vertical and goes right to the center of the log. The other is at a 30° angle. What is the volume of the notch?



11. Two cylinders of radius one intersect such that the axes through their centers are perpendicular to each other. What is the volume of the region common to the two cylinders? Think of each slice of a cylinder vertical (in the diagram)—what shape is it and what do they intersect in?



12. The rectangle whose corners are (0,0), (2,2), (6,-2), and (4,-4) is revolved around the x-axis. What is the volume of the resulting solid?

13. When check-in for a flight opens, there are already 160 people waiting in line. Additional people arrive on line according to the function $a(t) = 40 - 0.5t$ (measured in people per minute) on the interval $0 \le t \le 60$. Then people stop entering. People leave the line when they have the chance to check in with an agent. For the first 20 minutes, people leave the line at the rate of 35 people per minute, then some agents go on break and people leave the line at a rate of 15 people per minute until the line is empty. I encourage you to sketch a graph to help answer the questions below:

a. What is the total number of people that check in?

b. What is the average arrival time of those arriving in the hour after check-in opens?

c. At t=16, is the line getting shorter or longer? How about at t=26? Give the rates.

d. At what times is the line getting longer?

e. What is the longest the line ever gets?

f. When does the line empty out completely?

g. Write a piecewise function for the number of people in line at time t.

h. On average, over the first 60 minutes, how many people are in line? FnInt OK.

i. Elaine gets into line 10 minutes after check-in opens. For how long is she in line?

j. If someone got checked in at t=50, what time did they enter the line?

Answers

1a. $\int_0^2 (-x^2+x+2)dx = 3.33$ b. $\int_0^2 \left[\pi(x+4)^2 - \pi(x^2+2)^2\right]dx = 80.42$

c. $\int_0^2 \left[\pi(x+7)^2 - \pi(x^2+5)^2\right]dx = 143.26$ d. $\int_0^2 \left[\pi(-x^2+4)^2 - \pi(-x+2)^2\right]dx = 45.24$

e. $\int_2^4 \pi\left(\sqrt{y-2}\right)^2 dy + \int_4^6 \left[\pi\left(\sqrt{y-2}\right)^2 - \pi(y-4)^2\right]dy$ =16p*pi/3

2. $\int_{-2}^0 (2x^3-8x)dx + \int_0^2 (8x-2x^3)dx$

3a. $\int_0^2 \sqrt{x}dx + \int_2^4 (\sqrt{x}-(x-2))dx \ \ or \int_0^4 \sqrt{x}dx - 2$ = 10/3 b. $\int_0^2 \pi\left(\sqrt{x}\right)^2 dx + \int_2^4 \pi[\left(\sqrt{x}\right)^2 - (x-2)^2]dx$ =16π/3

c. $\int_0^2 \pi[\left(\sqrt{x}+3\right)^2 - 9]dx + \int_2^4 \pi[\left(\sqrt{x}+3\right)^2 - (x+1)^2]dx$ =76π/3 d. $\int_0^2 \left[\pi(y+5)^2 - \pi(y^2+3)^2\right]dy$ =32.267π

4a. $\int_0^4 s^2 dx = \int_0^4 (12-3x)^2 dx = 192$ b. $\int_0^4 \frac{\pi r^2}{2} dx = \frac{\pi}{2}\int_0^4 (6-1.5x)^2 dx = 24\pi$

5a. $\int_0^1 \pi(x^2-x^4)dx$ b. $\int_0^1 \pi(y-y^2)dy$ c. $\int_0^1 (x-x^2)^2 dx$

d. $\int_0^1 \frac{\sqrt{3}(x-x^2)^2}{4}dx$ since the area of an equilateral triangle with side s is $\frac{s^2\sqrt{3}}{4}$

6a. $\int_0^{60} (0.02(t-20)^2+5)dt = 780$

b. ppl in theater is $P(t) = \frac{0.02(t-20)^3}{3} + 5t + 160/3$ so average is $\frac{1}{60}\int_0^{60} P(t)dt = 270$

c. 780/60=13 ppl/minute

d. $\frac{1}{780}\int_0^{60} (60-t)(0.02(t-20)^2+5)dt = 20.77$ or $\frac{1}{780}\int_0^{60} P(t)dt = 20.77$ (ppl minutes / ppl)

7. the surface of the pond is $\int_0^4 (14-5\sqrt{x}-x)dx = 21.33$

a. total water in pond is $\int_0^4 (2x+5)(14-5\sqrt{x}-x)dx = 160$ so avg depth is 7.5 units

b. total water in pond is $\int_6^{10} (0.2y)(y-6)dy + \int_{10}^{20} (0.2y)(4-0.2y)^2 dy = 13.87 + 33.33 = 47.2$ so avg is 2.21

8. using shells for each cross-section I get $\int_0^{10} 2\pi r(density)dr = \int_0^{10} 2\pi r(3+0.1r^2)dr = 2513.27$ or 800π

so 5026.54 or 1600π for the whole cylinder (since it is 2 cm high)

9. circle is $x^2+y^2=64$ and subtract the part where $-2\le x\le 2$ and revolve around the y-axis.

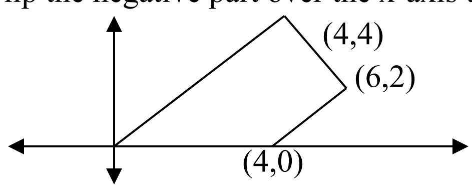
Using shells: $\int_2^8 2\pi x\cdot 2\sqrt{64-x^2}\,dx \approx 1946.77$

10. base is semi-circle of radius 20 cm and cross-sections perp to it are 30-60-90 triangles with side opposite 30 along the base so volume = $\int_{-20}^{20} 0.5(y)(y\sqrt{3})dx = \int_{-20}^{20} 0.5\sqrt{3}(400-x^2)dx = 9237.6$

11. $\int_{-1}^{1}(2r)^2\,dx = \int_{-1}^{1}(2\sqrt{1-x^2})^2\,dx = 4\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 16/3$

12. Flip the negative part over the x-axis to see what is highest…

Get:



So $V = \int_0^4 \pi x^2\,dx + \int_4^6 \pi\left((8-x)^2-(x-4)^2\right)dx = \frac{64\pi}{3} + \pi\int_4^6(48-8x)dx = \frac{64\pi}{3}+16\pi = \frac{112\pi}{3}$

13a. $160+\int_0^{60}(40-0.5t)dt = 1660$ b. at t=16, arrive at 32 per min and leave at 35 so getting shorter by 3 ppm at t=26 line is growing at the rate of 12 ppm   b. $\frac{1}{1500}\int_0^{60} t(40-0.5t)dt = 24$ min after check-in opens

c. for 0<t<20 line grows from $0\le t<10$; from 20<t<60 the line grows from 20<t<50 so [0,10) U (20,50)

d. line is longest at 10 or 50; at 10 it is $160+\int_0^{10}(40-0.5t)dt = 535$ minus 350 = 185 people

at t = 50 it is $160+\int_0^{50}(40-0.5t)dt = 1535$ minus (700+450) = 385 people so 385 (ugh!)

e. at t=60 there are 1660 entered and (700+600) served so 360 left, divide by 15 per minute means 24 more minutes, so when t=84

f. $L(t) = \begin{cases} -0.25t^2+5t+160 & 0\le t\le 20 \\ -0.25t^2+25t-240 & 20<t\le 60 \\ -15t+1260 & 60<t\le 84 \end{cases}$ g. $\frac{1}{60}\int_0^{60} L(t)dt = 276.67$

h. Elaine's place in line: $160+\int_0^{10} a(t)dt = 535$; gets checked in at 535/35 = 15.29 min, so only 5.29 min

i. Person number 700+15*30=1150 so $40t-0.25t^2+160 = 1150$ and t=30.6

Unit 5 Handout #1: Separable Differential Equations

1. Find $f(x)$ *or* y for each part below.

a. $f'(x) = -2x^2 + 4x - 1$ *and* $f(1) = 0$

b. $\dfrac{dy}{dx} = 2x - 3$ and it goes through the point (5,16)

c. $\dfrac{dy}{dx} = 0.05y$ and when x=2, y=50

d. $\dfrac{dy}{dx} = 0.2(y-1)$ and goes through (2,10)

e. $\frac{dy}{dx} = 0.3(2 - y)$ and goes through (1,5)

f. $\frac{dy}{dx} = xy$ and goes through (0,20)

g. $\frac{dy}{dx} = x^2 y + 2y$ and goes through (0,5)

h. $\frac{dy}{dx} = xy^2$ and goes through (1,2)

2. A function has a graph so that the slope of the tangent line at a point (x,y) is always equal to x/y^2. The function goes through the point (2,3). What is its equation?

3. A function's graph is such that the slope of the tangent line at any point (x,y) is equal to one half of the slope of the secant line from that point to the point (0,0) on the graph. It also goes through the point (1,1). Find its equation.

4. Write a differential equation for each scenario below. You do not need to solve it.

a. A population of fish increases at a continuous rate of 20% per year. The fish are being harvest at a constant rate of 10 million per year. If P is the fish population (in millions), what is dP/dt?

b. Money in the bank earns a 5% annual rate (compounded continuously). Additionally, money is added steadily at a rate of $1000 per year. How does the balance change over time (ie dB/dt=__)?

c. A pollutant spilled on the ground decays at a rate of 8% per day naturally. Additionally, a clean-up crew cleans up 30 gallons per day. How does the amount of pollutant change over time?

