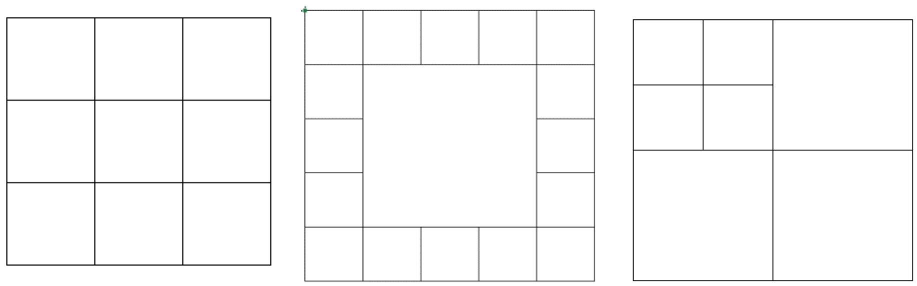
Unit 1 Handout #1

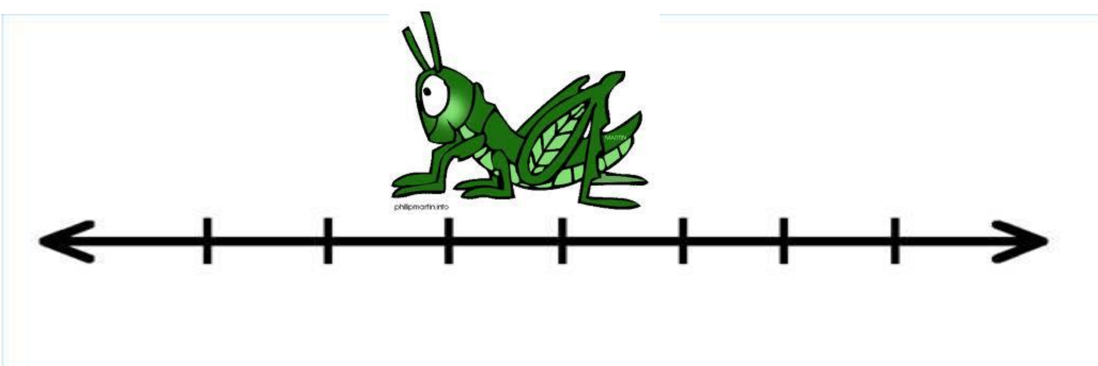
1. A square can be divided into smaller squares, not necessarily congruent to each other. In the diagrams below, squares are divided into nine, seventeen, and seven smaller squares respectively.



Give all possible numbers of squares that a square can be divided into. Explain how you can do so. While there might be numerous ways to divide a square into, say twelve smaller squares, you only need to describe one way to do so. Look for patterns.

2. Let n be a positive integer. Find the lowest value of n such that $3n+2$ is a perfect square or explain why it cannot be one.

3. A grasshopper is hopping along the number line (infinite in both directions). It starts at position 100 and takes 50 jumps, where the jumps can only be in the positive or negative direction. The first jump is 1 unit, the second jump is 2 units, third is 3 units…and the last one is 50 units. For example, the first three jumps might be right 1, left 2, and left 3, putting the grasshopper at position 96.



a. What two furthest right ending positions are possible?

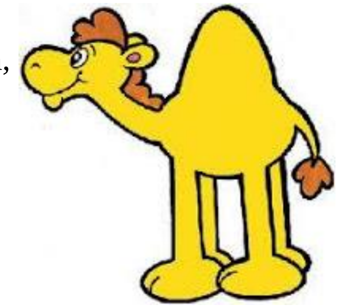
b. Describe one set of jumps that causes the bug to end at the origin or explain why this is not possible.

c. What is the greatest number of times it may return to its starting point of 100? ("Return" implies that the first time does not count)

d. Suppose the grasshopper changes direction exactly twice (for example, going right at least one jump, then left at least one jump, and then right some more—or left, right, then left). Give all ways that it may end up at space 99 or explain why this is not possible.

e. How many different series of jumps will cause it to end at 1345?

4. You have 3000 bananas at one edge of a 1000-mile wide desert and are trying to take as many as possible across to the other side. You cannot carry any bananas on your own, but the camel can carry up to 1000 at a time. The camel eats bananas continuously at a rate of one every mile—and assume bananas can be cut into fractions!



a. What is the most bananas you can carry across the desert, and how should you organize your trips to attain this (fractions are OK!)? You are allowed to take some into the desert, stockpile them, and come back for more.

b. Instead of a 1000-mile wide desert, what is the widest desert that you can cross with 3000 bananas?

c. Back to the 1000-mile wide desert: what is the least number of bananas you can start with in order to have 1000 bananas at the other side? Pat thinks you can take your answer to part *a* above and scale it (by multiplying it by a constant) to end with 1000 bananas. What do you think of Pat's idea?

d. Back. If instead of 3000 bananas, you have 3001. Now what is the most you can take across the desert? How about 2999 bananas?

e. Let B be your starting number of bananas (it was originally 3000) and E(B) be the number you end with on the other side, 1000 miles away. Sketch a graph of $E(B)$ in the domain $0 \le B \le 6000$.

f. What is $\lim_{B\to\infty} E(B)$? What is $\lim_{B\to\infty} E'(B)$?

5. On a regular 8x8 checkerboard, two squares are chosen at random.

a. What is the probability that they share an edge (sharing only a corner does not count)? Hint: maybe your first part of the tree should be whether the first square chosen is a corner, a non-corner side, or a non-side square.

b. What is the probability that they share a corner but not an edge?

c. Instead of 8x8, it is now n-by-n. Now what is the probability that two randomly-chosen squares share an edge? Your answer should be in terms of *n*. Simplify it fully.

6. You can buy a certain type of candy in bags of 5 or 7. What is the largest number of candies that you cannot buy? (for example, you can buy 12 by purchasing one bag of each size, but you cannot buy 13).

7. At McDonalds, Chicken McNuggets come in orders of 6, 9, or 20. What is the most you cannot buy?

8. Kate and Mark like to walk their dogs together. They each spend 20 minutes walking their dogs in the park. They each arrive sometime between 6:30 and 7:30 am and their arrival times are independent of each other. What is the probability that, on a given day, the dogs get to play together for at least a moment?

9. (continuation) Instead of staying 20 minutes, how long would they each have to stay to make the probability that they overlap exactly 50%?

10. How many positive integers exist that are greater than or equal to 10 so that the digits strictly decrease (zero is the lowest digit)? Some examples are 952, 21, and 754,210.

11. There are a bunch of slips of paper in a bag. One of them has the number 1 on it. Two others have the number 2; three others have the number 3…and so on until fifty of them have the number 50 on them. What is the least number you need to take out of the bag to guarantee that you have a set of ten slips with the same number on them?

12. A person initially has none of a particular medicine in her body. She takes a pill every hour that adds 50 mg of medicine. Her body also metabolizes 30% of the medicine in her body each hour, thus decreasing the amount in her body. Over time, how much medicine is in her body?

18. Two random numbers are selected between zero and one. A third number is equal to one. What is the probability that these three numbers can be the sides of:

a. A triangle

b. An obtuse triangle

14. Find the exact value of the following infinite sum:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\text{..........}\frac{n}{2^n}+\text{....}$$

15. A ball starts in the lower left corner of a rectangular billiards table with no pockets. It travels with a slope of one and bounces around until it hits a corner. When the ball bounces off of a side, the angle in is equal to the angle out, so the slope changes sign but not magnitude. The length and width are integers.

a. If the dimensions of the table and 2x3, how many bounces will it take before it hits a corner?

b. If the dimensions are 3x5, how many bounces until it hits a corner?

c. If the dimensions are *m*x*n* where *m* and *n* are relatively prime, then how many bounces?

16. The expression $n^{n^{n^{\cdots}}}$ has an infinite number of *n*'s. When *n*=1, the expression is clearly equal to 1. It is tempting to conclude that this sum is infinite for any value of *n* greater than 1.

a. Show that this expression converges when $n=\sqrt{2}$ and find the value it converges to.

b. What is the highest value of *n* for which it will converge?

Answers

1. All except 2, 3, and 5.

For any even integer above 2, you can split the square into a large square in the upper left and a set of smaller squares making an "L" around it. This L can be thought of as the difference between two consecutive squares, which are all odd numbers. Add the larger square in the upper-left and you get an even number. For example, 12 can be 1 big and 11 small, where the smaller squares have side length of 1/5 of the large square. So 5 of these are to the right, 5 below, and 1 in the lower corner.

Given that you can divide a square into n smaller squares, you can easily divide it into n+3 smaller squares by taking any of the n squares and splitting it into 4 congruent smaller squares. Thus any odd number above 5 can be attained by adding three to an even number.

2. never: you can write any integer as $3a$, $3a+1$, or $3a+2$ where a is a non-negative integer. If you square each of these, you will see that they are either divisible by 3 or have a remainder of one when divided by three. Therefore, no perfect square can have a remainder of 2 when divided by 3,

3a. sum of jumps is 1275 so furthest right is 1375. Next is 1373 if first jump is left and rest are right

b. impossible: must end one an odd location

c. 12 times: every 4 jumps it can get back by LRRL or RLLR pattern (can save a jump in the first set by going RRL or LLR)

d. 1-7 right; 8-36 left and remainder right or 1-11 left 12-37 right and remainder left. One way to solve this is to find an arithmetic series somewhere in the middle of 1+2+3…. That sums to or (total movement of 1275 results in one step to the left, so series in center goes left 638 or right 637)

e. 27 (not very elegant way)

4a. 533.333 a good way is to stockpile multiples of 1000 as far as you can into the desert—so when you start to move them the camel's capacity is fully utilized. To make a pile of 2000 you burn 1000 in five one-way trips – so 200 miles in. Then you make a pile of 1000 as far in as you can—three one-way trips to burn 1000 bananas means 333.33 miles so you have 1000 bananas at position 533.33. 466.67 are burned taking them out leaving 533.33 at the end.

b. 1533.333 c. 7673—ask Pat how the camel will get from the far end back to the near end! It takes more than 1000 bananas for this. d. for 3001: 533.333+1/7 for 2999: 533.333- 1/5

e. piecewise linear f. infinity; zero seems reasonable but computer simulation by students shows that it never gets there and you can do some cute calculus to show that it is approximately 1/e^2

5a. corner is 4/64 * 2/63 non-corner edge is 24/64 * 3/63 ; middle is 36/64 * 4/63 so (8+72+144)/(64*63) = 1/18

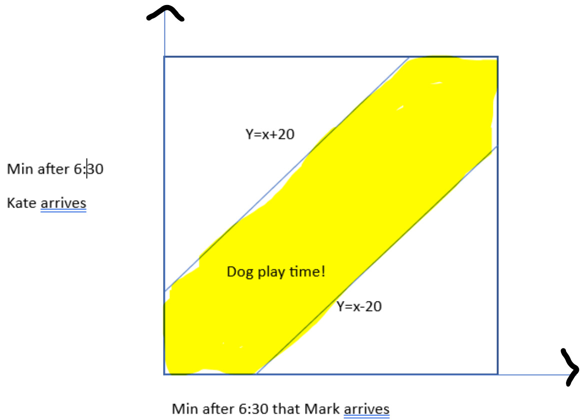
b. corner 4/64*1/63; non-corner edge is 24/64 * 2/63; middle is 36/64*4/63 so 196/(64*63)=7/144

c. corner / non-corner edge / center: $\frac{4}{n^2}\cdot\frac{2}{n^2-1}+\frac{4n-8}{n^2}\cdot\frac{3}{n^2-1}+\frac{(n-2)^2}{n^2}\cdot\frac{4}{n^2-1}$

$=\frac{4n^2-16n+16+12n-24+8}{n^2(n^2-1)}=\frac{4n^2-4n}{n^2(n^2-1)}=\frac{4n(n-1)}{n^2(n^2-1)}=\frac{4}{n(n+1)}$ checks by getting 1/18 when n=8

6. 23 7. 43

8. 5/9 : the ratio of yellow area to the area of the 60-by-60 square



9. $60-30\sqrt{2}$ minutes: make the sum of the areas of the right triangles where it does not work =1800

10. write the digits 9876543210 and keep some and eliminate others. So 2^10 but this includes 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 plus zero in another way so 1013

11. the worst possible outcome is all of numbers 1-9 and 9 each of #'s 10-50. These total 45+369 = 414. So you need 415 to make sure you get at least one set of 10 of a kind.

12. will reach equilibrium when the 30% out is equal to the 50 in.. 0.3x=50 so x=166.67.