5. A population of laboratory animals grows over time in the following way: its weekly growth is equal to 12% of its population, but its population is reduced continuously at the rate 60 organisms per week as the scientists take animals for experiments. Given that its initial population is 800, find its population at any time *t*. What is $\lim_{t\to\infty} P(t)$?

6. A population of laboratory animals grows over time in the following way: its weekly growth is equal to 12% of its population, but its population is reduced continuously at the rate 120 organisms per week as the scientists take animals for experiments. Given that its initial population is 800, determine its population at any time *t*. What is $\lim_{t\to\infty} P(t)$?

7. A potato with a temperature of 70° is placed into a 400° oven. Its temperature increases of time in the following way: it grows at the rate of 3 percent of the temperature difference (between where it is and 400°) per minute. Find the function describing the potato's temperature and determine when it reaches 380°.

8. Money left in a bank account (m) is growing at the instantaneous rate $\frac{dm}{dt} = 0.07m$, where t is the number of years since the account was opened. Seven years after the account was opened, the balance was $5200.

a. Write the function describing the amount of money in the account at time *t*.

b. What was the average balance over the first three years the account was open?

c. When did the balance reach $10000? Solve analytically, if you can.

9. When someone is doing a new job or task, the "learning curve" describes what percent effectiveness the person is (out of 100%) at any point in time in their tenure. Let *k* reflect a person's effectiveness after *t* months on the job.

a. In a typical learning curve, the learning (increase in *k*) is proportional to what is unknown. Let us assume that $\frac{dk}{dt} = 0.4(1 - k)$. Find the general solution to this differential equation.

b. Assume the person initially is 30% effective. Now find the equation for *k*.

c. How effective is the person after 6 months, and at what rate is he or she learning?

d. When is the person 95% effective?

e. On average, what percent effective was the person on his/her first 6 months?

Answers

1a. $f(x) = \frac{-2}{3}x^3 + 2x^2 - x - \frac{1}{3}$ b. $y = x^2 - 3x + 6$ c. $y = 45.24e^{0.05x}$

d. $y = 6.03e^{0.2x} + 1$ e. $y = 4.05e^{-0.3x} + 2$ f. $y = 20e^{0.5x^2}$ g. $y = 5e^{x^3/3+2x}$ h. $y = \frac{2}{2-x^2}$

2. $\frac{y^3}{3} = \frac{x^2}{2} + 7$ *so* $y = \sqrt[3]{1.5x^2 + 21}$ 3. $\ln y = 0.5 \ln x$ *so* $y = \sqrt{x}$

4a. $\frac{dP}{dt} = 0.2P - 10$ b. $\frac{dB}{dt} = 0.05B + 1000$ c. $\frac{dP}{dt} = -0.08P - 30$

5. $\frac{dP}{dt} = 0.12P - 60$ *so* $P = 300e^{0.12t} + 500$ $\lim_{t\to\infty} P(t)$ is infinity, as population grows and grows

6. $\frac{dP}{dt} = 0.12P - 120$ *so* $P = -200e^{0.12t} + 1000$ $\lim_{t\to\infty} P(t)$ is 0 (or negative infinity) as it dies out

7. $\frac{dP}{dt} = 0.03(400 - P)$ *so* $P = 400 - Ce^{-0.03t} = 400 - 330e^{-0.03t}$ hits 380 degrees when t=93.4

8a. $m = 3186e^{0.07t}$ b. $\frac{1}{3}\int_0^3 3186e^{0.07t}\,dt = 3545$ c 16.34 years

9a. $k = Ce^{-0.4t} + 1$ (you may have a –C, it doesn't matter) b. $k = -0.7e^{-0.4t} + 1$

c. when t=6, k=93.65% and dk/dt=0.4(1-k) which is 2.54% per month

d. about 6.6 months e. $\frac{1}{6}\int_0^6 (1 - 0.7e^{-0.4t})dt = 73.5\%$

Unit 5 Handout #2: Differential Equations and Slope Fields

1. Find $f(x)$ or y given the following.

a. $\dfrac{dy}{dx} = 2x + xy$ and goes through (0,7)

b. $\dfrac{dy}{dx} = \dfrac{2x}{y}$ and goes through (1,10)

c. $\dfrac{dy}{dx} = y + 2xy$ and goes through (0,12)

d. $\dfrac{dy}{dx} = 0.12y$ and when x=3, y=80

e. $\dfrac{dy}{dx} = 0.12(y-1)$ and when x=3, y=80

f. $\dfrac{dy}{dx} = \dfrac{2x-3}{y}$ and goes thru (1,-5)

g. $\dfrac{dy}{dx} = y^2 + 3x^2y^2$ and goes through (1,4)

2. Newton's law of cooling says that an object's temperature adjusts to room temperature at a rate that is proportional to the difference between its temperature and room temperature. Assuming room temperature is 20°C, we can say that $\frac{dH}{dt} = k(H - 20)$, where *t* is time, *H* is temperature, and *k* is a constant that depends on the object (a baked potato cools at a different rate than a cup of coffee).

a. Do you expect *k* to be positive or negative? Explain?

b. Solve this equation for *H*. It will be in terms of *k* and some constant *C*.

c. Assume an object's initial temperature is 50°C. This enables you to find one of the parameters (*C* or *k*). Find it.

d. After 30 minutes, the temperature is 35°C. Find the other parameter (use 2 significant digits). Now you know *H(t)*.

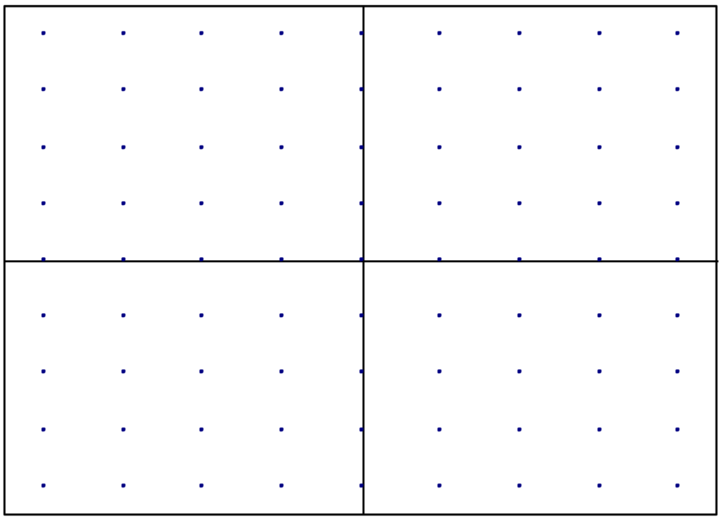
e. What is $\lim_{t\to\infty} H(t)$? What does this mean?

f. What is the object's temperature after 40 minutes?

g. Use logs to find when the object's temperature hits 25°C.

h. What is the average temperature over the first 30 minutes? Find analytically.

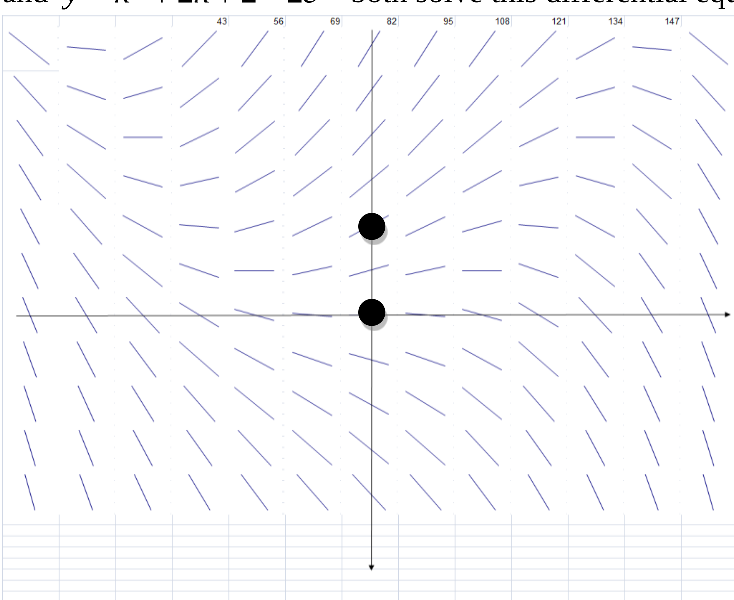
3. Draw a slope field showing $\frac{dy}{dx} = \frac{-x}{y}$. Then solve it analytically. Assume one point on the solution is (1,3). Then use derivatives to verify that the equation of the solution solves the differential equation.



4. The graph below shows slope fields for the equation $\frac{dy}{dx} = y - x^2$.

a. Draw solutions through (0,2) and (0,0). These points are marked for you.

b. One solution should be a parabola and the other is not. Use derivatives to show that $y = x^2 + 2x + 2$ and $y = x^2 + 2x + 2 - 2e^x$ both solve this differential equation.