13a. make a 1x1 square in the first quadrant where the x coordinate is the first random # and the y is the second. It will make a triangle as long as x+y>1, which is 50%

b. need x+y>1 but also the hypothenuse must be larger than the sum of the squares of the side so 1-0.25π

14. 2: write it as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+....$$
$$+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+....$$
$$+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+....$$
$$\frac{1}{16}+\frac{1}{32}+....$$

Each row is a geometric series: their sums are $1, 1/2, 1/4, 1/8....$ these sums constitute another geometric series whose sum is 2

15a. 3 b. 6 c. m + n -2

16a. since $1<\sqrt{2}<2$ and the square root function is increasing, we know (by taking square 2 to each of these powers) that $\sqrt{2}<\sqrt{2}^{\sqrt{2}}<2$. Thus when you keep on taking root 2 to this power more and more times, each term will be higher than the previous term but lower than 2. It will eventually converge to 2.

b. using calculus, the maximum value of $x^{1/x}$ occurs when x=e so the $\sqrt[e]{e}$ is the highest value for which it converges.

Unit 1 Handout #2

1. Mark is driving Milo home from their morning walk. Milo is in the passenger seat, with no seatbelt. The car senses weight in the seat and thus the seatbelt alarm goes off. It beeps regularly, meaning the same amount of time elapses between successive soundings. While it goes off, Mark looks at the digital clock, showing the time in hours and minutes. On the way home from the walk, the alarm sounds fifteen times, as listed below.

7:11
7:12
7:12
7:13
7:13
7:14
7:14
7:14
7:15
7:15
7:16
7:16
7:16
7:17

a. Is it possible to know exactly how much time elapses between successive beeps of the alarm? Make no assumptions about this being a natural number, or even a rational number!

b. Based on the information given, what is a reasonable estimate for the amount of time between successive beeps?

c. Based on the information given, find the smallest possible range in which the time between successive beeps must lie.

d. Is it possible that the next two beeps would also occur when the clock shows 7:17? Explain.

e. Assume the pattern between 7:12 and 7:16 repeats itself between 7:17 and 7:21 (so that the alarm sounds twice in 7:17, twice in 7:18, thrice in 7:19, twice in 7:20, and thrice in 7:21). Does this allow you to narrow the range of possible times between successive beeps? If so, what is the new range?

f. Now assume the pattern observed between 7:12 and 7:16 repeats itself forever. What, if anything, does that indicate about the range of possible times between successive beeps?

2. At a school, there are 1000 lockers, numbered 1 through 1000. Initially all are closed. The one thousand students line up. The first student opens every locker. The 2nd student closes every locker that is a multiple of 2. The third student changes the status of every locker that is a multiple of 3 (if open, she closes it; if closed, she opens it). The continues until the last student changes the status of the 1000th locker.

a. How many lockers are open at the end of this process, and which ones?

b. Now only the odd-numbered people go down the line changing the status of each locker (and the even-numbered people do nothing). Now what lockers are left open?

c. Instead, only the even-numbered people change the lockers. Now what lockers are left open?

3. A bug is initially at one end of a line segment of length one. It takes many jumps; each jump takes it half-way from its current position to one end of the segment. Its first jump must be to the right. For instance, if it hops right then right then left, its position is:

Right from 0 to 0.5

Right from 0.5 to 0.75

Left from 0.75 to 0.375

a. How many different places may it end up after 5 hops? Describe the set of such places.

b. Is there any place on the number line that the bug can get to in two different ways?

c. If the bug's final location is somewhere on the interval [0,0.25] then what do we know about its history of hops?

d. If the bug's final location is somewhere on the interval [0.5,0.75] then what do we know about its history of hops?

e. Is there a series of jumps that ends at 7/32? How about 115/128? Find them, if possible.

f. Can the bug get somewhere in the interval [0.39, 0.4] after five hops? If so, explain the sequence of hops. If not, what is the least number of hops the bug needs to get somewhere in that interval?

4. What is the exact value of x if $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+.......}}}}$. Write it as simply as possible. You may use your calculator to compute some values if you like.

5. Sketch the graph of $\cos y > \cos x$ on the coordinate plane.

6. When Steve walks up and down stairs, he can take steps one or two at a time. If a staircase has two steps, there are two ways he can proceed (2 or 1-1). If a staircase has 3 steps, there are 3 ways he can proceed (1-1-1, 1-2, 2-1). How many ways can Steve ascend a staircase that has 12 steps?

7. (continuation) Steve's sister Stephanie has longer legs than Steve. She can take steps one, two, or three at a time. How many ways can Stephanie ascend a staircase that has 12 steps?

8. Given that $\log 2 \approx 0.3010$, how many digits are in 5^{44} ?

9. If a set of items is shared equally with 2, 3, 4, 5, or 6 people, there is one left. But if shared with 7 people there are none left. What are the two smallest possible numbers of items in this set?

10. Simplify: $2+\cfrac{15}{2+\cfrac{15}{2+\cfrac{15}{2+\ldots}}}$

11. The positive integers are written in a grid as follows:

1	2	4	7	11
3	5	8	12	
6	9	13		
10	14			
15				

Let the position (x,y) reflect the xth column and yth row. So the position in (5,1) is 11 and in (2,4) is 14.

a. What is the number in position (1,10)?

b. What is the number in position (10,10)?

c. What is the number in position (3,13)?

d. What is the number in position (1,n) where n is a positive integer?

e. What is the number in position (n,n) where n is a positive integer?

f. What is the sum of the first n elements in the first column? Think degree!

g. What is the number in position (n,m) where n and m are positive integers?

12. A list of positive integers contains no perfect squares or cubes. What is the 1000^{th} number on the list?

13. The men’s NCAA basketball tournament (“March Madness”) used to be is a single elimination tournament with 64 teams. In the first round, there were 32 games. The winners advanced to the second round, where there were 16 games, etc. until the final two remaining teams played in the championship game.

a. How many games were played in the tournament?

b. If instead of 64 teams, there were 512 teams in a single-elimination tournament, then how many games would be played?

Answers

1a. no b. 12/5 so 2.4 beeps per minute or 1 every 25 seconds c. $24<x<180/7$ d. no

e. $420/17 < x< 480/19$ f. lowers until is approaches 25 seconds.