Answers

1a. $y = 9e^{.5x^2} - 2$ b. $y = \sqrt{2x^2 + 98}$ c. $y = 12e^{x^2+x}$ d. $y = 55.81e^{0.12x}$

e. $y = 55.12e^{0.12x} + 1$ f. $y = -\sqrt{2x^2 - 6x + 29}$ g. $y = \dfrac{-1}{x^3 + x - 2.25}$

2a. negative since when H>20 it falls (H'<0) and when H<20, H'>0

b. $H(t) = Ce^{kt} + 20$ c. C=30 so $H(t) = 30e^{kt} + 20$ d. k=-0.023 $H(t) = 30e^{-0.023t} + 20$

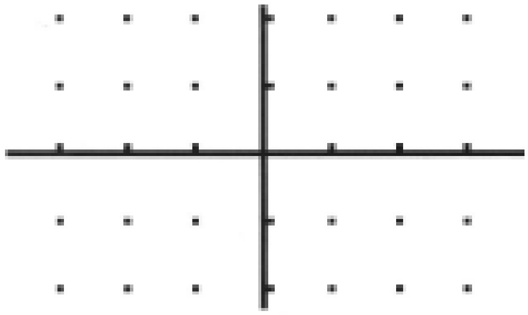
e. 20.. eventual temperature is 20, which should be no surprise.. f. $H(40) = 31.96$

g. $30e^{-0.23t} + 20 = 25$ *so* $e^{-0.023t} = 1/6$ *and* $t = \ln(1/6)/(-0.023) = 77.9$ minutes

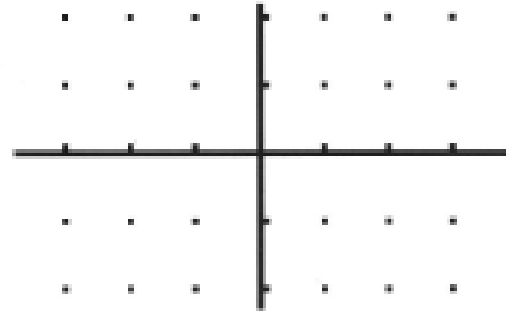
h. $\dfrac{1}{30}\int_0^{30}(30e^{-0.023t} + 20)dt = 41.67$ 3. $x^2 + y^2 = 10$ *or* $y = \pm\sqrt{10 - x^2}$

Draw a slope field for each of the following differential equations.

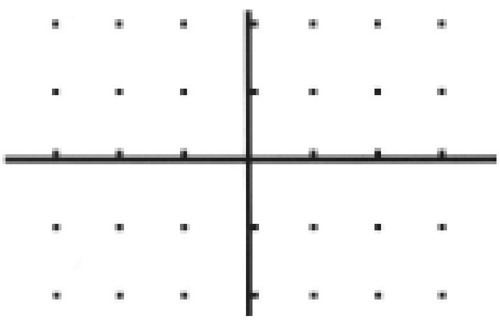
1. $\dfrac{dy}{dx} = x + 1$



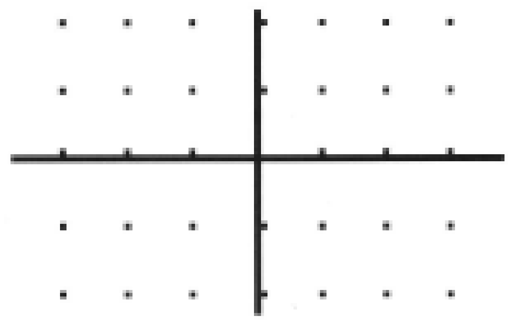
2. $\dfrac{dy}{dx} = 2y$



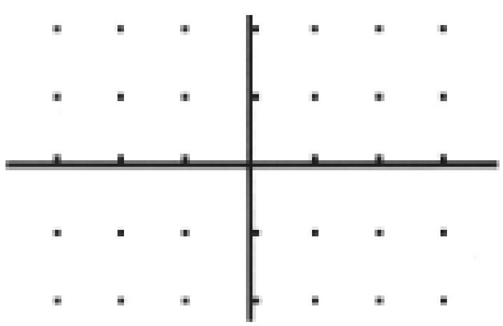
3. $\dfrac{dy}{dx} = x + y$



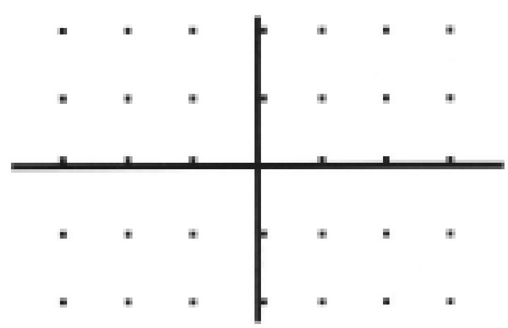
4. $\dfrac{dy}{dx} = 2x$



5. $\dfrac{dy}{dx} = y - 1$

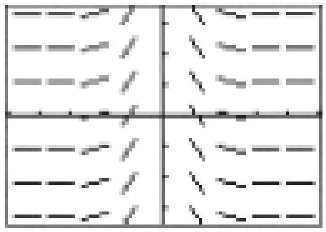


6. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

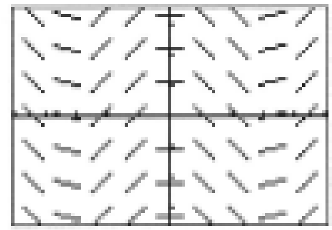


Match each slope field with the equation that the slope field could represent.

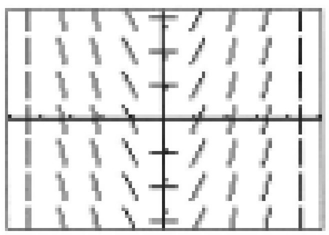
(A)



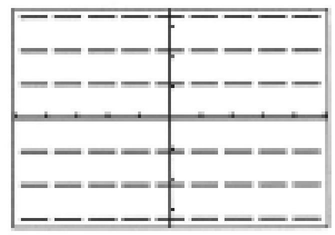
(B)



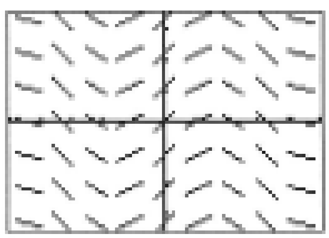
(C)



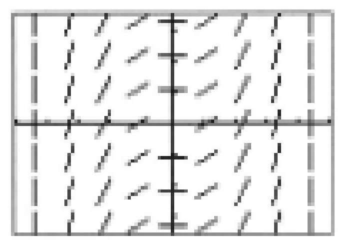
(D)



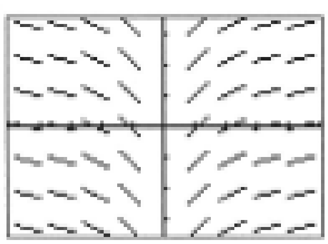
(E)



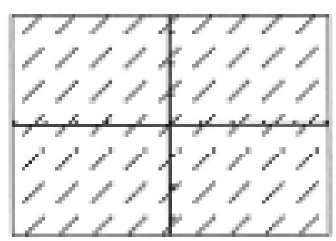
(F)



(G)



(H)



7. $y = 1$

8. $y = x$

9. $y = x^2$

10. $y = \frac{1}{6}x^3$

11. $y = \frac{1}{x^2}$

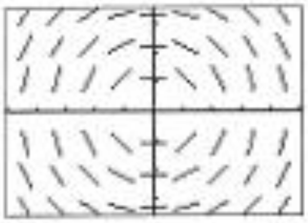
12. $y = \sin x$

13. $y = \cos x$

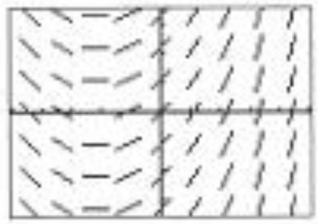
14. $y = \ln|x|$

Match the slope fields with their differential equations.

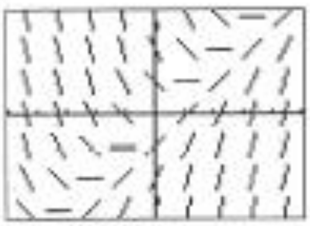
(A)



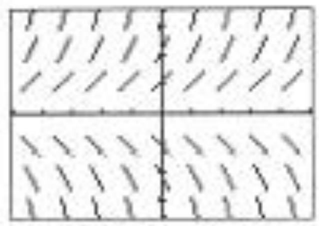
(B)



(C)



(D)



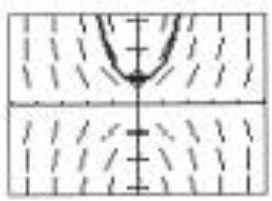
15. $\frac{dy}{dx}=\frac{1}{2}x+1$

16. $\frac{dy}{dx}=y$

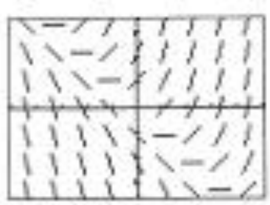
17. $\frac{dy}{dx}=x-y$

18. $\frac{dy}{dx}=-\frac{x}{y}$

19. The calculator drawn slope field for the differential equation $\frac{dy}{dx}=xy$ is shown in the figure below. The solution curve passing through the point (0, 1) is also shown.
(a) Sketch the solution curve through the point (0, 2).
(b) Sketch the solution curve through the point $(0,-1)$.

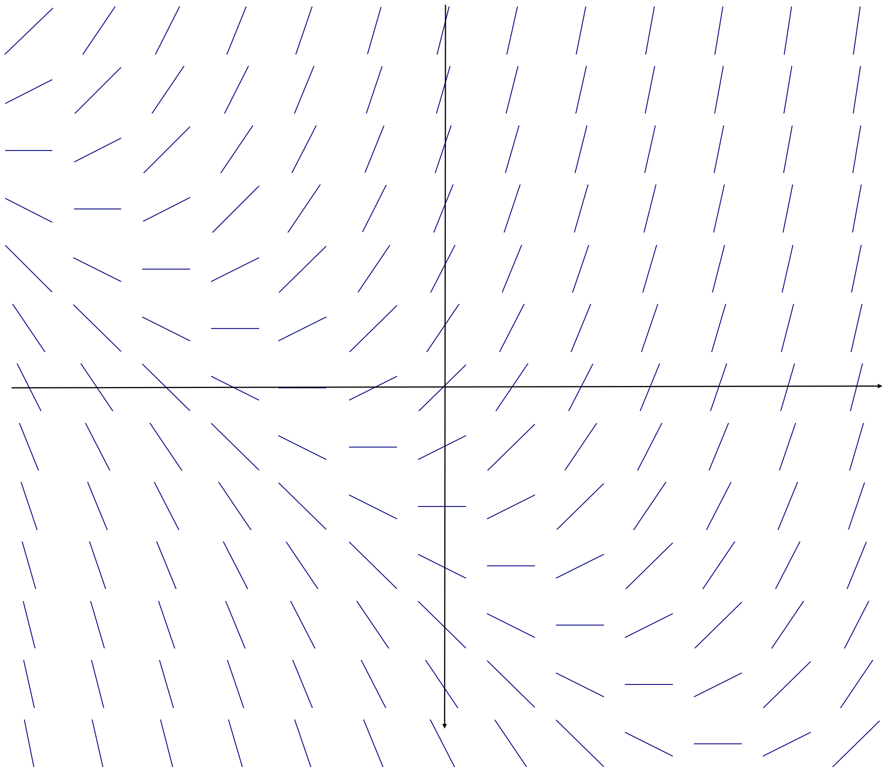


20. The calculator drawn slope field for the differential equation $\frac{dy}{dx}=x+y$ is shown in the figure below.
(a) Sketch the solution curve through the point (0, 1).
(b) Sketch the solution curve through the point $(-3,0)$.



Unit 5 Handout #4: Slope Fields and Euler's Method

1. This is the slope field for $\frac{dy}{dx} = \frac{x+y}{2} + 1$



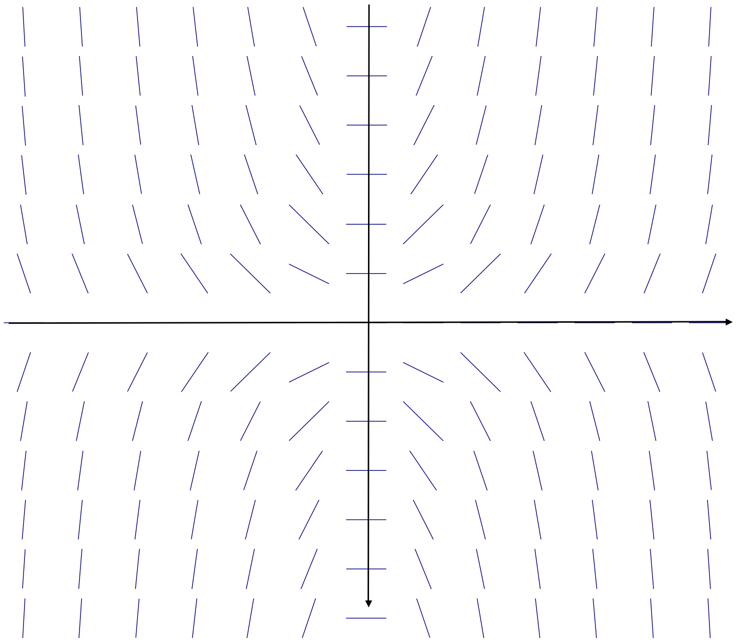
a. Sketch the solution through (-1,-2).

b. Use Euler's Method to approximate that solution. Use an increment(Δx) of 0.5. Euler's Method is just an application of linearization. You approximate successive points by linearizing the function from the last point. Fill in the table below to find the approximate *y* when *x*=2:

x	***y***	**Slope**	**Δ*x***	**Δ*y***
-1	-2	-0.5	0.5	-0.25
-0.5	-2.25			
0				
0.5				
1				
1.5				
2				

2a. This is the slope field for which of the following? Why?

i. $\frac{dy}{dx} = 0.3y$ ii. $\frac{dy}{dx} = 0.3x$ iii. $\frac{dy}{dx} = 0.5xy$ iv. $\frac{dy}{dx} = \frac{0.5x}{y}$



b. Trace the solution through (0,2)

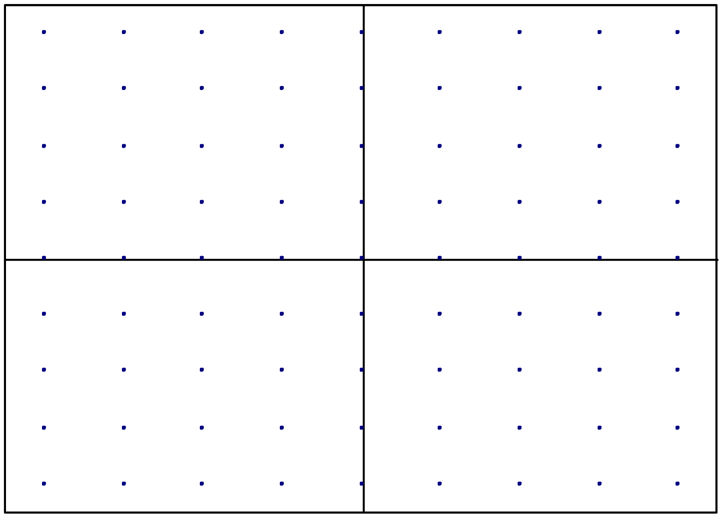
c. Use Euler's Method to approximate the solution to the equation with the initial value (0,2). Find the *y*-value when *x*=3 using increments of 0.5. The table below may help.

x	***y***	**Slope**	**Δ*x***	**Δ*y***
0				
0.5				
1				
1.5				
2				
2.5				

d. Solve the differential equation analytically and find the exact value of the solution through (0,2) when *x*=3.

3. You are trying to solve the differential equation $\frac{dy}{dx} = \frac{xy^2}{4}$

a. Draw a slope field below, for the integers between -4 and 4 for each *x* and *y*. Use symmetry to build it outside of the first quadrant.



b. Sketch solution curves through the points (1,-4) and (-3,2).

c. For the solution curve through (1,-4). Use Euler's method to approximate *y* when *x*=2 and *x*=3. Use an increment (Δx) of 0.5.

d. Solve the differential equation by separation of variables. Find the exact answers to the questions in part *c* above analytically.

4. The general solution to the differential equation $\frac{dy}{dx} = \frac{1-2x}{y}$ is a family of curves of what shape?

5. One point on the solution to the differential equation $\frac{dy}{dx} = -0.2xy^2$ is (5,2).

a. Use linearization ("tangent-line approximation") to approximate *y* when *x*=5.2.

b. Use implicit differentiation to find the second derivative $\frac{d^2y}{dx^2}$ at the point (5,2). Doe this tell you your approximation in part *a* above is likely to be too high or too low?

c. Solve the differential equation and find the exact *y* when *x*=5.2.

6. The rate of change of a population of wolves $N(t)$ is proportional to $350 - N(t)$. In other words, $\frac{dN}{dt} = k(350 - N)$ where *k* is a constant. When *t*=0 the population is 200 and when *t*=3 the population is 300. Write the equation describing the population at time *t* and find the population at *t*=5.

7. The thickness of ice growing on a lake's surface, $I(t)$ (measured in inches), satisfies the equation $I'(t) = \frac{3}{I(t)}$, where *t* is measured in hours. Given that the initial thickness is one inch, determine when the ice becomes two inches thick. Verify that your function solves the differential equation by taking derivatives.

8. A person is at a new job and is learning as fast as she can. The total “knowledge” that will enable her to do the job perfectly is 300 units. The instantaneous change in her knowledge $\left(\dfrac{dk}{dt}\right)$, in knowledge units per week, is equal to 10% of the knowledge which she does not yet have.

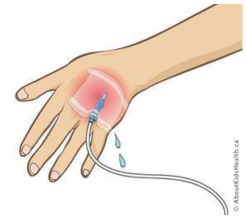
a. Assume that her initial knowledge is 60 units. Use Euler’s method to approximate her knowledge four weeks later. Use an increment of one week.

b. Write a differential equation showing how her knowledge changes with respect to time.

c. Solve the differential equation analytically and compare your answer to your approximation from part *c*.

d. Verify using derivatives.

9. A certain population increases at a rate proportional to the square root of the population (call this constant of proportionality k). The initial population of 2500 grew to 3600 in five years. Use this to find c and k and then determine when the population will hit 7500.



10. A body initially has 30 mg of a drug in it. The drug is administered intravenously at a rate of 10 mg per hour. The body also metabolizes the drug at a rate of 20% per hour, so that 20% of the hour's initial concentration leaves the bloodstream continuously over the hour.

a. Write a differential equation describing the change in the concentration of the drug (in mg/hour).

b. Use Euler's method to find the concentration over the first four hours; use a one-hour increment.

c. Solve it using separation of variables.

d. What is the long-term equilibrium amount of drug in the body? Why is this not surprising?

e. How would your answers to *c* and *d* change if the initial amount was 200 mg?

f. Draw a slope field on your calculator showing the concentration over time. Set your window to reasonable *x*'s and *y*'s before running the program.

g. For different positive initial values, what does your slope field show?