2a. all the perfect squares: 31 lockers : they have an odd number of factors so remain open

b. open lockers are powers of 2, odd perfect squares, and odd perfect squares times powers of 2

c. perfect squares times odd powers of 2 are open: 2, 8, 32, 128, 256, 18, 72, 288, 50, 200, 800, 98, 392, 162, 648, 242, 968, 338, 450, 578, 722, 882, 938

3a. 1st is right and each of next 4 can be r or l and no 2 ways to get to the same place so 16;

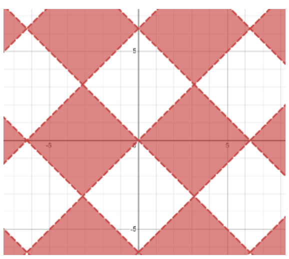
Odd numbers of 32nds: 1/32, 3/32, 5.32, ... 31/32

b. no c. last 2 hops are left d. last hop is right and next-to-last was left

e. rrrlll; rrllrrr use base-2 representations of numerator in reverse order

4. $\frac{1+\sqrt{5}}{2}$

5. cool!



6. Fibonacci: 233 7. 927 8. 32 9. 301, 721 10. $x=2+\frac{15}{x}$ *so* 5

11a. 55 b. 181 c. 118 d. $\frac{n(n+1)}{2}$ e. $2n^2-2n+1$ f. $\frac{n^3}{6}+\frac{n^2}{2}+\frac{n}{3}$

g. (n,1) is $F=\frac{n^2-n+2}{2}$ now sum F + ((n+1))+ ((n+2)) + …((n+m-1)) $=\frac{n^2-n+2}{2}+\frac{(m-1)}{2}(2n+m)$

which is $\frac{m^2+2mn+n^2-3n-m+2}{2}$

12. 1039 -- squares up to 1024 are 32 cubes not squares are 8, 27, 125, 216, 343, 512 1000

13a. 63 b. 511 one can add up powers of 2 to get the answer. Or just realize that one team loses each game and n-1 teams lose, so there are n-1 games!

Unit 1 Handout #3

1. Person A owes person B \$10. They have no cash, but person A has 14 (non-divisible) objects each valued at \$184 and person B has many (non-divisible) objects each valued at \$110. Can they exchange objects in a way that person A gives objects with a value of \$10 more than she receives?

2. Four whole numbers are added three at a time, giving sums of 180, 197, 208, and 222. What is the largest of the four numbers?

3. How many real solutions does the equation $\sin x = 0.01x$ have? In radians, of course!

4. A stick of length one is broken at two places chosen at random, creating three sticks. What is the probability that the three sticks can be the sides of a triangle?

5. A variant: the stick is broken into 2 pieces at a random location. And then the longer one is broken into two again at a random location. What is the probability that the three sticks can be lengths of a triangle?

6. Sketch graphs of the following:

a. $|x|+|y|=1$ b. $x^2+y^2=1$ c. $x^{2/3}+y^{2/3}=1$ d. $x^{100}+y^{100}=1$

e. $x^{2/99}+y^{2/99}=1$ f. $x^{99}+y^{99}=1$

7. A circle has a circumference of 100. Three arcs are randomly placed on it. The red arc is 60 units long, the blue arc is 50, and the yellow arc is 30.

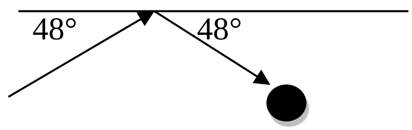
a. What is the probability that the union of the red and blue arcs is the entire circle?

b. What is the probability that there is at least part of the circle covered by all three arcs?

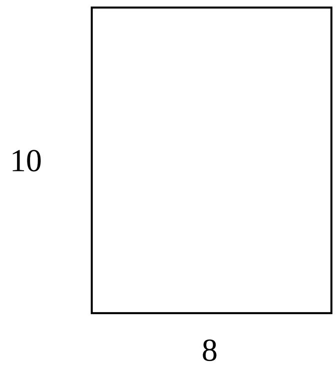
c. What is the probability that the union of the three arcs is the entire circle?

8. (from NCSSM Precalc Book). When watching basketball on television one day, I observed the following incident involving Michael Jordan: As he drove for the basket early in the game, he was fouled. According to the announcer, at that point in the season, "Michael Jordan has made 78% of his free throws this season". He misses the first shot and makes the second. Soon thereafter, Michael Jordan was again fouled. This time, as he came to the free throw line to take his shot, the announcer stated that "Michael Jordan has made 76% of his free throws this season". How many attempts and successes had Jordan had before he first got to the foul line that day?

9. Follow the bouncing ball: When a ball reflects off a wall, the angle at which it comes in is the same as the angle at which it leaves (note that if the ball has some spin on it, this will not be true). For example:



The rectangular box below is 8 units wide and 10 units high. A ball starts in the lower left-hand corner with a slope of 2 and bounces around the box for a while.



a. What happens when a ball with slope 2 enters a corner? Thinking about the corner as the limit of two sequential bounces may help!

b. What corner does the ball with slope 2 first enter (after leaving the lower left-hand corner), and how many bounces does it take before getting there?

c. A ball leaves the lower left-hand corner with a slope of 0.012. How many other times does it hit the left wall before first hitting the top?

d. A different ball starts at the bottom of the box 6 units to the right of the lower-left corner. It has a positive slope initially, and, after hitting the right, top, then left walls, returns to its starting place. In fact, it will continue to bounce around hitting the walls at the same four places forever (this repeating of its path is called a "fixed orbit"). What was its initial slope?

e. (continuation of *d*) There are other initial positive slopes the ball may have to give it a fixed orbit where it returns to the bottom wall only at the starting point 6 units along the bottom. These other fixed orbits will hit some walls more than once. Give some other initial slopes that result in fixed orbits.

f. If a ball leaves the bottom left corner with a slope of π, will it ever have a fixed orbit?

g. Give any three slopes a ball can leave the lower left corner with such that the first corner it hits is the upper left corner.

h. Is it possible for the ball to leave the lower-left corner such that the first corner it hits is the upper right corner where it bounces off a wall exactly 15 times before reaching the corner? If so, what is its initial slope?

i. Is it possible for the ball to leave the lower-left corner such that the first corner it hits is the upper right corner where it bounces off a wall exactly *n* times before reaching the corner? Are there some non-negative integral values of *n* that are not possible?

10a. A 5-by-14 rectangle is composed of 1x1 squares. How many of these squares does a diagonal go through?

b. How about if the rectangle was 7-by-31?

c. How about if the rectangle was 10-by-30?

d. Now imagine a 3x4x5 rectangular prism composed of 60 1x1x1 cubes. How many cubes does a diagonal of the prism go through?

11. Someone has 19 large packets of marbles and 3 small packets of marbles. She has 224 marbles. How many are in a large packet? A small packet?

12. A bug starts at one corner of a rectangular prism whose dimensions are 5-by-4-by-6. One corner is the origin, and the opposite corner is the point (5,4,6). With each step, it can increase one of its coordinates by 1 unit (no cutting diagonals!). What is the probability that it goes through the point (3,2,3)?

13. Explain why the sum of 4 consecutive positive integers cannot be a perfect square.

14. Many positive integers can be expressed as the sum of at least two consecutive positive integers. For instance, 15=4+5+6, 37=18+19, and 9=2+3+4. Describe the set of positive integers that cannot be expressed this way. Prove that integers in this set cannot be expressed this way.

15. The number $n!$ ends in exactly 20 zeros. List all possible positive integers n. What if, instead, it ended in exactly 30 zeros?

Answers

1. x(184)=y(110)+10   x must be a multiple of 5 not exceeding 14... 5 does not work nor does 10.. .so no?

2. add these 4 numbers up and get 3 times the sum, so sum is 269 and largest is 269-180=89

3. 31   4. ¼ using a unit square on the coordinate axes and finding what works

5. 1/3, I think (can use the same diagram as in #5 but remove the two corners where the smaller stick was broken)

6d. pretty much like a square om the [-2,2] [-2,2] region

7a. 10%   b. 60% c. I think 15%

8: several possibilities 18/23 21/27   25/32   28/36   31/40 and 38/49

9a. leaves with slope of 2   b. (40,80) so lower right   4 sides and 7 top/bot so 11?

c. dy/dx = 12/1000 if dy=10 then dx=833.33   hits at 16, 32, 16n   so 52   833.33/16 over 52

d. dy/dx = 20/16 so 1.25   e. 20/32   20/48   20/16n???   f. no

g. 10/16, 10/32, plus many many more   30/32

h. can't do it with vertical of 10… needs even # of bounces…so no???   i. n must be even???

10a. it must cross 13 vertical dividers and 4 horizontal dividers so 17 plus the first one or 18… since 14 and 5 are relatively prime it never hits a vertical and horizontal simultaneously.

b. 30 + 6 + 1 initial one is 37

c. 29 + 9 + 1 minus 9 lattice points so 30   (can think of this as going thru 10 different 1x3 rectangles)

d. 1 + 2+ 3 + 4=10: every time it crosses an integer plane in any direction: no lattice points

11. 19a + 3b=224 where a and b are integers and a>b: for 224-19a to be divisible by 3, a=2, 5, 8, etc.

A=2 and b=62 but a<b   a=5 b=43   a=8 b=24   a=11 b=5   so 11 and 5

12. all ways it can go are ways to arrange RRRRRUUUUBBBBBB which is $\frac{15!}{4!5!6!}$

Through (3,2,3) means arrange RRRUUBBB to get there and then RRUUBBB to finish which is

$$\frac{8!}{3!3!2!}\cdot\frac{7!}{2!3!2!} \quad so \quad \frac{\frac{8!}{3!3!2!}\cdot\frac{7!}{2!3!2!}}{\frac{15!}{4!5!6!}}$$

13. sum of 4 consecutive positive integers is a+a+1 + a+2 + a+3 = 4a+6

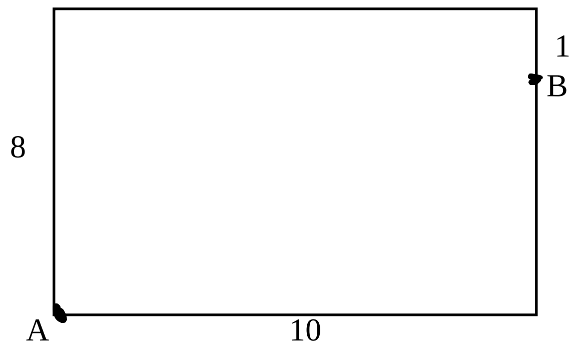
odd integers squared must be odd, which 4a+6 isn't; even integers squared must be a multiple of 4, which 4a+6 isn't

14. does not work for powers of 2; using sum of n consecutive integers beginning with a, one can see that the sum needs to have an odd factor (greater than 1), which powers of 2 do not have

15. 20 zeros: 85, 86, 87, 88, 89 // 30 zeros is impossible since 124! ends in 28 and 125! ends in 31 (cool!)

Unit 1 Handout #4

1. In the 8x-by-10 box below, a ball starts at point A in the lower-left corner. After bouncing off sides exactly twice it arrives at point B, located one unit below the upper-right corner. Give all possible slopes that will achieve this.



2. All of the integers beginning with 1 are written out like 0.12345678910111213141516…. Let the function $z(n)$ represent the percentage of the first n digits to the right of the decimal place that are zero. What is $z(1002)$?

3. Two numbers are selected independently and at random from the interval [0,1]. What is the probability that the ratio of one to the other is between 1/3 and 3? (For example, the pair 0.1 and 0.6 does not qualify, but the pair 0.35 and 0.80 does).

4. A group of 200 people are arrayed in a rectangular grid of 20 columns and 10 rows. From each of the 20 columns, the shortest student is selected. The tallest of these 20 selected students is Sam. Separately, from each of the 10 rows, the tallest student is selected. The shortest of these 10 selected students is Gwen. Can we know for sure which person is taller, Gwen or Sam?

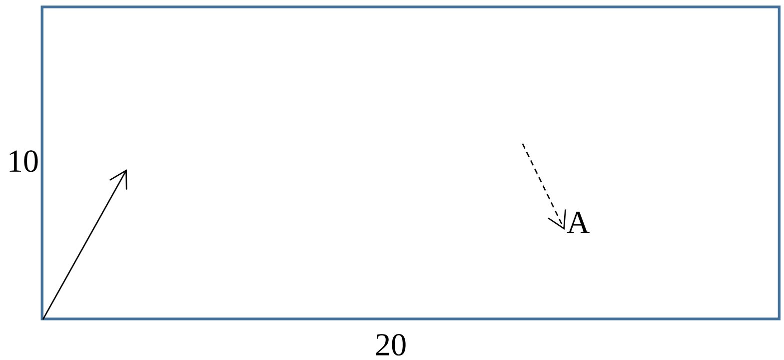
5. Answer the following questions about the region defined by set the of points where $2|x|+|y| \leq 40$.

a. Make a quick sketch of the region. Think efficiency!

b. How many lattice points are in that region? (A lattice point being a point whose coordinates are both integers).

c. Now look at the part of the region where $y \geq 0.$ Is it possible to fit a circle of radius 18 inside the region. Support your answer.