11. The growth in W is proportional to 1000-W. When t=0, W=200 and when t=5, W=350. What is W when t=13?

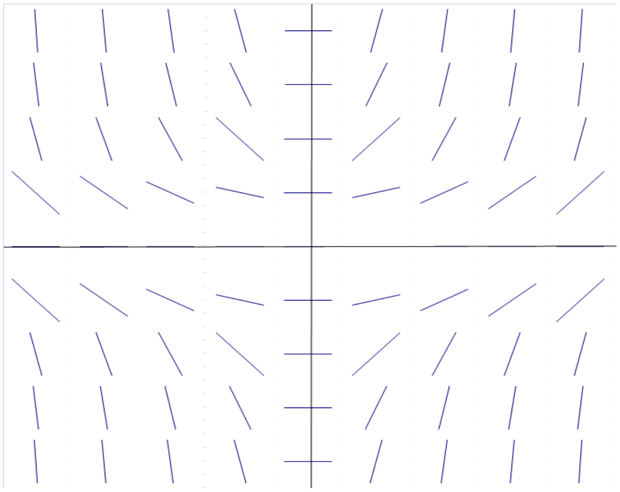
12. A tank contains 100 L of pure water (with no salt in it). Brine that contains 0.1kg of salt per liter enters the tank at a rate of 10 L/min. The contents of the tank are mixed together and water drains from the tank at the same rate, 10L/min. How much salt is in the tank after 6 minutes? Hint: define S as the amount of salt (in kg) at any point in time.

Answers

1b. I got (2,-2.19)… the number in the lower-left corner is the y that corresponds to x=2.5 not x=2.

2a. it is iii c. I get 9.43 d. $y = 2e^{0.25x^2}$ which goes through (2,18.98)… the approximation in part c was way off because the slope is changing so quickly.

3a.



c. I get (2,-1.25) and (3,-0.63) d. $y = \dfrac{8}{-1-x^2}$ which goes through (2,-1.6) and (3,-0.8)

4. these are ellipses

5a. $f(5.2) \approx f(5) + 0.2f'(5) \approx 2 + 0.2(-4) = 1.2$

b. $\dfrac{d^2y}{dx^2} = -0.2\dfrac{dx}{dx}y^2 - 0.2x \cdot 2y\dfrac{dy}{dx} = -0.8 - 4(-4) = 15.2$; concave up so approx. of 1.2 is too low

c. $\dfrac{-1}{y} = -0.1x^2 + C \;\; so\ C = 2\ and\ y = \dfrac{1}{0.1x^2 - 2}$ so when x=5.2 y=1.42

6. $N(t) = 350 - 150e^{-0.366t}$ so $N(5) \approx 326$ 7. $I = \sqrt{6t+1}$ so I=2 when t= ½ hour

8a about 142.5 b. $\dfrac{dk}{dt} = 0.1(300 - k)$ c. $k = 300 - 240e^{-0.1t}$ so after 4 weeks k is 139.12

9. $\dfrac{dP}{dt} = k\sqrt{P}$ so $2\sqrt{P} = kt + C$ when t=0 P=2500 so C=100 so $2\sqrt{P} = kt + 100$

And when t=5 P=3600 so k=4 so $2\sqrt{P} = 4t + 100$ and $P = (2t+50)^2$ and P=7500 when t≈18.3

10a. $\dfrac{dy}{dt} = -0.2y + 10$ b. (0,30) (1,34) (2,37.2) (3,39.76) (4,41.808) c. $y = -20e^{-0.2t} + 50$

d. 50, since at equilibrium the 20% leaving the body equals the 10 mg coming in… 0.2x=10 so x=50

e. $y = 150e^{-0.2t} + 50$ still approaches equilibrium of 50 g. always asymptotically approaches 50

11. $\dfrac{dW}{dt} = k(1000 - W)\ so\ \dfrac{dW}{1000 - W} = kdt\ \ and\ -\ln(1000 - W) = kt + C$

So $W = 1000 - e^{-kt-C} = 1000 - C'e^{-kt}$ since (0,200) C=800

So $W = 1000 - 800e^{-kt}$ and (5,350) indicates that -5k=ln(650/800) so k=0.0415 and When t=13 W=533.57

12. Rate of inflow of salt is 1 kg per minute and rate of outflow of salt is 1/10 of S

$\dfrac{dS}{dt} = 1 - 0.1S = -0.1(S - 10)$ given initial S is zero we get $S = 10 - 10e^{-0.1t}$ so when t=6 S=4.512 kg

Unit 5 Handout #5: Review Problems

1. The graph of an equation has an interesting property: the normal line through any point (x,y) on the graph happens to go through the point (2,0) also. The graph goes through the point (1,1). What is the equation of the graph? Do you recognize the shape of the graph from its equation?

2. The growth in a population is proportional to the cube-root of the population. When t=5, the population is 100 and when t=10 then population is 200. When does the population reach 500?

3. A town's population is initially 5000. There are two sources of growth of the population: it grows by an annual rate of 5% (continuously compounded) and an additional 100 people move into the town each year.

a. Write the differential equation that describes how the population changes.
b. Solve the differential equation, using the initial value of 5000.
c. Exactly when will the population reach 10000?

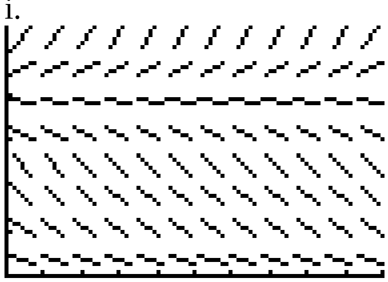
4. Newton's Law of Cooling states that the rate of change in an object's temperature is a constant proportion of the difference between its temperature and the ambient temperature. Assume a 180° cup of coffee is placed in a 70° room. Ten minutes later the coffee's temperature is 120°.

a. Write a differential equation in terms of k (the "cooling proportion") describing the temperature.
b. Solve the differential equation in terms of k and some constant C. Then find their values using the temperature data given.
c. Exactly when does the temperature hit 100°?

5. A population grows in a way that growth is proportional to current size of the population and to the "room to grow"—how far below capacity the population is. For this particular population, changes are modeled by the function $\frac{dP}{dt} = 0.05P(8-P)$, with an initial value of 2. The units are in 1000s per year.

a. Use Euler's Method to approximate the population after 1 year, using an increment of 0.5 years.

b. Which of the following slope fields represents this scenario? All have windows of [0,20] for x and [0,11] for y. Why?

i.


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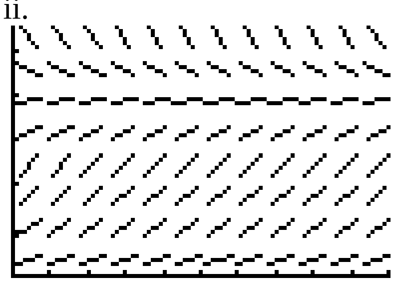

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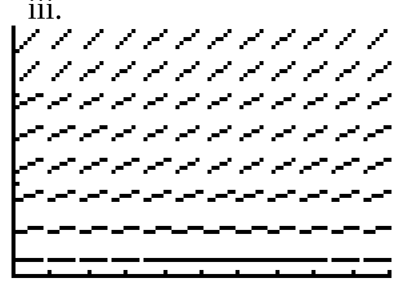


c. Derivatives and algebra review (optional): verify that the solution is $P = \frac{8}{1+3e^{-0.4t}}$. Check that it satisfies the initial value and the differential equation.

6. Find the specific solutions to the following differential equations.

a. $\dfrac{dy}{dx}=\dfrac{1}{1+2x}$ through (0,4)

b. $\dfrac{dy}{dx}=\dfrac{x+2}{y}$ through (1,8)

c. $\dfrac{dy}{dx}=\dfrac{1+2y}{x}$ through (1,5)

d. $\dfrac{dy}{dx}=\dfrac{y}{2x}$ through (5,12)

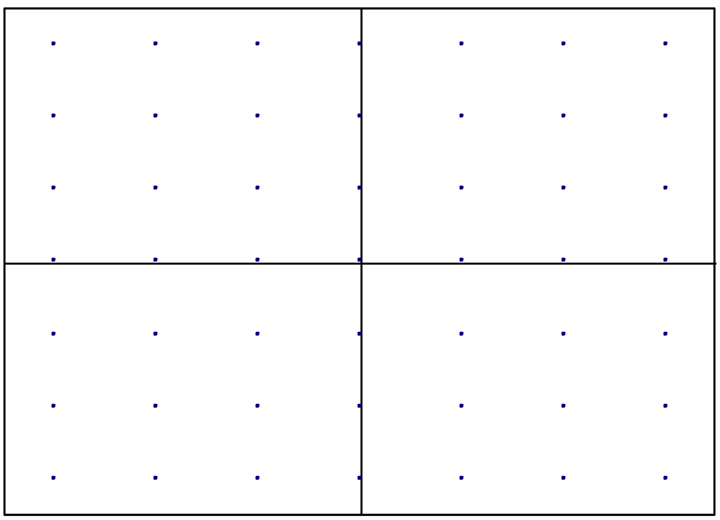
e. $\dfrac{dy}{dx}=\dfrac{-xe^{x^2}}{4}$ through (0,10)

f. $\dfrac{dy}{dx}=\dfrac{\sqrt{y}}{x}$ through (9,5)

g. $\dfrac{dy}{dx}=\dfrac{y}{x}-y$ through (1,4)

7. Answer the following questions about the differential equation $\dfrac{dy}{dx}=\dfrac{2x}{y^2}$.

a. Draw a slope field below for the equation from [-3,3] and [-3,3].



b. Draw solution curves for initial values of (-2,3) and (3,2).