6. A ball leaves the lower left-hand corner of the box below (10-by-20) and bounces until it gets to a corner, at which time it stops.



a. If the initial slope of the ball is 2/5, then in which corner does it finish?

b. (continuation of *a*) How long was the path it took to get there?

c. Give any three different initial slopes that will cause the ball to end in the upper left corner.

d. Point A is 14 units over and 3 units up from the lower-left corner. Give any slope such that the ball leaves from its initial corner, goes through point A on a downward swing, and ends up in one of the right corners in its first pass of the box (ie, it never bounces off of the right wall). Indicate which corner it ends at. (Note: you do not have to give the lowest possible slope for this to occur.)

7. Positive integers are arranged as follows:

1
2 3
4 5 6
7 8 9 10
11 12 13 14 15

a. What number is in the center of the 19^{th} row?

b. What number is in the center of the nth row (assuming n is odd)

c. A row contains the number 2017. What row is it? And what is the sum of the entries in that row?

8. A stick of length 1 is broken in two places at random, creating three smaller sticks. There are many ways that at least one of the resulting sticks is less than 0.15 units long. Find the probability that this occurs. [For example, if the breaks are at 0.32 and 0.81, then the three pieces are 0.32, 0.49, and 0.19 long and it did not occur. But if the breaks are at 0.1 and 0.6 then it did occur. It also occurs if the breaks happen to be at 0.8 and 0.9- since two segments are shorter than 0.15 units]

9. There are many 5-digit numbers contains the digits 2, 3, 5, and 7, and 9 one time each. A few examples are 27,935 and 92,357.

a. What is the 33th highest such number?

b. Find the sum of all such numbers.

10. Three points are selected randomly on a circle (not inside it). A triangle is constructed using those points as vertices. What is the probability that the circle's center is NOT inside this triangle?

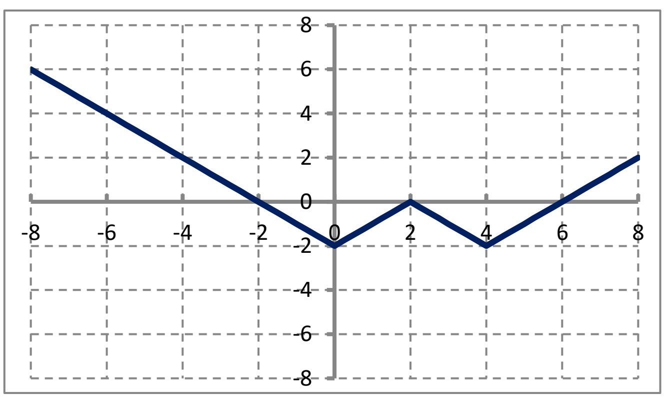
11. Coin flipping game: Ariel and Ben are playing a game. A coin is flipped until one of the following two sequences are observed:

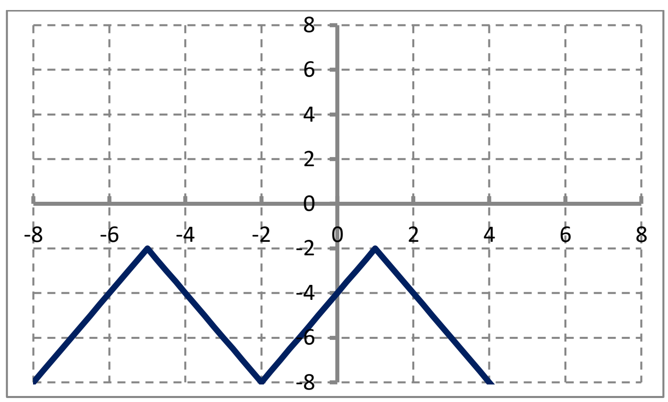
If the last two flips are both tails, then Ariel wins.

If the last two flips are a head and then a tail, Ben wins.

Otherwise the coin continues to be flipped until either TT or HT occurs. What is the probability that Ariel wins?

12. The equations graphed below each have two absolute values in them. Can you determine them? You may try to attack it with algebra, or you may think about what the absolute value function does graphically and "unfold" this graph.





13. A triangle is inscribed in a circle. Its vertices divide the circumference of the circle into arcs of length 3, 4, and 5. Find the area and perimeter of the triangle without using your calculator.

14. If *n* has exactly 7 positive divisors, then how many does n^2 have?

15. There are three boxes of books into which twelve books are randomly placed. One holds 3 books, one holds 4, and one holds 5. Three of the books are math books. What is the probability that they end up in the same box?

16. What is the highest number under 50 that cannot be written as MN+(M+N) where M, N are positive integers (they may or may not be equal).

17. Show that among any seven distinct positive integers not greater than 126, one can find two of them, say *x* and *y*, where $1 < \frac{x}{y} \le 2$.

18. Three lines can divide a plane into at most 7 regions. How about 4, 5, or *n* lines?

19. Let [x] = greatest integer function, in other words: the greatest integer not greater than x. For example $[1.5] = 1,\ [-2.8] = -3,\ and\ [7] = 7$.

a. Find $[y] - [1 - y]$.

b. Let $f(x) = \frac{[x]}{x}$. What is $f(x)$ when *x* is an integer? A positive non-integer? A negative non-integer? Is $\lim_{x \to \infty} f(x)$ defined? How about $\lim_{x \to 7} f(x)$? How about $\lim_{x \to 7^+} f(x)$?

c. Solve x[x]=11.

20. For positive integers *x*, *y*, and *z*, how many solutions are there to $x + y + z = 7$? $x + y + z = 9$? To $x + y + z = A$? (where *A*>3)

21. There are many squares on a *n*x*n* checkerboard. How many? Note, this includes all 2x2 squares, 3x3 squares… etc., not just the 1x1 squares!