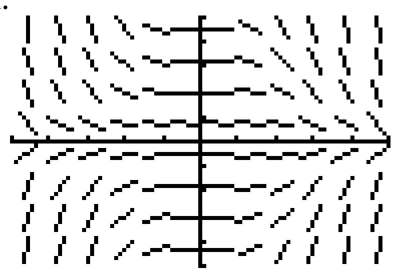
c. Given a starting point if (-2,3), use Euler's Method to find the value of y when x=0. Use an x-increment of 0.5.

d. Solve the equation analytically for the exact solutions with the initial points (-2,3) and (3,2).

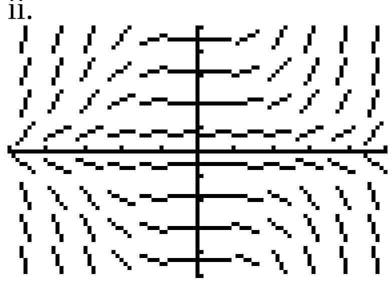
8. Given that $\dfrac{dy}{dx}=-0.1x^2y$.

a. Which slope field below represents this problem? Why? All are [-5,5] for both x & y.

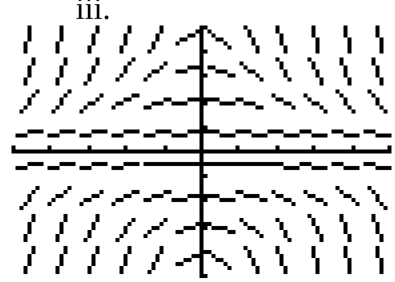
i.



ii.



iii.



b. On the correct one, sketch solutions with an initial value of (-4,3) and with an initial value of (-2,-4).

c. Solve the equation by separation of variables. Use the starting value of (-4,3).

9. Answer the following questions about the differential equation $\frac{dy}{dx} = \frac{xy}{x+3}$. Assume the solution goes through the point (2,1).

a. Without solving the differential equation, approximate *y* when *x*=2.5. (use linearization).

b. Find $\frac{d^2y}{dx^2}$ at the point (2,1). Does this make you feel that your approximation in part *a* above is too low or too high?

c. Solve the differential equation and find the actual *y* when *x*=2.5.

10. Pure water is flowing into a tank at a rate of 10 liters per minute. The tank initially has 30 kg of salt in 200 liters of water. The salt is well mixed and water leaves the tank at a rate of 10 liters per minute. When does the tank have exactly 20 kg of salt in it? 10 kg of salt?

11. Challenge: A tank has pure water flowing into it at 12 liters/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liters/min. Initially, the tank contains 10 kg of salt in 100 liters of water. How much salt is in the tank after 30 minutes? Note that the amount of water in the tank is increasing! Your differential equation should have both *t* and *S* in it!

Answers

1. At (x,y) the normal line has slope of y/(x-2) so the tangent line has slope -(x-2)/y so $\frac{dy}{dx} = \frac{-(x-2)}{y}$

→ $y^2 = -x^2 + 4x - 2$. This is a circle with center (2,0) and radius $\sqrt{2}$.

2. $\frac{dP}{dt} = k\sqrt[3]{P}$ so $1.5P^{2/3} = kt + C$ given that P=200 when t=10 and P=100 when t=5 we get

5k=1.5($\sqrt[3]{200^2} - \sqrt[3]{100^2}$) and k≈3.80 and C≈13.3 so $1.5P^{2/3} = 3.8t + 13.3$ & when P=500 t=21.37

3a. $\frac{dP}{dt} = 0.05P + 100$ b. $P = 7000e^{0.05t} - 2000$ c. 10.78 years

4a. $\frac{dH}{dt} = k(H - 70)$ (where k<0) b. $H = 70 + Ce^{kt}$ so C=110 and k=-0.0788 c. about 16.5 min

5a. (0,2) $\frac{dP}{dt} = 0.05(2)(6) = 0.6$ since Δt=0.5, ΔP=(0.5)(0.6)=0.3 – increased P by 0.3 so (0.5, 2.3)

(0,5, 2.3) $\frac{dP}{dt} = 0.05(2.3)(5.7) = 0.66$; Δt=0.5, ΔP=(0.5)(0.66)=0.33 – so P inc by 0.33 (1, 2.63)

b. it must be (ii) because it should be positive between y-axis is 0 and 8 and negative elsewhere

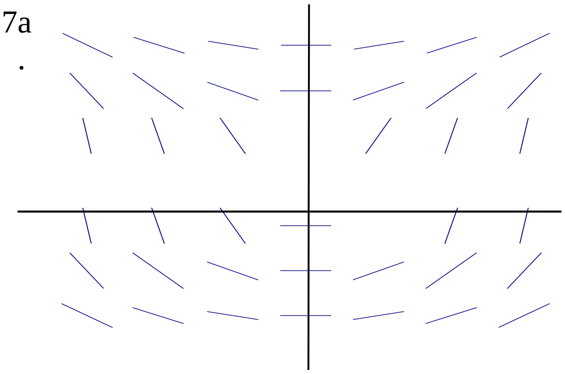
c. $-8(1+3e^{-0.4t})^{-2}(-1.2e^{-0.4t}) = (0.05)\left(\frac{8}{1+3e^{-0.4t}}\right)\left(8 - \frac{8}{1+3e^{-0.4t}}\right)$ checks with some work…

6a. $y = 0.5\ln(2x+1)+4$ b. $y = \sqrt{x^2+4x+59}$ c. $y = \dfrac{cx^2-1}{2}$ thru (1,5) instead it is $y = \dfrac{11x^2-1}{2}$

d. $y = \dfrac{12\sqrt{x}}{\sqrt{5}}$ e. $y = -0.125e^{x^2} + 10.125$ f. $y = \left(\dfrac{\ln x + C}{2}\right)^2$ $y = \left(\dfrac{\ln x + 2.27}{2}\right)^2$

g. $y = Cxe^{-x}$ *so* $y = 4exe^{-x}$ *or* $y = 4xe^{-x+1}$

7a.



c. (-2,3) → (-1.5, 2.78) → (-1,2.58) → (-0.5, 2.43) → (0,2.35)

d. $y = \sqrt[3]{3x^2 + C}$ (-2,3) is $y = \sqrt[3]{3x^2+15}$ (3,2) $y = \sqrt[3]{3x^2-19}$

8a. i: when y>0, slope < 0 and vice versa c. $y = Ce^{-0.1x^3/3}$ specific solution is $y = 0.355e^{-0.1x^3/3}$

9a. $f(2.5) \approx f(2) + 0.5f'(2) = 1 + 0.5(0.4) = 1.2$

b. $\dfrac{d^2y}{dx^2} = \dfrac{(x+3)\left[x\dfrac{dy}{dx}+y\right]-xy}{(x+3)^2} = \dfrac{5(1.8)-2}{25} = 0.28$; 1.2 is probably (not definitely) too low for $f(2.5)$

c. $\dfrac{dy}{y} = \dfrac{x}{x+3}dx = \dfrac{x+3-3}{x+3}dx = \left(1-\dfrac{3}{x+3}\right)dx$ so $\ln y = x - 3\ln(x+3)+C$

$0 = 2 - 3\ln 5 + C$ *so* $C \approx 2.828$ and $\ln y = x - 3\ln(x+3) + 2.828$

So $y = \dfrac{16.917e^x}{(x+3)^3}$ and $f(2.5) = 1.239$; the 1.2 approximation was a bit low….

10. $\dfrac{dS}{dt} = -0.05S$ so $S = Ce^{-0.05t}$ *and* $S = 30e^{-0.05t}$

so there are 20kg when $t = -20\ln(2/3) \approx 8.11$ min and 10kg when $t = -20\ln(1/3) \approx 21.97$ min

11. Volume of water in tank is V=100+2t so portion of water leaving is $\dfrac{10}{100+2t}$ and rate that salt leaves is $\dfrac{10S}{100+2t}$. So $\dfrac{dS}{dt} = \dfrac{-10S}{100+2t} = \dfrac{-5S}{50+t}$ and thus $\dfrac{dS}{S} = \dfrac{-5dt}{50+t}$ so $\ln S = -5\ln(50+t)+C$

And thus $S(50+t)^5 = C$ *so* $S = \dfrac{C}{(50+t)^5}$ and $C = 10\cdot 50^5$ so $S = \dfrac{10(50)^5}{(50+t)^5}$

And when t=30 S=0.954

Cal B Final Exam Review

Topics

-Area: "under curve", between curves, exact and approximation with rectangles / trapezoids
-Anti-derivatives: expect a lot using u-substitution of logs, exponentials, trig, etc.
-Definite integrals
-Average values
-Applications / rates problems: inflow & outflow rates, position-velocity-acceleration
-Volume of revolutions: using discs, washers, and shells
-Differential equations: including writing them from a textual description
-Slope fields
-Euler's Method
-Functions defined by integrals—including taking derivatives, graphical analysis