22. Given that $f(x) = ax^7 + bx^5 + cx^3 - 6$ and $f(5) = 18$, what is $f(-5)$?

23. Find the exact value of (no calculators!) $\frac{1}{3} - \frac{2}{9} + \frac{3}{27} - \frac{4}{81} + \frac{5}{243} + \ldots.. \frac{(-1)^{n+1} n}{3^n}$

24. All of the integers beginning with 1 are written out like 12345678910111213141516…. What digit occupies the one millionth place?

25. A moving particle starts at the point (4,4) and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (*a*,*b*), it moves at random to one of the points (*a*-1,*b*), (*a*,*b*-1), or (*a*-1,*b*-1), with equal probability, independently of its previous moves. Find the probability that it will hit the coordinate axes at (0,0). [old AIME question]

Answers

1. off top and bottom: vert change =23 and hor=10 so 23/10

Off left and right: vert chg is 7 and horizo is 30 so 7/30

2. I get numbers up to and including 370 so 67 zeros over 1002 digits

3. 2/3

4. Gwen is taller than at least one person in each of the other columns… so the shortest person in each column should be shorter (or = to Gwen…)

5a. a rhombus whose vertices are (-20,0), (20,0), (0,40), and (0,-40)

b. 1+5+9+13+…+81+77+….+1 =1640

6a. Slope means it goes 20 horizontally and 8 vertically so it needs to go across 5 times (since 40 is LCM of 8 and 10) and ends it lower right corner

b. "unfold" the right triangle: 100 across and 40 vertically so hypotenuse is 10*sqrt(116)=20*sqrt(29)

c. Rise of 10 and run of 40 so ¼ ; rise of 30 and run of 40 so ¾; rise of 10 and run of 80 so 1/8….

d. to come down to A is must have run of 14 and "rise" of 17, 37, 57. 77, 97,….

From A to upper right corner is run of 6 and rise of 13, 33, 53,…

From A to lower right corner is run of 6 and rise of 3, 23, 43…

We want slopes to be equal so $\frac{20n-3}{14}=\frac{10m+3}{6}$ where m and n are integers. 77/14 = 33/6 so the slope can be 77/14=5.5

7a. right most of nth row is triangular numbers... 1+2+3+... n

Rightmost of 19^{th} row is 1+2+...19 =190 but we want the 10^{th} number in that row which is 181

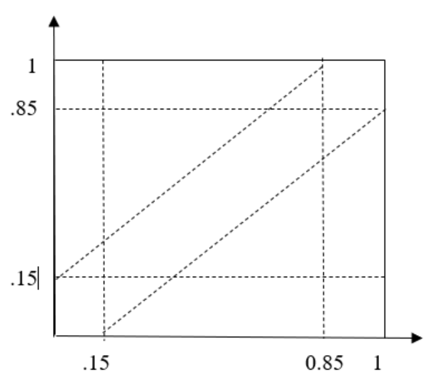
b. last number of nth row is 1+2+3+...n = $\frac{n(n+1)}{2}$ then subtract $\frac{n-1}{2}$ to get center so $\frac{n^2+1}{2}$

c. Last entry in each row is triangular numbers 1, 1+2, 1+2+3… which are (n)(n+1)/2

so where is n(n+1)/2 first over 2017? N^2+n=4034 (n+0.5)^2 = 4034.5 so need 64^{th} row

64^{th} row ends in 2080 and has 64 entries, so it starts with 2017 and sum is 32*(4097)=131,104

8.



The prob it does not happen is the two triangles that are 0.55 by 0.55 so 0.55^2 is not and yes is 1-(0.55^2)

9a. top 24 start with 9 next 24 with 7 so it starts with 7

Of 24 numbers starting with 7, it is the ninth highest … 6 start with 79 then 6 with 75 so it is the third highest starting with 75   which is 75392

b. there are 4!=24 with 2 in the one's digit and 24 with 3 in the one's digit etc…

so sum of ones digit is 24*(2+3+5+7+9)=624 and same with 10's digit etc.

so 24*26*11111=6,933,264

10. I get ¾

11. ¼ → she can only with if the first 2 flips are TT.. else there must be HT before TT

12a. $y = ||x-2|-2|-2$ b. $y = -|-2|x+2|+2|-2$

13. r=6/π area =0.5*r^2(sin150+sin90+sin120)= $\dfrac{27+9\sqrt{3}}{\pi^2}$ perim= $\dfrac{108+36\sqrt{3}}{\pi^2}$

14. 13   15. $\dfrac{{}_3C_3}{{}_{12}C_3} + \dfrac{{}_3C_3 \cdot {}_9C_1 \cdot}{{}_{12}C_4} + \dfrac{{}_3C_3 \cdot {}_9C_2 \cdot}{{}_{12}C_5}$ 16. 46

17. 2^7=128 so can't work.. they cannot all be more than 2 times the next smaller one

18. 11, 16, $(n+1)\cdot\dfrac{n}{2}+1$

19a. 2y-1 if y is an integer, else 2[y]

b. 1 ; <1; >1; yes (to infinity); not to 7; yes to 7 from the positive side…   c. 11/3

20. for 7: if x=1 there are 5; if x=2 there are 4… if x=5 there;s 1 so 5+4+3+2+1 = 15

For 9: 7+6+5+4+3+2+1= 28

For A: (A-2)+ (A-3) + …. 1 = $\dfrac{(A-1)(A-2)}{2}$

21. n^2 are 1x1; $(n-1)^2$ are 2x2.. $(n-2)^2$ are 3x3.. … 1 is nxn. So this is $\sum_{i=1}^{n} i^2$. Which, if you recall from Cal B (or accel precalc) is $\dfrac{n(n+1)(2n+1)}{6}$

22. -30 (besides for the -6, all of the other coefficients would flip signs)

23.

$$\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\ldots=\frac{1/3}{1-(-1/3)}=\frac{1}{4}$$

write as $-\dfrac{1}{9}+\dfrac{1}{27}-\dfrac{1}{81}+\ldots=\dfrac{-1/9}{1-(-1/3)}=-1/12$ so sum of these sums is $\dfrac{1/4}{1-(-1/3)}=3/16$

$$+\frac{1}{27}-\frac{1}{81}+\ldots=\frac{1/27}{1-(-1/3)}=\frac{1}{36}$$

24. the numbers with 1-5 digits take a total of 488,889 places so it is a 6-digit number

1,000,000-488,889 = 511,111 / 6 = 85185.167 so the first 85185 6-digit #s bring us to a total of 999,999 digits

So it is the first digit of the 85,186th 6-digit number… which must be a 1 (I think!)