1. Find the following anti-derivatives: "A" level

a. $\int (x^2 - 3x + 7)dx$ b. $\int (5x^{3/2} - 6x^{-2/3})dx$ c. $\int \frac{x^2 - 5x + 7}{2x} dx$

d. $\int (2x-1)^{11} dx$ e. $\int (x^2 - 6)^5 xdx$ f. $\int (x^2 - 3)^2 dx$

g. $\int \frac{-2dx}{\sqrt[3]{7x-15}}$ h. $\int e^{3\cos x - 1} \sin xdx$ i. $\int \frac{dx}{e^{5x-7}}$

j. $\int (xe + \frac{x}{e} + \frac{e}{x} + e^3)dx$ k. $\int 6x\sqrt{5x^2 - 9}dx$ l. $\int 3xe^{2x^2+7} dx$

m. $\int (\sin(5x) + 5\cos x)dx$ n. $\int (3x - 7\cos x)dx$ o. $\int \sin^3 x \cdot \cos xdx$

p. $\int \sqrt{5 - 7\cos x} \sin xdx$ q. $\int (2 + 3\tan x)^8 \sec^2 xdx$ r. $\int \frac{6x}{x^2 - 7} dx$

s. $\int \frac{x-3}{x^2 - 6x + 4} dx$ t. $\int \frac{x^4}{x^5 + 1} dx$ u. $\int \frac{x^5 + 1}{x^4} dx$

v. $\int \sin(x^2) xdx$

2. Find the following: "B level"

a. $\int x\sqrt{x-2}dx$ b. $\int \cot(2x)dx$ c. $\int \frac{x^2 + 3}{x - 2} dx$

d. $\int \sin^3 xdx$ (hint: substitute for $\sin^2 x$) e. $\int \frac{6}{x^2 + 9} dx$ f. $\int \frac{\ln x}{x} dx$

3. Evaluate the following:

a. $\int_1^4 \frac{6}{\sqrt{x}}dx$

b. $\int_{-2}^{1.5} \sqrt{2x+13}dx$

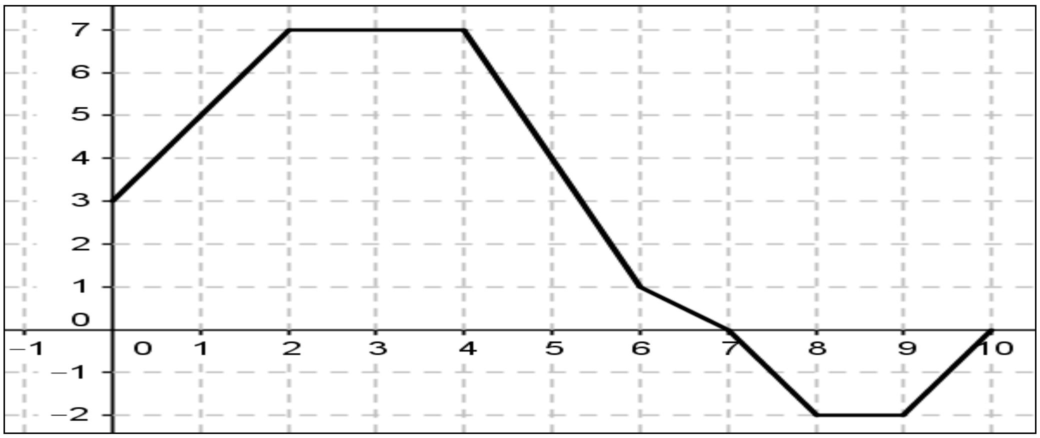
c. $\int_0^{\pi/2} 3\cos(2x)dx$

d. $\int_2^5 \frac{4}{2x-3}dx$

e. $\int_0^1 4xe^{x^2}dx$

f. $\int_1^3 (2x-5)^4 dx$

4. Use the graph of $f(x)$ below to find the following:



a. $\int_2^4 f(x)dx$

b. $\int_0^5 f(x)dx$

c. $\int_7^{10} f(x)dx$

d. $\int_0^{10} f(x)dx$

e. If $\int_{10}^{15} f(x)dx = 14$ then what is $\int_0^{15} f(x)dx$?

Let the function $g(t)$ be defined as $g(t) = 6 + \int_1^t f(x)dx$, where $f(x)$ is defined by the graph above.

Also $h(t) = \int_{t-2}^t f(x)dx$ on the interval [2,10]

f. What are the coordinates of all local maxima of $g(t)$ *and* $h(t)$ on this interval?

g. When is $g'(t) = 0$?

h. When is $h'(t) = 0$? Approximately when is $h(t) = 0$?

i. Sketch a graph of $g(t)$

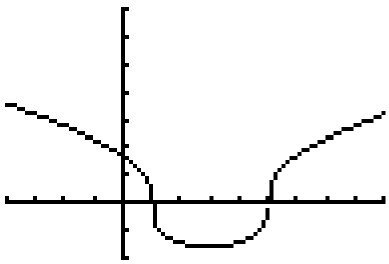
j. Write the equation of the line tangent to $h(t)$ where t=10.

5. A rumor spreads based on how many people have heard it. The growth in the number of people "in the know" is proportional to the square of the number of people "in the know". When t=2, 20 people know it. When t=8, 80 people know it.

a. When does the 300th person hear the rumor?

b. On the interval [0,9] what is the average number of people "in the know"?

6. Given the graph of $f(x)$ below, sketch the graphs of the derivative of $f(x)$ and (any) anti-derivative of $f(x)$.



7. Find the derivative of each function below:

a. $f(x)=6+\int_3^x \sin(t^2)dt$

b. $f(x)=6x+\int_{x-1}^{3x}\frac{dt}{1+t^2}$

8. For the function $f(x)=3x+\int_2^{2x} e^{t^2}dt$, write the equation of the tangent line when x=1.

9. Given the function $f(x)=\int_1^{2x}\frac{\ln t}{t}dt$ for $x\geq 1$, find all x values where $f(x)$ has inflection points.

10. Find any local max and local mins of the function $f(x)=\int_{0.1}^{x} t\ln t\,dt$

11. Sketch the graphs of $f(x)=\sqrt{x}$ *and* $g(x)=x$. Define R as the region in the first quadrant between the two curves.

a. Write an integral expressing this area. You must find the end-points algebraically.

b. Evaluate this intergral (no using fnInt!)

c. Write an integral showing the volume that results from this area being revolved around the x-axis. Evaluate it with fnInt.

d. Write an integral showing the volume that results from this area being revolved around the line y = -2. Evaluate it with fnInt.

e. Write an integral showing the volume that results from this area being revolved around the line x=3. Evaluate it with fnInt.

12. Find the area of the region bounded by the graph of $f(x) = x^3 - 2x^2 - 7x + 3$ and $g(x) = x + 3$. You must find the intersection points algebraically. Note: this means the **positive** area of the **two** regions.

13. A region R in the 1st quadrant is bounded by the x-axis, the line x=4, the line $y = x + 2$ *and* $y = x^2$. (It is touching the x-axis, if you are unclear which region it it). You may use fnInt but set up integrals clearly first.

a. What is the area of R?

b. What is the volume when R is revolved around the *x*-axis?

c. What is the volume when R is revolved around the line *y*=10?

d. Region R is the base of a solid. The cross-sections perpendicular to the *x*-axis are squares. What is the volume of this solid?

e. Region R is the base of a solid. The cross-sections perpendicular to the *y*-axis are semi-circles. What is the volume of this solid?

14 Write an integral that describes the average value of the function $f(x) = x^3 + x - 2$ on the interval [3,8].

15. A bowl is shaped like a hemisphere of radius 20 cm. The water level (from the very bottom of the bowl to the surface) is 8 cm. What is the volume of water in the bowl?

16. A helicopter is rising straight up into the air. Its velocity *t* seconds after it starts is given by the equation $v(t) = 3t + 1$. (in meters per second). Its initial height is 2 meters.

a. Write the function describing the helicopter's height at any given time.

b. When is its height 50 meters?

c. What is its average velocity over the first 4 seconds?

d. What is its average height over the first four seconds?

17. Find the function $f(x)$ given $f''(x) = 4x - 7$ $\quad f'(1) = 11$ $\quad f(0) = 31$

18. A cylinder's height is 10 cm and its radius is 3 cm. The density of the material depends on the distance from the line through each circular cross-section's center according to the function $d(x) = (x+1)^{1/3}$. What is the average density of the cylinder?

19. A bank account initially has $20,000. Money is being taken out continuously at a rate of $400 per month. Interest on the money in the account accrues continuously at a rate of 1% per month (this is a great rate!).

a. Write a differential equation showing how the account balance (B) changes over time. (so $\frac{dB}{dt} =$)

b. Solve it.

c. When does the money run out?

d. What is the average balance over the time from the beginning until the money is all gone?

20. The instantaneous rate at which fans enter Fenway Park for a Red Sox game is given by the function $f(t) = -0.1t^2 + 6t + 210$ on the interval [0,80]. Here, *t* is the number of minutes **after 6 pm** and the rate is people per minute. FnInt is OK, but write the integral first.

a. Write an integral showing how many fans entered between 6:10 pm and 6:20 pm and evaluate it.

b. What was the average rate that fans entered Fenway Park over the [0,80] interval, and what was the total number of fans entering the park during this 80-minute period?

c. Assuming that there were already 2,500 fans in the ballpark at 6 pm, write a function $F(t)$ that shows how many fans were in the ballpark *t* minutes after 6 pm (assume no fans left the ballpark).

d. Exactly when did the 10,000$^{\text{th}}$ fan enter the ballpark?

e. On average, how many fans were in the park between 6 and 7:20 pm?

f. Of those fans arriving between 6 and 7:20, what was the average arrival time?

21. An item accelerates at a constant rate from an initial velocity of zero. After 4 seconds, the object has traveled 100 meters. How long will it take to go the next 50 meters? No physics formulas! Hint: write the equation for acceleration given it is k, then integrate to find velocity and position functions in terms of acceleration.

22. The average value of $f(x) = x^2$ on the interval [0,k] is 11. What is k?

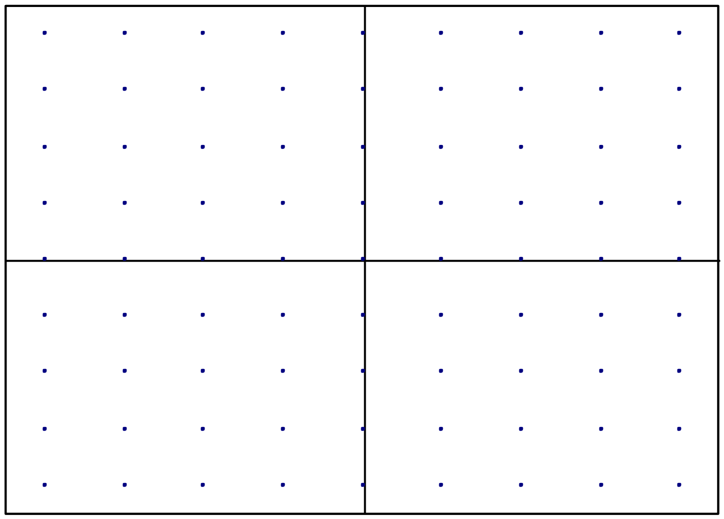
23. Solve the following differential equations given an initial point:

a. $\frac{dy}{dx} = 2x\sqrt{y}$ (2,9)

b. $\frac{dy}{dx} = 0.05(8 - y)$ (0,4)

24. Answer the following questions about the differential equation $\frac{dy}{dx} = \frac{\sqrt{x}}{y}$

a. Sketch a slope field on the window $-3 \leq y \leq 3$ and $0 \leq x \leq 3$



b. Sketch the particular solution through (1,1).

c. Use Euler's Method to estimate the value of *y* when *x*=2 given the solution in part *b* above. Use an *x*-increment of 0.5.

d. Find the exact value of *y* when *x*=2 by solving the differential equation analytically.

ANSWERS

1a. $\frac{x^3}{3}-\frac{3x^2}{2}+7x+C$ b. $2x^{5/2}-18x^{1/3}+C$ c. $\frac{x^2}{4}-\frac{5x}{2}+\frac{7\ln|x|}{2}+C$ d. $\frac{(2x-1)^{12}}{24}+C$

e. $\frac{(x^2-6)^6}{12}+C$ f. $\frac{x^5}{5}-2x^3+9x+C\ (FOIL)$ g. $\frac{-3}{7}(7x-15)^{2/3}+C$ h. $\frac{-1}{3}e^{3\cos x-1}+C$

i. $\frac{-1}{5}e^{-5x+7}+C$ j. $\frac{ex^2}{2}+\frac{x^2}{2e}+e\ln|x|+xe^3+C$ k. $\frac{2}{5}(5x^2-9)^{3/2}+C$ l. $\frac{3}{4}e^{2x^2+7}+C$

m. $-\frac{1}{5}\cos(5x)+5\sin x+C$ n. $\frac{3x^2}{2}-7\sin x+C$ o. $\frac{(\sin x)^4}{4}+C$ p. $\frac{2}{21}(5-7\cos x)^{3/2}+C$

q. $\frac{(2+3\tan x)^9}{27}+C$ r. $3\ln|x^2-7|+C$ s. $\frac{1}{2}\ln|x^2-6x+4|+C$ t. $\frac{1}{5}\ln|x^5+1|+C$

u. $\frac{x^2}{2}-\frac{1}{3x^3}+C$ v. $\frac{-1}{2}\cos(x^2)+C$

2a. subst u=x-2 and get $\frac{2}{5}(x-2)^{2.5}+\frac{4}{3}(x-2)^{1.5}+C$ b. cot=cos/sin so $0.5\ln(\sin(2x))+C$

c. divide so $0.5x^2+2x+7\ln|x-2|+C$ d. get $\int(1-\cos^2 x)\sin x dx=-\cos x+\frac{\cos^3 x}{3}+C$

e. divide all by 9 and get $2\tan^{-1}(x/3)+C$ f. $0.5(\ln|x|)^2+C$

3a. 12 b. $\frac{37}{3}$ c. 0 d. $2\ln 7$ e. $2e-2$ f. 244/10

4a. 14 b. 29.5 c. -4 d. 28.5 e. 42.5

f. g(t) has a maximum at (7,34.5) h(t) has a maximum at (4,14) g. at t=7 or 10.

h. h'(t)=0 at t=4 and at 9.5 and h(t)=0 around 7.75 j. h(10)=-3 and h'(10)=2 so y+3=2(x-10)

5. $\frac{dP}{dt}=kP^2\ \ so\ \ P=\frac{-1}{kt+C}=\frac{-1}{(t/160)-(1/16)}\ \ or\ \ \frac{160}{10-t}$ so P=300 when t=9.47 b. 40.93

7a. $\sin(x^2)$ b. $6+\frac{3}{1+9x^2}-\frac{1}{1+(x-1)^2}$

8. $f(1)=3\ and\ f'(1)=3+2e^4\ \ so\ \ y-3=(3+2e^4)(x-1)$ 9. at x=0.5e

10. $f'(x)=x\ln x$ is zero when x=0 or 1; zero is not in the domain and at x=1 the derivative changes from negative to positive so it is a min

11 a. $\int_0^1(\sqrt{x}-x)dx$ b. 1/6 c. $\int_0^1[\pi x-\pi x^2]dx=0.52$ d. $\int_0^1[\pi(\sqrt{x}+2)^2-\pi(x+2)^2]dx=2.62$

e. shells: $\int_0^1 2\pi(3-x)(\sqrt{x}-x)dx=2.73$. 12. $\int_{-2}^0(f(x)-g(x))dx+\int_0^4(g(x)-f(x))dx=49.33$

13a. $\int_0^2 x^2dx+\int_2^4(x+2)dx=\frac{38}{3}$ b. $\int_0^2\pi x^4dx+\int_2^4\pi(x+2)^2dx=179.28$

c. $\int_0^2\pi[10^2-(10-x^2)^2]dx+\int_2^4\pi[10^2-(10-(x+2))^2]dx=616.59$

d. $\int_0^2 x^4 dx + \int_2^4 (x+2)^2 dx = 57.07$ e. $\int_0^4 \frac{\pi}{2}\left(\frac{4-\sqrt{y}}{2}\right)^2 dy + \int_4^6 \frac{\pi}{2}\left(\frac{4-(y-2)}{2}\right)^2 dy = 4\pi$

14. $\frac{1}{5}\int_3^8 (x^3 + x - 2)dx = 204.25$

15. make bowl lower semicircle of $x^2 + y^2 = 400$ and using discs: $\int_{-20}^{-12} \pi(400 - y^2)dy = 3485$

Using shells: $\int_0^{\sqrt{256}} 2\pi x\left(\sqrt{400-x^2} - 12\right)dx = 3485$

16a. $h(t) = 1.5t^2 + t + 2$ b. 5.33 seconds c. $\frac{1}{4}\int_0^4 (3t+1)dt = 7$ OR $\frac{1}{4}[h(4) - h(0)] = 7$

d. $\frac{1}{4}\int_0^4 (1.5t^2 + t + 2)dt = 12$ 17. $f(x) = \frac{2}{3}x^3 - \frac{7}{2}x^2 + 16x + 31$

18. total weight of cylinder = 10cm times cross-sectional weight, so it is $10\int_0^3 2\pi x\sqrt[3]{x+1}dx = 404.9$ and volume is $\pi r^2 h = 90\pi$ so avg density is 1.432 units per cubic cm (this is different than the average value of that cube-root function on the interval [0,3] because more volume has the higher density than the lower density)

19a. $\frac{dB}{dt} = 0.01B - 400$ b. $B = 40000 - 20000e^{0.01t}$ c. 69.3 months d. about \$11,148

20a. $\int_{10}^{20} f(t)dt = 2767$ b. total = $\int_0^{80} f(t)dt = 18933$ avg is this over 80 = 236.67

c. $F(t) = \frac{-0.1}{3}t^3 + 3t^2 + 210t + 2500$ d. $F(t) = 10000$ $when\ t = 28\,\text{min}$ e. $\frac{1}{80}\cdot\int_0^{80} F(t)dt = 13033.3$

f. total of minutes after 6 pm is $\int_0^{80} t(-0.1t^2 + 6t + 210)dt$ = 672000

divide by number of fans 18933 and get 35.5 minutes, so about 6:35 or 6:36

one alternative way to think: avg t with weights of $f(t)$ $so\ \frac{\int tf(t)dt}{\int f(t)dt}$

you could also calculate the average wait time and subtract that for 7:20…

21. $s''(t) = k$ $s'(t) = kt + 0$ $s(t) = 0.5kt^2 + 0$ (since s(0)=0)… so $s(4) = 0.5k(16) = 100$ and k=12.5
So $s(t) = 6.25t^2$ and s(t)=150 when t=sqrt(24) so sqrt(24)-4= about 0.90 seconds

22. $\frac{1}{k}\int_0^k x^2 dx = 11$ $so\ 11k = \frac{k^3}{3}$ $and\ k = \sqrt{33}$ 23a. $y = (0.5x^2 + 1)^2$ b. $y = -4e^{-0.05x} + 8$

24c. 1.94 d. $y = \sqrt{\frac{4}{3}x^{3/2} - \frac{1}{3}}$ so when x=2 y=1.854