25. 245/3^7

Unit 1 Handout #5: Practice for the Test

1. (Warm up): 20 camels eat 20 bananas in 20 minutes. How long will it take 80 camels to eat 100 bananas? (assuming camels share well, bananas are easily divided, and camels' appetites are such that they do not fill up until they've had at least hundreds of bananas!)

2. Pick two numbers at random from the interval [0,1]. Break a stick of length one in these two places. What is the probability that the longest piece is at least 0.6 units long?

3. Find the highest value of n such that $\frac{200!}{9^n}$ is an integer. No calculator.

4. A bug starts at the origin on a number line and takes 6 hops. Each hop may be in either direction. The first hop is one unit, the next 2 units, the next 4, then 8, then 16, and the last hop is 32.

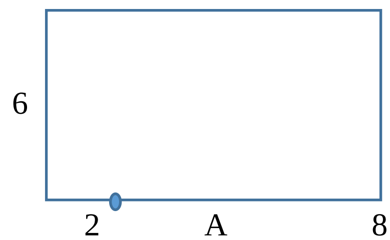
a. Describe the set of points it can finish on.

b. List all possible hops that lead to an ending position of 17 or explain why there are none.

c. Now the bug starts at the origin of a coordinate plane and does the same series of 6 hops, but each can be in any of the four directions (up, down, left, or right). Describe the set of possible ending points.

5. Many pairs of prime numbers above ten are two apart from each other: for example, 11 and 13, 29 and 31, 101 and 103. Explain why the number between these pairs of primes must be divisible by six.

6. A ball starts at point A on the bottom of a 10x6 box, 2 units from the corner. It leaves with a positive slope and bounces forever, only hitting the bottom at 2 distinct places (point A is one of them). Give any 2 slopes that make this possible, and, for each slope, give the distance the ball travels between successive visits to point A.



7. All positive integers are written in a two-dimensional grid, spiraling clockwise around away from the center. The beginning is this: (the grid grows as necessary!)

	7	8	9	10	
	6	1	2	11	
	5	4	3	12	
			14	13	

a. Go eight columns to the right of the 1 and two rows up. What number is there?

b. What number occupies the position to the left of the number 500?

c. For a positive integer *n*, what number is *n* units left of the number 1 in the grid?

8. Answer the following questions about the region defined by $|x-2|+|y| \le 20$

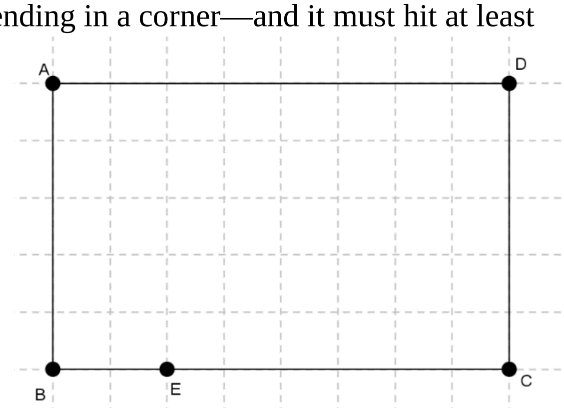
a. What are the range and domain of this relation?

b. A lattice point is defined as a point whose coordinates are both integers (positive, negative, or zero). How many lattice points are in the region?

9.A ball bounces across a rectangle 8 units long and 5 units wide. It starts at point E on the bottom, 2 units from a corner. The ball hits a wall exactly five times before ending in a corner—and it must hit at least three different walls.

a. If it ends at corner A, given on possible slope.

b. If it ends at corner B, given on possible slope.



10. Positive integers except those that are multiples of 9 or 7 (or both) are listed from low to high. What is the 10,000th such integer?

11. Players A and B are playing a game where the first player to make a shot wins. Player A goes first and player A has a 30% chance of making each shot.

a. If Player B's chance of making a shot is also 30%, then what is the probability that player A wins?

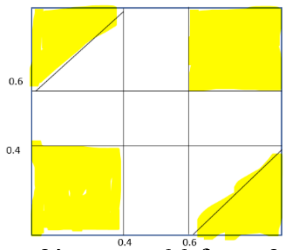
b. Explain why the equation p+0.7p=1 can be used to answer part a above.

c. Instead of player B's probability of making a shot being 30%, what would it have to be so that the players are equally likely to win the game?

Answers

1. 25 minutes

2. 48%



3. how many 3's …. 66 from 3 plus 22 more from 9 plus 7 more from 27 plus 2 more from 81

97 factors of 3 so 48.5 nines and answer is 48

4a. odd integers [-63,63] b. LLLRLR c. lattice pts in |x|+|y|<=63 where one coordinate is even and one is odd

5. any 3 consec needs one to be divisible by 3; in any 3 consec needs at least one / by 2 so must be middle one for both

6. 24/20 and 24/60 are two distances are sqrt(976) and sqrt(4176)

7a. 231 b. 593 c. 1 + (1+4) + (1+4+9) + lower-eft corner is avg of consec odd squares

2 to left avg of 3^2 and 5^2 ,… 3 to left avg of 5^2 and 7^2

N to left is avg if (2n+1)^2 and (2n-1)^2 which is 4n^2 +1 so 4n^2 + 1 + then add n so 4n^2+n+1

8. -18<= x <=22 b. 841

9a. ½ after hitting RTLBR b. -10/9 after hitting LTBRT or 1/3 (there are other solutions)

10. 13124? Of each 63 there are 48 OK… so 10000/48=208.xxx

So 208*63=13104… after that there are 208*48=9984 numbers.. need 16 more which is 20 counting numbers so 13124

11a. A wins on first shot is 0.3; A wins on her 2nd shot is (0.7)(0.7)(0.3)… geometric series

$$S = \frac{a}{1-r} = \frac{0.3}{1-0.49} = 10/17$$

b. if A misses (70% of the time) the B has a p chance of winning (since it is the same game with B going first). So p+0.7p=1 and p=10/17… cool!

c. A wins on first shot is 0.3; let's B's chance be 1-x. A wins on first shot is 0.3; on her 2nd shot is

(0.7)(x)(0.3); on her 3rd shot is (0.7)(x)(0.7)(x)(0.3) geometric whose sum is $\frac{0.3}{1-0.7x}$. That is equal to 0.5 when x is (4/7) so B needs to have 3/7 chance of making a shot.